FORMULAS OF RAMANUJAN FOR THE POWER SERIES COEFFICIENTS OF CERTAIN QUOTIENTS OF EISENSTEIN SERIES

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In Memory of Robert A. Rankin

1. Introduction

In their last published paper [9], [16, pp. 310–321], G. H. Hardy and S. Ramanujan derived infinite series representations for the coefficients of certain modular forms of negative weight which are not analytic in the upper half-plane. In particular, they examined in detail the coefficients of the reciprocal of the Eisenstein series $E_6(\tau)$.

While confined to the sanitarium, Matlock House, in 1918, Ramanujan wrote several letters to Hardy about the coefficients in the power series expansions of certain quotients of Eisenstein series. These letters are photocopied in [18, pp. 97–126], and printed versions with commentary can be found in [6, pp. 175–191]. In these letters, Ramanujan recorded formulas for the coefficients of several quotients of Eisenstein series not examined by Hardy and him in [9]. These claims fall into two related classes. In the first class are formulas for coefficients that arise from the main theorem of Hardy and Ramanujan, or a slight modification of it, and these results have been proved in a paper by Berndt and Bialek [5]. Those in the second class, which we prove in this paper, are much harder to prove. To establish the first main result, we need an extension of Hardy and Ramanujan's theorem due to H. Petersson [11]. To prove the second primary result, we need to first extend work of H. Poincaré [14], Petersson [11], [12], [13], and J. Lehner [10] to cover double poles. In all cases, the formulas have a completely different shape from those arising from modular forms analytic in the upper half-plane, such as the famous infinite series for the partition function $p(n)$ arising from the reciprocal of the Dedekind eta-function. As we shall see in the sequel, the series examined in this paper are very rapidly convergent, even more so than those arising from modular forms analytic in the upper half-plane, so that truncating a series, even with a small number of terms, provides a remarkable approximation. Using *Mathematica*, we calculated several coefficients and series approximations for the two primary functions $1/B(q)$ and $1/B^2(q)$ (defined below) examined by Ramanujan. As will be seen from the first table, the coefficient of q^{10} in $1/B(q)$, for example, has 17 digits, while just two terms of Ramanujan's infinite series representation calculate this coefficient with

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an error of approximately .0003. Although we will not provide details, we calculated the coefficients of $1/B^2(q)$ up to $n = 50$. To demonstrate the rapid convergence of Ramanujan's series, we remark that for $n = 20, 30, 40$, and 50, the coefficients have, respectively, 29, 43, 57, and 70 digits, while two-term approximations give, respectively, 29, 42, 55, and 66 of these digits. The purpose of this paper is to provide proofs for these remarkable formulas from Ramanujan's letters. There are two primary formulas; three further formulas follow from one of the two key formulas.

Different kinds of formulas for coefficients of modular forms have recently been established by J. H. Bruinier, W. Kohnen, and K. Ono [8].

In Ramanujan's notation, the three relevant Eisenstein series are defined for $|q| < 1$ by

$$P(q) := 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}, \tag{1.1}$$

$$Q(q) := 1 + 240\sum_{k=1}^{\infty} \frac{k^3q^k}{1-q^k}, \tag{1.2}$$

and

$$R(q) := 1 - 504\sum_{k=1}^{\infty} \frac{k^5q^k}{1-q^k}. \tag{1.3}$$

Central for our proofs are Ramanujan's differential equations [15, eqs. (30)], [16, p. 142]

$$q\frac{dP}{dq} = \frac{P^2(q) - Q(q)}{12}, \tag{1.4}$$

$$q\frac{dQ}{dq} = \frac{P(q)Q(q) - R(q)}{3}, \tag{1.5}$$

and

$$q\frac{dR}{dq} = \frac{P(q)R(q) - Q^2(q)}{2}. \tag{1.6}$$

In more contemporary notation, the Eisenstein series $E_{2j}(\tau)$ is defined for $j > 1$ and $\operatorname{Im}\tau > 0$ by

$$E_{2j}(\tau) := \frac{1}{2}\sum_{\substack{m_1,m_2\in\mathbf{Z}\\(m_1,m_2)=1}} (m_1\tau + m_2)^{-2j} = 1 + \frac{(2\pi)^{2j}(-1)^j}{\zeta(2j)(2j-1)!}\sum_{k=1}^{\infty} \frac{k^{2j-1}e^{2\pi ik\tau}}{1-e^{2\pi ik\tau}}, \tag{1.7}$$

where $\zeta(s)$ denotes the Riemann zeta-function. Thus, for $q = \exp(2\pi i\tau)$, $E_4(\tau) = Q(q)$ and $E_6(\tau) = R(q)$, which have weights 4 and 6, respectively [20, p. 50]. Since (1.7) does not converge for $j = 1$, the Eisenstein series $E_2(\tau)$ must be defined differently. First let

$$E_2^*(\tau) := P(q), \qquad q = e^{2\pi i\tau}. \tag{1.8}$$

Then $E_2(\tau)$ is defined by

$$E_2(\tau) := E_2^*(\tau) - \frac{3}{\pi \operatorname{Im} \tau}. \tag{1.9}$$

Then $E_2(\tau)$ satisfies the functional equation of a modular form of weight 2 [20, pp. 67–68].

Now define

$$B(q) := 1 + 24 \sum_{k=1}^{\infty} \frac{(2k-1)q^{2k-1}}{1 - q^{2k-1}}, \qquad |q| < 1, \tag{1.10}$$

and define the coefficients b_n by

$$\frac{1}{B(q)} = \sum_{n=0}^{\infty} b_n q^n, \tag{1.11}$$

where $|q| < q_0 < 1$, for q_0 sufficiently small. As will be seen, $B(q)$ is the (unique) modular form of weight 2 with multiplier system identically equal to 1 on the modular group $\Gamma_0(2)$. Next, define the coefficients δ_n by

$$\frac{Q(q)}{R(q)} = \sum_{n=0}^{\infty} \delta_n q^n, \tag{1.12}$$

where $|q|$ is sufficiently small. An infinite series representation for the coefficients δ_n can be found with the publication of the lost notebook [18, pp. 102–104], [6, pp. 179–182], and was first proved by Berndt and Bialek [5, Thm. 3.1]. One of the unresolved claims of Ramanujan is a formula for b_n relating it to δ_n. (See Section 3 for a precise statement.) The second major claim is a formula for the coefficients of the series

$$\frac{1}{B^2(q)}$$

relating them to the coefficients of the series

$$\frac{1}{R(q)} \qquad \text{and} \qquad \frac{P(q)}{R(q)}. \tag{1.13}$$

The principal theorem of Hardy and Ramanujan as well as the extensive generalizations by Poincaré [14], Petersson [11], [12], [13], and Lehner [10] do not provide formulas when poles are of order greater than or equal to 2. In order to prove Ramanujan's second claim, we first then need to prove a corresponding theorem for double poles.

Ramanujan claims that his assertions follow from eight identities for Eisenstein series and theta-functions which he states without proofs at the beginning of his letter [6, pp. 189–190]. Indeed, these eight identities are central to our proofs.

In Section 2, we prove the eight identities cited above. Section 3 contains a proof of the first main theorem. In Section 4, we show that three of Ramanujan's claims are consequences of the claim proved in Section 3. Lastly, in Section 5, we first prove an analogue for double poles of Hardy and Ramanujan's chief theorem, after which we prove Ramanujan's formula for the coefficients of $1/B^2(q)$.

Throughout the paper, we consider quotients of Eisenstein series which are not analytic in the upper half-plane. Each quotient is analytic in some disc, $|q| < q_0 < 1$, where q_0 is not necessarily the same at each appearance. The residue of a function

$f(\tau)$ at a pole α is denoted by $\mathrm{Res}(f, \alpha)$. The full modular group is denoted by $\Gamma(1)$, and the modular subgroup $\Gamma_0(2)$ of $\Gamma(1)$ is defined by

$$\Gamma_0(2) = \left\{ T(\tau) = \frac{a\tau + b}{c\tau + d} \;\middle|\; a, b, c, d \in \mathbf{Z}; ad - bc = 1; c \text{ even} \right\}.$$

2. Eight Identities for Eisenstein Series and Theta-functions

Two of the identities involve the classical theta-functions (in Ramanujan's notation),

$$\varphi(q) := \sum_{k=-\infty}^{\infty} q^{k^2} \qquad \text{and} \qquad \psi(q) := \sum_{k=0}^{\infty} q^{k(k+1)/2}. \tag{2.1}$$

To establish the eight identities, we need to use evaluations of theta-functions and Eisenstein series from Chapter 17 of Ramanujan's second notebook [17], [3, pp. 122–138]. If

$$q := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \qquad |x| < 1,$$

where ${}_2F_1$ denotes the ordinary hypergeometric function, these evaluations are given in terms of, in Ramanujan's notation,

$$z := {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x), \qquad |x| < 1, \tag{2.2}$$

and x.

Theorem 2.1. *Recall that $Q(q)$ and $R(q)$ are defined by* (1.2) *and* (1.3)*, respectively, that $B(q)$ is defined by* (1.10)*, and that $\varphi(q)$ and $\psi(q)$ are defined in* (2.1)*. Then*

$$\text{(i)} \qquad B(\sqrt{q}) + B(-\sqrt{q}) = 2B(q),$$

$$\text{(ii)} \qquad B(\sqrt{q})B(-\sqrt{q})B(q) = R(q),$$

$$\text{(iii)} \qquad 5B^2(q) = Q(q) + 4Q(q^2),$$

$$\text{(iv)} \qquad 7B^3(q) = -R(q) + 8R(q^2),$$

$$\text{(v)} \qquad \frac{2}{3}\left(\frac{1}{B(\sqrt{q})} + \frac{1}{B(-\sqrt{q})}\right) - \frac{1}{3B(q)} = \frac{Q(q)}{R(q)},$$

$$\text{(vi)} \qquad \frac{11}{24}\left(\frac{1}{B(\sqrt{q})} + \frac{1}{B(-\sqrt{q})}\right) + \frac{1}{12B(q)} = \frac{Q(q^2)}{R(q)},$$

$$\text{(vii)} \qquad 15\varphi^8(-q) = -Q(q) + 16Q(q^2),$$

$$\text{(viii)} \qquad 240q\psi^8(q) = Q(q) - Q(q^2).$$

Proof of (i). The proof is straightforward, with only the definition of $B(q)$ in (1.10) needed in the proof. □

Proof of (ii). By Part (i), with the replacement of $\sqrt{q}$ by q, we can rewrite (ii) in the form

$$B(q)\left\{2B(q^2)-B(q)\right\}B(q^2)=R(q^2). \tag{2.3}$$

By (1.10), we easily see that

$$2P(q^2)-P(q)=B(q). \tag{2.4}$$

¿From Entries 13(viii), 13(ix), and 13(ii) in Chapter 17 of Ramanujan's second notebook [3, pp. 126–127],

$$B(q)=2P(q^2)-P(q)=z^2(1+x), \tag{2.5}$$

$$B(q^2)=2P(q^4)-P(q^2)=z^2(1-\tfrac{1}{2}x), \tag{2.6}$$

$$R(q^2)=z^6(1+x)(1-\tfrac{1}{2}x)(1-2x), \tag{2.7}$$

where z is defined in (2.2). Using the evaluations (2.5)–(2.7) in (2.3), we find that each side of (2.3) equals

$$z^6(1+x)(1-\tfrac{1}{2}x)(1-2x),$$

which completes the proof of (ii). □

Proof of (iii). By Entries 13(iii) and 13(i) in Ramanujan's second notebook [3, pp. 126–127],

$$Q(q)=z^4(1+14x+x^2) \quad \text{and} \quad Q(q^2)=z^4(1-x+x^2). \tag{2.8}$$

Using (2.5) and (2.8) in (iii), we find that each side of (iii) reduces to

$$z^4(5+10x+5x^2),$$

thus establishing the truth of (iii). □

Proof of (iv). By Ramanujan's work in Chapter 17 of his second notebook [3, Entry 13(iv), p. 127],

$$R(q)=z^6(1+x)(1-34x+x^2). \tag{2.9}$$

Thus, by (2.9), (2.5), and (2.7), each side of (iv) can be written in the form

$$7z^6(1+x)^3.$$

This completes the proof of (iv). □

Proof of (v). Replacing $\sqrt{q}$ by q and using (i), we can rewrite (v) in the form

$$\frac{2}{3}\left(\frac{1}{B(q)}+\frac{1}{2B(q^2)-B(q)}\right)-\frac{1}{3B(q^2)}=\frac{Q(q^2)}{R(q^2)}. \tag{2.10}$$

By (2.5) and (2.6),

$$\frac{1}{B(q)}+\frac{1}{2B(q^2)-B(q)}=\frac{1}{z^2(1+x)}+\frac{1}{z^2(1-2x)}. \tag{2.11}$$

Utilizing (2.11) and (2.6) and employing a heavy dosage of elementary algebra, we find that the left side of (2.10) reduces to

$$\frac{1-x+x^2}{z^2(1+x)(1-2x)(1-\frac{1}{2}x)}. \tag{2.12}$$

On the other hand, by (2.7) and (2.8), the right side of (2.10) also reduces to (2.12). This completes then the proof of (2.10), and hence of (v). ☐

Proof of (vi). Replacing $\sqrt{q}$ by q and using (i), we can rewrite (vi) in the form,

$$\frac{11}{24}\left(\frac{1}{B(q)}+\frac{1}{2B(q^2)-B(q)}\right)+\frac{1}{12B(q^2)}=\frac{Q(q^4)}{R(q^2)}. \tag{2.13}$$

Using (2.11) and (2.6), we find that the left side of (2.13) takes the shape

$$\frac{1-x+\frac{1}{16}x^2}{z^2(1+x)(1-2x)(1-\frac{1}{2}x)}. \tag{2.14}$$

Again, from Ramanujan's work [3, p. 127, Entry 13(v)],

$$Q(q^4)=z^4(1-x+\tfrac{1}{16}x^2), \tag{2.15}$$

so that, by (2.15) and (2.7), the right side of (2.13) also reduces to (2.14). This completes the proof of (vi). ☐

Proof of (vii). By Entry 10(ii) in Chapter 17 of Ramanujan's second notebook [3, p. 122],

$$\varphi(-q)=\sqrt{z}(1-x)^{1/4}. \tag{2.16}$$

Thus the left side of (vii) takes the shape

$$15z^4(1-x)^2, \tag{2.17}$$

while by (2.8) the right side of (vii) also equals (2.17). ☐

Proof of (viii). Appealing again to Chapter 17 of Ramanujan's second notebook [3, p. 123, Entry 11(i)], we have

$$\psi(q)=\sqrt{\frac{1}{2}z}(x/q)^{1/8}. \tag{2.18}$$

Thus, by (2.18), the left side of (viii) equals

$$240(\tfrac{1}{16}z^4x)=15z^4x, \tag{2.19}$$

while, by (2.8), the right side of (viii) equals (2.19) as well. ☐

3. The Coefficients of $1/B(q)$

We begin this section by giving a complete statement of one of the theorems from [5, Thm. 4.1].

Theorem 3.1. *Let, for* $|q| < q_0$,

$$\frac{Q(q^2)}{R(q^2)} =: \sum_{n=0}^{\infty} \delta_n q^{2n} \tag{3.1}$$

and

$$J := Q(e^{-2\pi}) = 1 + 240 \sum_{k=1}^{\infty} \frac{k^3}{e^{2\pi k} - 1} = 1.45576\ldots. \tag{3.2}$$

Then, if $n \geq 0$,

$$\begin{aligned}\delta_n =& \frac{2}{J} \left\{ e^{2n\pi} - \frac{(-1)^n}{2^2} e^{n\pi} + \frac{2\cos\left(\frac{4\pi n}{5} + 4\tan^{-1} 2\right)}{5^2} e^{2n\pi/5} \right. \\ & \left. + \frac{2\cos\left(\frac{3\pi n}{5} + 4\tan^{-1} 3\right)}{10^2} e^{2n\pi/10} + \cdots \right\} \\ :=& \frac{2}{J} \sum_{(\mu)} \frac{v_\mu(n)}{\mu^2} e^{2n\pi/\mu}. \end{aligned} \tag{3.3}$$

Here, μ *runs over the integers of the form*

$$\mu = 2^a \prod_{j=1}^{r} p_j^{a_j}, \tag{3.4}$$

where $a = 0$ *or* 1, p_j *is a prime of the form* $4m+1$, *and* a_j *is a nonnegative integer,* $1 \leq j \leq r$. *Furthermore,*

$$v_1(n) = 1, \tag{3.5}$$

$$v_2(n) = (-1)^{n+1}, \tag{3.6}$$

and, for $\mu \geq 5$,

$$v_\mu(n) = \sum_{c,d} 2\cos\left((ac+bd)\frac{2n\pi}{\mu} + 4\tan^{-1}\frac{c}{d}\right), \tag{3.7}$$

where the sum is over all pairs (c,d), *where* (c,d) *is a distinct solution to* $\mu = c^2 + d^2$ *and* (a,b) *is any solution to* $ad - bc = 1$. *Also, distinct solutions* (c,d) *to* $\mu = c^2 + d^2$ *give rise to distinct terms in the sum in* (3.7). *If* $n < 0$, *then the sum on the far right side of* (3.3) *equals* 0.

The definition of a *distinct solution* is given prior to the statement of Theorem 4.1 in [5]. Roughly, if (c, d) is a distinct solution, then another solution obtained by changing the order of (c, d) or the sign of either c or d would not be regarded as distinct from the original solution (c, d).

We are now ready to state the first main theorem, which establishes an assertion of Ramanujan from his letter to Hardy containing Theorem 3.1 [18, p. 117], [6, p. 190]. For the sake of brevity, we write

$$\delta_n = \sum_{(\mu)} V_\mu(n), \tag{3.8}$$

where

$$V_\mu(n) = \frac{2}{J}\frac{v_\mu(n)}{\mu^2}e^{2n\pi/\mu}, \tag{3.9}$$

where $v_\mu(n)$ is defined by (3.5)–(3.7).

Theorem 3.2. *Recall that the coefficients b_n are defined by* (1.11)*. Then, with $V_\mu(n)$ defined by* (3.9)*,*

$$b_n = -3\sum_{(\mu_e)} V_{\mu_e}(n), \tag{3.10}$$

where μ_e runs over the even values of μ. In other words, μ_e runs over the even integers of the form (3.4)*.*

Set $\mathbf{B}(\tau) = B(q)$, where $q = \exp(2\pi i\tau)$. Then $\mathbf{B}(\tau)$ is a modular form on $\Gamma_0(2)$, as we show in the next lemma. We remark that $\mathbf{B}(\tau)$ is not a modular form on $\Gamma(1)$, because the dimension of the space of modular forms of weight 2 with multiplier system identically equal to 1 on $\Gamma(1)$ is zero [19, p. 103].

Lemma 3.3. *The function $\mathbf{B}(\tau)$ is a modular form of weight 2 and multiplier system identially equal to 1 on the group $\Gamma_0(2)$. That is,*

$$\mathbf{B}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\mathbf{B}(\tau),$$

where a, b, c, $d \in \mathbf{Z}$; $ad - bc = 1$; and c is even.

Proof. Recall that $E_2^*(\tau)$ is defined by (1.8). Thus, by (2.4),

$$2E_2^*(2\tau) - E_2^*(\tau) = \mathbf{B}(\tau). \tag{3.11}$$

Recall [20, pp. 50, 68] that for any modular transformation $(a\tau + b)/(c\tau + d)$,

$$E_2^*\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2E_2^*(\tau) - \frac{6ci}{\pi}(c\tau+d), \tag{3.12}$$

and so

$$E_2^*\left(2\frac{a\tau+b}{c\tau+d}\right) = E_2^*\left(\frac{a(2\tau)+2b}{\frac{c}{2}(2\tau)+d}\right) = (c\tau+d)^2E_2^*(2\tau) - \frac{3ci}{\pi}(c\tau+d), \tag{3.13}$$

for c even. Thus, by (3.12) and (3.13),

$$2E_2^*\left(2\frac{a\tau+b}{c\tau+d}\right)-E_2^*\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2\left(2E_2^*(2\tau)-E_2^*(\tau)\right). \tag{3.14}$$

By (3.14) and (3.11), we complete the proof. □

Lemma 3.4. *The function* $1/\mathbf{B}(\tau)$ *has a pole at* $\tau=\frac{1+i}{2}$.

By Lemmas 3.3 and 3.4, it follows that $1/\mathbf{B}(\tau)$ has poles at the points $\tau=(a\frac{1}{2}(1+i)+b)/(c\frac{1}{2}(1+i)+d)$, where a, b, and d are integers, c is an even integer, and $ad-bc=1$. By the valence formula [19, p. 98], there are no further poles of $1/\mathbf{B}(\tau)$ in a fundamental region of $\Gamma_0(2)$.

Proof of Lemma 3.4. By (3.11) and (1.9), we easily see that

$$\frac{1}{\mathbf{B}(\tau)}=\frac{1}{2E_2(2\tau)-E_2(\tau)}. \tag{3.15}$$

We show that both functions in the denominator of (3.15) vanish at $\tau=(1+i)/2$. First, by (1.9),

$$E_2(i)=1-24\sum_{k=1}^{\infty}\frac{ke^{-2\pi k}}{1-e^{-2\pi k}}-\frac{3}{\pi}=0, \tag{3.16}$$

by a well-known result established many times in the literature. See, for example, [1, p. 159] or [2, p. 256], where references to several proofs can be found. Thus, by periodicity and (3.16),

$$E_2\left(2\frac{1+i}{2}\right)=E_2(1+i)=E_2(i)=0,$$

and, since $E_2(\tau)$ is a modular form of weight 2,

$$E_2\left(\frac{1+i}{2}\right)=E_2\left(\frac{i}{1+i}\right)=E_2\left(\frac{0+1i}{1+1i}\right)=(1+i)^2E_2(i)=0.$$

Thus, both terms in the denominator of (3.15) vanish, and the proof of the lemma is therefore complete. □

The main theorem of Hardy and Ramanujan [9, Thm. 1], [16, p. 312], [5, Thm. 2.1] is therefore inapplicable. However, Poincaré [14, pp. 210–215, 432–462, 606–614], Lehner [10], and particularly Petersson [11, pp. 460–461, Satz 3], [12], [13] have extensively generalized Hardy and Ramanujan's theorem. We only need the special case for the subgroup $\Gamma_0(2)$, which we state below.

Theorem 3.5. *Suppose that* $f(q)=f(e^{\pi i\tau})=\phi(\tau)$ *is analytic for* $q=0$, *is meromorphic in the unit circle, and satisfies the functional equation*

$$\phi(\tau)=\phi\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^n, \tag{3.17}$$

where a, b, c, $d \in \mathbf{Z}$; $ad - bc = 1$; c is even; and $n \in \mathbf{Z}^+$. If $\phi(\tau)$ has only one pole in a fundamental region for $\Gamma_0(2)$, a simple pole at $\tau = \alpha$ with residue A, then

$$f(q) = -2\pi i A \sum \frac{1}{(c\alpha + d)^{n+2}} \frac{1}{1 - (q/\underline{q})^2}, \qquad |q| < q_0, \tag{3.18}$$

where

$$\underline{q} = \exp\left(\left(\frac{a\alpha + b}{c\alpha + d}\right)\pi i\right), \tag{3.19}$$

and the summation runs over all pairs of coprime integers (c, d) (with c even) which yield distinct values for the set $\{\underline{q}, -\underline{q}\}$, and a and b are any integral solutions of

$$ad - bc = 1. \tag{3.20}$$

We are now prepared to prove Theorem 3.2.

Proof of Theorem 3.2. Using (2.4) and (1.4), we find that

$$\begin{aligned} B'(q) =& 4qP'(q^2) - P'(q) \\ =& \frac{P^2(q^2) - Q(q^2)}{3q} - \frac{P^2(q) - Q(q)}{12q} \\ =& \frac{1}{12q}\left\{\left(2P(q^2) - P(q)\right)\left(2P(q^2) + P(q)\right) - 4Q(q^2) + Q(q)\right\} \\ =& \frac{1}{12q}\left\{B(q)\left(2P(q^2) + P(q)\right) - 4Q(q^2) + Q(q)\right\}. \end{aligned} \tag{3.21}$$

By Lemma 3.4, $B(-e^{-\pi}) = 0$, and therefore from Theorem 2.1 (iii), we deduce that

$$Q(-e^{-\pi}) = -4Q(e^{-2\pi}). \tag{3.22}$$

Hence, setting $q = -e^{-\pi}$ in (3.21), we find that

$$B'(-e^{-\pi}) = \frac{1}{12(-e^{-\pi})}\left\{-8Q(e^{-2\pi})\right\} = \frac{2}{3}e^{\pi}Q(e^{-2\pi}), \tag{3.23}$$

which is explicitly calculated in Proposition 3.8 below.

We now apply Theorem 3.5 to $\phi(\tau) = f(q) := 1/B(q^2)$, where $q = e^{\pi i\tau}$ and $\alpha := (1+i)/2$. By the chain rule,

$$A = \mathrm{Res}(\phi, \alpha) = \frac{\mathrm{Res}(f, e^{\pi i\alpha})}{\pi i e^{\pi i\alpha}}, \tag{3.24}$$

and, by (3.23),

$$\begin{aligned} \mathrm{Res}(f, e^{\pi i\alpha}) =& \mathrm{Res}\left(\frac{1}{B(q^2)}, e^{\pi i\alpha}\right) = \left.\frac{1}{dB(q^2)/dq}\right|_{q=e^{\pi i\alpha}} \\ =& \left.\frac{1}{2qB'(q^2)}\right|_{q=e^{\pi i\alpha}} = \frac{1}{2e^{\pi i\alpha}}\frac{3e^{-\pi}}{2Q(e^{-2\pi})}. \end{aligned} \tag{3.25}$$

Hence, combining (3.24) and (3.25), we deduce that

$$-2\pi i A = -\frac{2\pi i}{\pi i e^{\pi i\alpha}}\frac{3e^{-\pi}}{4e^{\pi i\alpha}Q(e^{-2\pi})} = \frac{3}{2Q(e^{-2\pi})}. \tag{3.26}$$

We next calculate $\underline{q}$. Recall that $ad - bc = 1$ with c even. Thus, d is odd. Hence,

$$\begin{aligned}\underline{q} &= \exp\left(\pi i \frac{a\alpha + b}{c\alpha + d}\right) = \exp\left(\pi i \frac{a + 2b + ai}{c + 2d + ci}\right)\\ &= \exp\left(\pi i \frac{(a + 2b + ai)(c + 2d - ci)}{c^2 + 4cd + 4d^2 + c^2}\right)\\ &= \exp\left(\pi i \frac{(a + b)(c + d) + bd + i}{(c + d)^2 + d^2}\right).\end{aligned}$$

Let $\mu = c'^2 + d^2$, where $c' = c + d$, and let $a' = a + b$. Note that $a'd - bc' = 1$ and that μ is even. Thus,

$$\underline{q} = \exp\left(\pi i \frac{a'c' + bd + i}{\mu}\right) = \exp\left(-\frac{\pi}{\mu}\right) \exp\left(\pi i \frac{a'c' + bd}{\mu}\right). \tag{3.27}$$

Next,

$$c\alpha + d = c\frac{1}{2}(1 + i) + d = \frac{1}{1 + i}(ci + d + di) = \frac{1}{1 + i}(c'i + d), \tag{3.28}$$

where $c' = c + d$.

The requisite calculations have now been made in order to apply Theorem 3.5. By (1.11), (3.26), and (3.28), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} b_n q^{2n} = \frac{1}{B(q^2)} &= \frac{3}{2Q(e^{-2\pi})} \sum_{c',d} \frac{1}{(1 + i)^{-4}(c'i + d)^4} \frac{1}{1 - (q/\underline{q})^2}\\ &= -\frac{6}{Q(e^{-2\pi})} \sum_{c',d} \frac{1}{(c'i + d)^4} \frac{1}{1 - (q/\underline{q})^2},\end{aligned} \tag{3.29}$$

where the sum is over all pairs c', d with $a'd - bc' = 1$ and c' odd (since c is even and d is odd), and where $\underline{q}$ is given by (3.27). Now from [5, eq. (4.14)]

$$\sum_{n=0}^{\infty} \delta_n q^{2n} = \frac{Q(q^2)}{R(q^2)} = \frac{2}{Q(e^{-2\pi})} \sum_{c,d} \frac{1}{(ci + d)^4} \frac{1}{1 - (q/\underline{q})^2}, \tag{3.30}$$

where the sum is over *all* pairs c, d with $ad - bc = 1$, and where

$$\underline{q} = \exp\left(\pi i \frac{ai + b}{ci + d}\right) = \exp\left(-\frac{\pi}{\mu}\right) \exp\left(\pi i \frac{ac + bd}{\mu}\right).$$

A comparision of (3.29) and (3.30) shows that the right sides of (3.29) and (3.30) are identical except in two respects. First, in (3.29), there is an extra factor of -3 on the right side. Second, upon expanding the summands in geometric series on the right sides of (3.29) and (3.30), we see that the sum in (3.29) is over only even μ. In other words,

$$b_n = -3 \sum_{(\mu_e)} V_{\mu_e}(n),$$

where $V_{\mu_e}(n)$ is defined by (3.9). This completes the proof. □

The series in (3.10) converges very rapidly to b_n. Using *Mathematica*, we calculated $b_n, 1 \leq n \leq 10$, and the first two terms of (3.10). As the following table shows, only two terms of the series give extraordinary approximations.

n	b_n	$-3(V_2(n)+V_{10}(n))$
1	−24	−23.971586
2	552	551.992987
3	−12,768	−12,768.016604
4	295,464	295,463.973727
5	−6,837,264	−6,837,264.003874
6	158,219,040	158,219,040.077478
7	−3,661,298,112	−3,661,298,112.002170
8	84,724,974,120	84,724,974,119.926326
9	−1,960,594,584,504	−1,960,594,584,504.044556
10	45,369,516,658,032	45,369,516,658,031.999703

Using Theorem 2.1 (v), we can easily establish a formula for δ_n in terms of b_n, but we were unable to use this relation to prove Theorem 3.2.

Proposition 3.6. *For each positive integer n,*

$$\delta_n = \frac{4}{3}b_{2n} - \frac{1}{3}b_n.$$

Proof. By Theorem 2.1 (v), (3.1), and (1.11),

$$\begin{aligned}\sum_{n=0}^{\infty}\delta_n q^n = \frac{Q(q)}{R(q)} &= \frac{2}{3}\left(\frac{1}{B(\sqrt{q})}+\frac{1}{B(-\sqrt{q})}\right)-\frac{1}{3B(q)}\\ &= \frac{2}{3}\left(\sum_{n=0}^{\infty}b_n(\sqrt{q})^n+\sum_{n=0}^{\infty}b_n(-\sqrt{q})^n\right)-\frac{1}{3}\sum_{n=0}^{\infty}b_nq^n\\ &= \frac{4}{3}\sum_{n=0}^{\infty}b_{2n}q^n-\frac{1}{3}\sum_{n=0}^{\infty}b_nq^n.\end{aligned}$$

□

In our previous attempts to prove Theorem 3.2, we showed that $V_\mu(n)$ for μ odd is a multiple of $V_\mu(2n)$ when μ is even. Although we were not able to use this result in our goal, we think the formula is very interesting by itself and so prove it now.

Theorem 3.7. *For each positive integer n,*

$$V_{\mu_o}(n) = -4V_{2\mu_o}(2n), \tag{3.31}$$

where μ_o is an odd integer of the form (3.4).

Proof. We first easily establish the case $\mu_o = 1$. From (3.9), (3.5), and (3.6), we see that

$$V_1(n) = \frac{2}{J}e^{2n\pi},$$

while

$$-4V_2(2n) = \frac{-8}{J}\frac{(-1)^{2n+1}}{2^2}e^{4n\pi/2} = \frac{2}{J}e^{2n\pi}.$$

Now we assume that $\mu_0 > 1$. Suppose that $(a_1,\, b_1,\, c_1,\, d_1)$ satisfies

$$ad - bc = 1, \tag{3.32}$$

with $c^2 + d^2 = \mu_0$. We can assume, without loss of generality, that $a_1 > b_1 > 0$ and that a_1 and b_1 are odd. Since

$$(c_1 + d_1)^2 + (c_1 - d_1)^2 = 2\mu_0,$$

we find that, if we let

$$c_2 = c_1 + d_1 \qquad \text{and} \qquad d_2 = c_1 - d_1, \tag{3.33}$$

then (c_2, d_2) satisfies

$$c^2 + d^2 = 2\mu_0.$$

Similarly, if we let

$$a_2 = -\frac{1}{2}(a_1 + b_1) \qquad \text{and} \qquad b_2 = -\frac{1}{2}(a_1 - b_1), \tag{3.34}$$

then (a_2, b_2, c_2, d_2) satisfies (3.32). By (3.9) and (3.7), in order to prove (3.31), it suffices to prove that

$$\begin{aligned}&\frac{2}{J}\frac{2\cos\left\{(a_1c_1 + b_1d_1)\dfrac{2n\pi}{\mu_0} + 4\operatorname{Arctan}\dfrac{c_1}{d_1}\right\}}{\mu_0^2}e^{2n\pi/\mu_0}\\ &= -\frac{8}{J}\frac{2\cos\left\{(a_2c_2 + b_2d_2)\dfrac{4n\pi}{2\mu_0} + 4\operatorname{Arctan}\dfrac{c_2}{d_2}\right\}}{(2\mu_0)^2}e^{4n\pi/(2\mu_0)},\end{aligned}$$

or equivalently that

$$\begin{aligned}&\cos\left\{(a_1c_1 + b_1d_1)\frac{2n\pi}{\mu_0} + 4\operatorname{Arctan}\frac{c_1}{d_1}\right\}\\ &= -\cos\left\{(a_2c_2 + b_2d_2)\frac{2n\pi}{\mu_0} + 4\operatorname{Arctan}\frac{c_2}{d_2}\right\}.\end{aligned} \tag{3.35}$$

By the identities (3.33) and (3.34), we can rewrite the right-hand side as

$$\begin{aligned}&-\cos\left\{\left(-\frac{1}{2}(a_1 + b_1)(c_1 + d_1) - \frac{1}{2}(a_1 - b_1)(c_1 - d_1)\right)\frac{2n\pi}{\mu_0} + 4\operatorname{Arctan}\frac{c_1 + d_1}{c_1 - d_1}\right\}\\ &= -\cos\left\{(-a_1c_1 - b_1d_1)\,\frac{2n\pi}{\mu_0} + 4\operatorname{Arctan}\frac{c_1 + d_1}{c_1 - d_1}\right\}.\end{aligned} \tag{3.36}$$

Note that

$$\operatorname{Arctan}\frac{c_1 + d_1}{c_1 - d_1} = \frac{k\pi}{4} - \operatorname{Arctan}\frac{c_1}{d_1}, \tag{3.37}$$

where $k \equiv 3 \pmod 4$. Using (3.37), we can rewrite the latter expression in (3.36) as

$$\begin{aligned}&-\cos\left\{-(a_1c_1+b_1d_1)\frac{2n\pi}{\mu_0}-4\operatorname{Arctan}\frac{c_1}{d_1}+k\pi\right\}\\&=\cos\left\{(a_1c_1+b_1d_1)\frac{2n\pi}{\mu_0}+4\operatorname{Arctan}\frac{c_1}{d_1}\right\}.\end{aligned}$$

Thus, (3.35) has been proved, and hence (3.31) as well. □

We close this section by showing that $Q(e^{-2\pi})$ in Theorem 3.1 can be evaluated in closed form. Then in a corollary, we evaluate another interesting series.

Proposition 3.8. *We have*

$$Q(e^{-2\pi})=\frac{3\pi^2}{4\Gamma^8(\frac34)}. \tag{3.38}$$

Proof. We apply Entry 13(i) in Chapter 17 of Ramanujan's second notebook [3, p. 126]. In [3], $M(q)=Q(q)$, in the present notation. Set $y=\pi$ there, and note that $x=\frac12$. We find immediately that

$$Q(e^{-2\pi})=\tfrac34\,{}_2F_1^4(\tfrac12,\tfrac12;1;\tfrac12)=\frac{3\pi^2}{4\Gamma^8(\frac34)},$$

where we have employed a special case of a well-known theorem of Gauss, rediscovered by Ramanujan [2, p. 42, Entry 34]. (See also [3, p. 103, eq. (6.15)].) Hence, (3.38) has been shown. □

Corollary 3.9. *We have*

$$\sum_{n=0}^{\infty}\frac{(2n+1)^2}{\cosh^2\{(2n+1)\pi/2\}}=\frac{\pi^2}{12\Gamma^8(\frac34)}.$$

Proof. From the definition of $B(q)$ in (1.10), we easily find that

$$B'(q)=24\sum_{n=0}^{\infty}\frac{(2n+1)^2q^{2n}}{(1-q^{2n+1})^2}.$$

Setting $q=-e^{-\pi}$ and using (3.23), we find that

$$\begin{aligned}\frac23 e^{\pi}Q(e^{-2\pi})=B'(-e^{-\pi})&=24e^{\pi}\sum_{n=0}^{\infty}\frac{(2n+1)^2e^{-(2n+1)\pi}}{(1+e^{-(2n+1)\pi})^2}\\&=6e^{\pi}\sum_{n=0}^{\infty}\frac{(2n+1)^2}{\cosh^2\{(2n+1)\pi/2\}}.\end{aligned}$$

If we now use (3.38) on the left side above and simplify, we complete the proof. □

Corollary 3.9 might be compared with further explicit evaluations of series containing the hyperbolic function cosh given by Ramanujan in Entry 16 of Chapter 17 in his second notebook [3, p. 134], in Entry 6 of Chapter 18 in his second notebook [3, p. 153], and in two particular results in his first notebook [4, pp. 398, 402, Entries 76, 78].

4. Formulas for the Coefficients of Further Eisenstein Series

Ramanujan [18, pp. 117–118], [6, pp. 190–191] concludes his letter to Hardy with three identities which are similar to (3.10). We show how each of the identities follows from (3.10), but first we need to make several definitions.

Let, for $|q| < q_0$,

$$\frac{Q(q^2)}{R(q)} =: \sum_{n=0}^{\infty} \sigma_n q^n, \quad \frac{\varphi^8(-q)}{R(q)} =: \sum_{n=0}^{\infty} v_n q^n, \quad \text{and} \quad \frac{q\psi^8(q)}{R(q)} =: \sum_{n=0}^{\infty} \chi_n q^n. \tag{4.1}$$

Theorem 4.1. *Suppose that* (3.10) *holds. Then*

(i) $$\sigma_n = \frac{11}{16} \sum_{(\mu_0)} V_{\mu_0}(n) - \frac{1}{4} \sum_{(\mu_e)} V_{\mu_e}(n),$$

where μ_0 and μ_e run over the odd and even integers of the form (3.4)*,respectively;*

(ii) $$v_n = \frac{2}{3} \sum_{(\mu_0)} V_{\mu_0}(n) - \frac{1}{3} \sum_{(\mu_e)} V_{\mu_e}(n);$$

and

(iii) $$\chi_n = \frac{1}{768} \sum_{(\mu_0)} V_{\mu_0}(n) + \frac{1}{192} \sum_{(\mu_e)} V_{\mu_e}(n).$$

Proof of (i). By (4.1), Theorem 2.1 (vi), (v), (3.1), and (1.11),

$$\begin{aligned}
\sum_{n=0}^{\infty} \sigma_n q^n &= \frac{11}{24}\left(\frac{1}{B(\sqrt{q})} + \frac{1}{B(-\sqrt{q})}\right) + \frac{1}{12B(q)} \\
&= \frac{11}{16}\left(\frac{2}{3}\left(\frac{1}{B(\sqrt{q})} + \frac{1}{B(-\sqrt{q})}\right) - \frac{1}{3B(q)}\right) + \frac{5}{16}\frac{1}{B(q)} \\
&= \frac{11}{16}\frac{Q(q)}{R(q)} + \frac{5}{16}\frac{1}{B(q)} \\
&= \frac{11}{16}\sum_{n=0}^{\infty} \delta_n q^n + \frac{5}{16}\sum_{n=0}^{\infty} b_n q^n.
\end{aligned}$$

Thus, by (3.8) and (3.10),

$$\begin{aligned}
\sigma_n &= \frac{11}{16}\delta_n + \frac{5}{16} b_n \\
&= \frac{11}{16}\sum_{(\mu)} V_\mu(n) + \frac{5}{16}\left(-3\sum_{(\mu_e)} V_{\mu_e}(n)\right) \\
&= \frac{11}{16}\sum_{(\mu_0)} V_{\mu_0}(n) - \frac{1}{4}\sum_{(\mu_e)} V_{\mu_e}(n)\,.
\end{aligned}$$

□

Proof of (ii). By (4.1), Theorem 2.1 (vii), and (3.1),

$$\sum_{n=0}^{\infty} v_n q^n = -\frac{1}{15}\frac{Q(q)}{R(q)} + \frac{16}{15}\frac{Q(q^2)}{R(q)} = -\frac{1}{15}\sum_{n=0}^{\infty}\delta_n q^n + \frac{16}{15}\sum_{n=0}^{\infty}\sigma_n q^n.$$

Thus, by (3.8) and part (i),

$$\begin{aligned} v_n &= -\frac{1}{15}\delta_n + \frac{16}{15}\sigma_n \\ &= -\frac{1}{15}\sum_{(\mu)} V_\mu(n) + \frac{16}{15}\left(\frac{11}{16}\sum_{(\mu_0)} V_{\mu_0}(n) - \frac{1}{4}\sum_{(\mu_e)} V_{\mu_e}(n)\right) \\ &= \frac{2}{3}\sum_{(\mu_0)} V_{\mu_0}(n) - \frac{1}{3}\sum_{(\mu_e)} V_{\mu_e}(n). \end{aligned}$$

□

Proof of (iii). By (4.1), Theorem 2.1 (viii), and (3.1),

$$\sum_{n=0}^{\infty} \chi_n q^n = \frac{1}{240}\frac{Q(q)}{R(q)} - \frac{1}{240}\frac{Q(q^2)}{R(q)} = \frac{1}{240}\sum_{n=0}^{\infty}\delta_n q^n - \frac{1}{240}\sum_{n=0}^{\infty}\sigma_n q^n.$$

Thus, by (3.8) and part (i)

$$\begin{aligned} \chi_n &= \frac{1}{240}\delta_n - \frac{1}{240}\sigma_n \\ &= \frac{1}{240}\sum_{(\mu)} V_\mu(n) - \frac{1}{240}\left(\frac{11}{16}\sum_{(\mu_0)} V_{\mu_0}(n) - \frac{1}{4}\sum_{(\mu_e)} V_{\mu_e}(n)\right) \\ &= \frac{1}{768}\sum_{(\mu_0)} V_{\mu_0}(n) + \frac{1}{192}\sum_{(\mu_e)} V_{\mu_e}(n). \end{aligned}$$

□

5. The Coefficients of $1/B^2(q)$

In another letter to Hardy [18, pp. 105–109], [6, pp. 185–188], Ramanujan offers a formula for the coefficients of $1/B^2(q)$. By Lemma 3.4, $1/\mathbf{B}^2(\tau)$ has a double pole at $(1+i)/2$. To the best of our knowledge, the generalizations of the principal theorem of Hardy and Ramanujan [9, Thm. 1], [16, p. 312] that we cited earlier do not consider double poles, mainly because of calculational difficulties. In fact, after stating his main theorem, Lehner [10, Thm. 1, p. 65] writes, "Poles of higher order can be treated in an analogous manner, but the algebraic details, into which we do not enter here, become rather complicated." Since $1/\mathbf{B}^2(\tau)$ has only one double pole on a fundamental region for $\Gamma_0(2)$, we confine ourselves to stating our theorem for $\Gamma_0(2)$ only and proving it for modular forms with only one double pole on a fundamental region for $\Gamma_0(2)$.

Theorem 5.1. *Suppose that $f(q) = f(e^{\pi i\tau}) = \phi(\tau)$ is analytic for $q = 0$, is meromorphic in the unit circle, and satisfies the functional equation*

$$\phi(\tau) = \phi\left(\frac{a\tau + b}{c\tau + d}\right)(c\tau + d)^n, \tag{5.1}$$

where a, b, c, d $\in \mathbf{Z}$*;* $ad-bc=1$*; c is even; and* $n \in \mathbf{Z}^+$*. Assume that* $\phi(\tau)$ *has only one pole in a fundamental region for* $\Gamma_0(2)$*, a double pole at* $\tau=\alpha$*. Suppose that* $f(q)$ *and* $\phi(\tau)$ *have the Laurent expansions,*

$$\phi(\tau)=\frac{r_2}{(\tau-\alpha)^2}+\frac{r_1}{\tau-\alpha}+\cdots=\frac{\ell_2}{(q-e^{\pi i\alpha})^2}+\frac{\ell_1}{q-e^{\pi i\alpha}}+\cdots=f(q). \tag{5.2}$$

Then

$$\begin{aligned} f(q)=&2\pi i\sum_{c,d}\left\{\frac{cr_2(n+2)}{(c\alpha+d)^{n+3}}-\frac{r_1}{(c\alpha+d)^{n+2}}\right\}\frac{1}{1-(q/\underline{q})^2}\\ &-4\pi^2r_2\sum_{c,d}\frac{1}{(c\alpha+d)^{n+4}}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}, \qquad |q|<q_0, \end{aligned} \tag{5.3}$$

where

$$\underline{q}=\exp\left(\left(\frac{a\alpha+b}{c\alpha+d}\right)\pi i\right), \tag{5.4}$$

and the summation runs over all pairs of coprime integers (c,d) *(with c even) which yield distinct values for the set* $\{\underline{q},-\underline{q}\}$*, and a and b are any integral solutions of*

$$ad-bc=1. \tag{5.5}$$

Furthermore,

$$r_1=-\frac{i\ell_1}{\pi e^{\pi i\alpha}}+\frac{i\ell_2}{\pi e^{2\pi i\alpha}} \qquad \text{and} \qquad r_2=-\frac{\ell_2}{\pi^2e^{2\pi i\alpha}}. \tag{5.6}$$

Proof. For brevity, set

$$T:=T(\tau):=\frac{a\tau+b}{c\tau+d} \qquad \text{and} \qquad Z:=\frac{a\alpha+b}{c\alpha+d}. \tag{5.7}$$

We want to calculate the Laurent expansion of ϕ as a function of T in a neighborhood of Z. Since

$$\tau=\frac{dT-b}{-cT+a}, \tag{5.8}$$

we easily find that

$$\begin{aligned} \tau-\alpha=&\frac{T(d+c\alpha)-(b+a\alpha)}{-cT+a}=\frac{T-\dfrac{a\alpha+b}{c\alpha+d}}{-cT+a}(c\alpha+d)\\ =&\frac{T-Z}{-cT+a}(c\alpha+d)=\frac{T-Z}{-cZ+a-c(T-Z)}(c\alpha+d). \end{aligned} \tag{5.9}$$

However, by (5.7) and (5.5), we easily find that

$$-cZ+a=\frac{1}{c\alpha+d}. \tag{5.10}$$

Employing (5.10) in (5.9), we find that

$$\tau-\alpha=\frac{T-Z}{1-c(c\alpha+d)(T-Z)}(c\alpha+d)^2,$$

or

$$\frac{1}{\tau-\alpha}=\frac{1-c(c\alpha+d)(T-Z)}{(T-Z)(c\alpha+d)^2} \tag{5.11}$$

and

$$\frac{1}{(\tau-\alpha)^2}=\frac{1-2c(c\alpha+d)(T-Z)+c^2(c\alpha+d)^2(T-Z)^2}{(T-Z)^2(c\alpha+d)^4}. \tag{5.12}$$

We next seek the expansion of $(c\tau+d)^n$ in powers of $(T-Z)$. By (5.8), (5.5), and (5.10),

$$\begin{aligned}(c\tau+d)^n =&(-cT+a)^{-n}\\ =&\left((-cZ+a)-c(T-Z)\right)^{-n}\\ =&\left(\frac{1}{c\alpha+d}-c(T-Z)\right)^{-n}\\ =&(c\alpha+d)^n\left(1-c(c\alpha+d)(T-Z)\right)^{-n}.\end{aligned} \tag{5.13}$$

Thus, from (5.2), (5.11)–(5.13), and (5.1),

$$\begin{aligned}\phi(\tau)=&\left(\frac{1}{(c\alpha+d)^4(T-Z)^2}-\frac{2c}{(c\alpha+d)^3(T-Z)}+\frac{c^2}{(c\alpha+d)^2}\right)r_2\\ &+\left(\frac{1}{(c\alpha+d)^2(T-Z)}-\frac{c}{c\alpha+d}\right)r_1+\cdots\\ =&(c\tau+d)^n\phi(T)\\ =&(c\alpha+d)^n\left(1-c(c\alpha+d)(T-Z)\right)^{-n}\left(\frac{R_2}{(T-Z)^2}+\frac{R_1}{T-Z}+\cdots\right),\end{aligned} \tag{5.14}$$

where R_1 and R_2 are the coefficients in the principal part of $\phi(T)$ about Z. Thus, rearranging (5.14), we easily find that

$$\begin{aligned}\frac{R_2}{(T-Z)^2}+\frac{R_1}{T-Z}+\cdots=&(c\alpha+d)^{-n}\left(1-cn(c\alpha+d)(T-Z)+\cdots\right)\\ &\times\left(\left(\frac{1}{(c\alpha+d)^4(T-Z)^2}-\frac{2c}{(c\alpha+d)^3(T-Z)}+\frac{c^2}{(c\alpha+d)^2}\right)r_2\right.\\ &\left.+\left(\frac{1}{(c\alpha+d)^2(T-Z)}-\frac{c}{c\alpha+d}\right)r_1+\cdots\right).\end{aligned} \tag{5.15}$$

It follows from above that

$$R_1=(c\alpha+d)^{-n}\left(-\frac{2cr_2}{(c\alpha+d)^3}-\frac{cnr_2}{(c\alpha+d)^3}+\frac{r_1}{(c\alpha+d)^2}\right) \tag{5.16}$$

and

$$R_2=\frac{r_2}{(c\alpha+d)^{n+4}}. \tag{5.17}$$

We now proceed as in [9, Thm. 1], [16, p. 312], or [5, eq. (2.34)]. Recall that the definition of $\underline{q}$ is given in (3.19). Accordingly,

$$f(q) = -\sum \operatorname{Res}\left(\frac{f(z)}{z-q}, \pm\underline{q}\right), \tag{5.18}$$

where the sum is over all poles $\pm\underline{q}$. If

$$g(z) := \frac{1}{z-q},$$

then, by Taylor's theorem,

$$g(z) = \frac{1}{\pm\underline{q}-q} - \frac{1}{(\pm\underline{q}-q)^2}(z \mp \underline{q}) + \cdots . \tag{5.19}$$

Let us write $f(z)$ as

$$f(z) = \frac{P_2}{(z-z_0)^2} + \frac{P_1}{z-z_0} + \cdots .$$

Then

$$\operatorname{Res}\left(\frac{f(z)}{z-q}, z_0\right) = P_2\left(\frac{1}{z-q}\right)'\bigg|_{z=z_0} + P_1\frac{1}{z-q}\bigg|_{z=z_0}. \tag{5.20}$$

We need to find P_1 and P_2 for $z_0 = \pm\underline{q}$.

Next, we take the Laurent expansion (5.15) and convert it into a Laurent expansion in powers of $(z-\underline{q})$. Observe that $e^{\pi i(Z+1)} = -\underline{q}$. Thus, the Laurent expansion in powers of $(z+\underline{q})$ arises from (5.15) with Z replaced by $Z+1$. Since the arguments in the two cases $+\underline{q}$ and $-\underline{q}$ are identical, we consider only the poles $+\underline{q}$. Set $z = e^{\pi i\tau}$ and remember that $\underline{q} = e^{\pi i Z}$. Also, put

$$h(z) = \log z - \log \underline{q},$$

where the principal branch of log is chosen. Then, by Taylor's theorem,

$$h(z) = \frac{1}{\underline{q}}(z-\underline{q}) - \frac{1}{2\underline{q}^2}(z-\underline{q})^2 + \cdots ,$$

and so

$$\frac{1}{h(z)} = \frac{\underline{q}}{z-\underline{q}}\left(1 + \frac{1}{2\underline{q}}(z-\underline{q}) + \cdots\right)$$

and

$$\frac{1}{h^2(z)} = \frac{\underline{q}^2}{(z-\underline{q})^2}\left(1 + \frac{1}{\underline{q}}(z-\underline{q}) + \cdots\right).$$

Hence,

$$\begin{aligned}\frac{R_2}{(\tau-Z)^2}+\frac{R_1}{\tau-Z}+\cdots &= -\frac{R_2\pi^2}{(\log z-\log \underline{q})^2}+\frac{R_1\pi i}{\log z-\log \underline{q}}+\cdots\\ &= -\frac{R_2\pi^2\underline{q}^2}{(z-\underline{q})^2}\left(1+\frac{1}{\underline{q}}(z-\underline{q})+\cdots\right)\\ &\quad+\frac{R_1\pi i\underline{q}}{z-\underline{q}}\left(1+\frac{1}{2\underline{q}}(z-\underline{q})+\cdots\right)+\cdots\\ &= -\frac{R_2\pi^2\underline{q}^2}{(z-\underline{q})^2}+\frac{1}{z-\underline{q}}\left(-R_2\pi^2\underline{q}+R_1\pi i\underline{q}\right)+\cdots . \end{aligned} \tag{5.21}$$

Therefore, by (5.20), (5.19), and (5.21),

$$\operatorname{Res}\left(\frac{f(z)}{z-q},\underline{q}\right)=\frac{1}{\underline{q}-q}\left(-R_2\pi^2\underline{q}+R_1\pi i\underline{q}\right)+\frac{R_2\pi^2\underline{q}^2}{(\underline{q}-q)^2}. \tag{5.22}$$

By a similar calculation,

$$\operatorname{Res}\left(\frac{f(z)}{z-q},-\underline{q}\right)=\frac{1}{-\underline{q}-q}\left(R_2\pi^2\underline{q}-R_1\pi i\underline{q}\right)+\frac{R_2\pi^2\underline{q}^2}{(-\underline{q}-q)^2}. \tag{5.23}$$

Hence, by (5.18), (5.22), (5.23), (5.16), and (5.17),

$$\begin{aligned}
f(q) =& -\sum_{c,d}\Bigg(\bigg(\frac{-\pi^2\underline{q}r_2}{(c\alpha+d)^{n+4}} - \frac{\pi i r_2 c(n+2)\underline{q}}{(c\alpha+d)^{n+3}} + \frac{\pi i r_1 \underline{q}}{(c\alpha+d)^{n+2}}\bigg)\frac{1}{\underline{q}-q} \\
&+ \frac{r_2\pi^2\underline{q}^2}{(c\alpha+d)^{n+4}(\underline{q}-q)^2} \\
&+ \bigg(\frac{\pi^2\underline{q}r_2}{(c\alpha+d)^{n+4}} + \frac{\pi i r_2 c(n+2)\underline{q}}{(c\alpha+d)^{n+3}} - \frac{\pi i r_1 \underline{q}}{(c\alpha+d)^{n+2}}\bigg)\frac{1}{-\underline{q}-q} \\
&+ \frac{r_2\pi^2\underline{q}^2}{(c\alpha+d)^{n+4}(\underline{q}+q)^2}\Bigg) \\
=& 2\pi i \sum_{c,d}\left\{\frac{cr_2(n+2)}{(c\alpha+d)^{n+3}} - \frac{r_1}{(c\alpha+d)^{n+2}}\right\}\frac{1}{1-(q/\underline{q})^2} \\
&+ 2\pi^2 r_2 \sum_{c,d}\frac{1}{(c\alpha+d)^{n+4}}\frac{1}{1-(q/\underline{q})^2} \\
&- 2\pi^2 r_2 \sum_{c,d}\frac{1}{(c\alpha+d)^{n+4}}\frac{1+(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2} \\
=& 2\pi i \sum_{c,d}\left\{\frac{cr_2(n+2)}{(c\alpha+d)^{n+3}} - \frac{r_1}{(c\alpha+d)^{n+2}}\right\}\frac{1}{1-(q/\underline{q})^2} \\
&+ 2\pi^2 r_2 \sum_{c,d}\frac{1}{(c\alpha+d)^{n+4}}\left\{\frac{1}{1-(q/\underline{q})^2} - \frac{1+(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}\right\} \\
=& 2\pi i \sum_{c,d}\left\{\frac{cr_2(n+2)}{(c\alpha+d)^{n+3}} - \frac{r_1}{(c\alpha+d)^{n+2}}\right\}\frac{1}{1-(q/\underline{q})^2} \\
&- 4\pi^2 r_2 \sum_{c,d}\frac{1}{(c\alpha+d)^{n+4}}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2},
\end{aligned}$$

where the sum on c, d is as stated in Theorem 5.1. This proves (5.3).

We next prove (5.6). From (5.2), since $q = e^{\pi i\tau}$,

$$\begin{aligned}
f(q) =& \frac{\ell_2}{e^{2\pi i\alpha}(e^{\pi i(\tau-\alpha)}-1)^2} + \frac{\ell_1}{e^{\pi i\alpha}(e^{\pi i(\tau-\alpha)}-1)} + \cdots \\
=& \frac{\ell_2}{e^{2\pi i\alpha}(\pi i(\tau-\alpha)+\frac{1}{2}(\pi i)^2(\tau-\alpha)^2+\cdots)^2} + \frac{\ell_1}{e^{\pi i\alpha}(\pi i(\tau-\alpha)+\cdots)} + \cdots \\
=& -\frac{\ell_2}{e^{2\pi i\alpha}\pi^2(\tau-\alpha)^2(1+\pi i(\tau-\alpha)+\cdots)} + \frac{\ell_1}{e^{\pi i\alpha}(\pi i(\tau-\alpha))} + \cdots \\
=& -\frac{\ell_2}{e^{2\pi i\alpha}\pi^2(\tau-\alpha)^2} - \frac{\ell_2}{e^{2\pi i\alpha}\pi i(\tau-\alpha)} + \frac{\ell_1}{e^{\pi i\alpha}\pi i(\tau-\alpha)} + \cdots. \qquad (5.24)
\end{aligned}$$

If we now compare the far right side of (5.24) with the right side of (5.2), we deduce (5.6). □

Lemma 5.2. *As in the general setting* (5.2)*, put*

$$\frac{1}{B^2(q)} = \frac{\ell_2}{(q-\underline{q})^2} + \frac{\ell_1}{q-\underline{q}} + \cdots, \tag{5.25}$$

where $\underline{q} = e^{\pi i \alpha}$*, and where now* $\alpha = 1+i$*. Then*

$$\ell_1 = -\frac{B''(\underline{q})}{(B'(\underline{q}))^3} \qquad \text{and} \qquad \ell_2 = \frac{1}{(B'(\underline{q}))^2}. \tag{5.26}$$

Proof. Since $B(\underline{q}) = 0$,

$$\begin{aligned}\frac{1}{B^2(q)} &= \frac{1}{\left\{B'(\underline{q})(q-\underline{q}) + \frac{1}{2}B''(\underline{q})(q-\underline{q})^2 + \cdots\right\}^2}\\ &= \left\{\frac{1}{B'(\underline{q})(q-\underline{q})} - \frac{B''(\underline{q})}{2B'(\underline{q})^2} + \cdots\right\}^2\\ &= \frac{1}{B'(\underline{q})^2(q-\underline{q})^2} - \frac{f''(\underline{q})}{B'(\underline{q})^3(q-\underline{q})} + \cdots. \end{aligned} \tag{5.27}$$

The values (5.26) now follow from (5.25) and (5.27). □

The coefficients of $1/B^2(q)$ are closely related to those for $1/R(q)$, which were established in Hardy and Ramanujan's paper [9, Thm. 3], [16, p. 319].

Theorem 5.3. *Define the coefficients* p_n *by*

$$\frac{1}{R(q^2)} =: \sum_{n=0}^{\infty} p_n q^{2n}, \qquad |q| < q_0. \tag{5.28}$$

Then, for $n \geq 0$*,*

$$p_n = \sum_{(\mu)} T_\mu(n), \tag{5.29}$$

where μ *runs over all integers of the form* (3.4)*, and where*

$$T_1(n) = \frac{2}{Q^2(e^{-2\pi})} e^{2n\pi}, \tag{5.30}$$

$$T_2(n) = \frac{2}{Q^2(e^{-2\pi})} \frac{(-1)^n}{2^4} e^{n\pi}, \tag{5.31}$$

and, for $\mu > 2$*,*

$$T_\mu(n) = \frac{2}{Q^2(e^{-2\pi})} \frac{e^{2n\pi/\mu}}{\mu^4} \sum_{c,d} 2\cos\left((ac+bd)\frac{2\pi n}{\mu} + 8\tan^{-1}\frac{c}{d}\right), \tag{5.32}$$

where the sum is over all pairs (c,d)*, where* (c,d) *is a distinct solution to* $\mu = c^2+d^2$ *and* (a,b) *is any solution to* $ad - bc = 1$*. Also, distinct solutions* (c,d) *to* $\mu = c^2+d^2$ *give rise to distinct terms in the sum in* (5.32)*.*

We are now ready to state Ramanujan's theorem on the coefficients of $1/B^2(q)$.

Theorem 5.4. *Define the coefficients* b'_n *by*

$$\frac{1}{B^2(q^2)} =: \sum_{n=0}^{\infty} b'_n q^{2n}, \qquad |q| < q_0.$$

Then,

$$b'_n = 18 \sum_{(\mu_e)} \left(n + \frac{3\mu_e}{2\pi} \right) T_{\mu_e}(n), \tag{5.33}$$

where the sum is over all even integers μ *of the form* (3.4)*, and where* $T_{\mu_e}(n)$ *is defined by* (5.30), (5.31)*, and* (5.32)*.*

Proof. Throughout the proof we frequently and tacitly use the equalities,

$$P(-e^{-\pi}) = 2P(e^{-2\pi}), \quad Q(-e^{-\pi}) = -4Q(e^{-2\pi}), \quad \text{and} \quad R(-e^{-\pi}) = R(e^{-2\pi}) = 0,$$

where the first equality follows from (2.4) and the equality $B(-e^{-\pi}) = 0$; the second comes from Theorem 2.1 (iii) (or (3.22)); and the third arises from Theorem 2.1 (ii), the equality $B(-e^{-\pi}) = 0$, and the fact that $e^{-2\pi}$ is a zero of $R(q)$ [19, p. 198].

By (2.4) and (1.4),

$$\begin{aligned} B'(-e^{-\pi}) =& \frac{d}{dq}\left(2P(q^2) - P(q)\right)\Big|_{q=-e^{-\pi}} \\ =& \frac{P^2(q^2) - Q(q^2)}{3q} - \frac{P^2(q) - Q(q)}{12q}\Big|_{q=-e^{-\pi}} = \frac{2Q(e^{-2\pi})}{3e^{-\pi}}. \end{aligned} \tag{5.34}$$

Next, by (5.34), (1.4), (1.5), and (3.16),

$$\begin{aligned} B''(-e^{-\pi}) =& \frac{1}{12q^2}\Bigg\{\Bigg(16qP(q^2)\frac{P^2(q^2) - Q(q^2)}{12q^2} - 8q\frac{P(q^2)Q(q^2) - R(q^2)}{3q^2} \\ & -2P(q)\frac{P^2(q) - Q(q)}{12q} + \frac{P(q)Q(q) - R(q)}{3q}\Bigg) q \\ & - 4P^2(q^2) + 4Q(q^2) + P(q^2) - Q(q)\Bigg\}\Bigg|_{q=-e^{-\pi}} \\ =& \frac{1}{12q^2}\left\{-4P(q^2)Q(q^2) + \frac{P(q)Q(q)}{2} + 8Q(q^2)\right\}\Bigg|_{q=-e^{-\pi}} \\ =& \frac{1}{12e^{-2\pi}}\left\{-8P(e^{-2\pi})Q(e^{-2\pi}) + 8Q(e^{-2\pi})\right\} \\ =& \frac{2Q(e^{-2\pi})}{3e^{-2\pi}}\left(1 - \frac{3}{\pi}\right). \end{aligned} \tag{5.35}$$

By the chain rule and (5.34) and (5.35), respectively, it follows that

$$\frac{B(q^2)}{dq}\Big|_{q=ie^{-\pi/2}} = 2qB'(q^2)\big|_{q=ie^{-\pi/2}} = \frac{4iQ(e^{-2\pi})}{3e^{-\pi/2}}$$

and

$$\begin{aligned}\left.\frac{d^2B(q^2)}{dq^2}\right|_{q=ie^{-\pi/2}} &= 2B'(q^2) + 4q^2B''(q^2)\big|_{q=ie^{-\pi/2}} \\ &= \frac{4Q(e^{-2\pi})}{3e^{-\pi}} - \frac{8e^{-\pi}Q(e^{-2\pi})}{3e^{-2\pi}}\left(1-\frac{3}{\pi}\right) \\ &= \frac{4Q(e^{-2\pi})}{3e^{-\pi}}\left(\frac{6}{\pi}-1\right).\end{aligned}$$

It follows from (5.26) that

$$\ell_2 = -\frac{9e^{-\pi}}{16Q^2(e^{-2\pi})}$$

and

$$\ell_1 = -\frac{27ie^{-3\pi/2}}{64Q^3(e^{-2\pi})}\frac{4Q(e^{-2\pi})}{3e^{-\pi}}\left(\frac{6}{\pi}-1\right) = -\frac{9ie^{-\pi/2}}{16Q^2(e^{-2\pi})}\left(\frac{6}{\pi}-1\right).$$

Using the calculations above in (5.6), we further find that

$$r_2 = -\frac{9}{16\pi^2Q^2(e^{-2\pi})} \tag{5.36}$$

and

$$r_1 = -\frac{i}{\pi e^{-\pi}}\left(-\frac{9e^{-\pi}}{16Q^2(e^{-2\pi})}\right) + \frac{9i}{16\pi Q^2(e^{-2\pi})}\left(\frac{6}{\pi}-1\right) = \frac{27i}{8\pi^2Q^2(e^{-2\pi})}. \tag{5.37}$$

We now apply Theorem 5.1 to $1/B^2(q^2)$. Note that $n = 4$ and that $\underline{q}$ is defined by (3.19). Accordingly,

$$\begin{aligned}\frac{1}{B^2(q^2)} &= 2\pi i \sum_{\substack{c,d \\ c \text{ even}}} \left\{\frac{6cr_2}{(c\frac{1+i}{2}+d)^7} - \frac{r_1}{(c\frac{1+i}{2}+d)^6}\right\}\frac{1}{1-(q/\underline{q})^2} \\ &\quad - 4\pi^2 r_2 \sum_{\substack{c,d \\ c \text{ even}}} \frac{1}{(c\frac{1+i}{2}+d)^8}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}.\end{aligned}$$

We use the calculations (3.27) and (3.28) with $c' = c + d$. Since c is even and d is odd, then c' is odd and $\mu = c'^2 + d^2$ is even. Replacing c' by c, we find that

$$\begin{aligned}\frac{1}{B^2(q^2)} &= 2\pi i \sum_{\substack{c,d \\ c^2+d^2 \text{ even}}} \left\{\frac{(1+i)^7 6(c-d)r_2}{(ci+d)^7} - \frac{(1+i)^6 r_1}{(ci+d)^6}\right\}\frac{1}{1-(q/\underline{q})^2} \\ &\quad - 4\pi^2 r_2 \sum_{\substack{c,d \\ c^2+d^2 \text{ even}}} \frac{(1+i)^8}{(ci+d)^8}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}.\end{aligned}$$

Using (5.36) and (5.37), we deduce that

$$
\begin{aligned}
\frac{1}{B^2(q^2)} =& 2\pi i \sum_{\substack{c,d\\ c^2+d^2 \text{ even}}} \left\{ \frac{-8i}{(ci+d)^7} \left(-\frac{6\cdot 9(c-d)(1+i)}{16\pi^2 Q^2(e^{-2\pi})} - \frac{27i(ci+d)}{8\pi^2 Q^2(e^{-2\pi})} \right) \right\} \frac{1}{1-(q/\underline{q})^2} \\
&+ \frac{36}{Q^2(e^{-2\pi})} \sum_{\substack{c,d\\ c^2+d^2 \text{ even}}} \frac{1}{(ci+d)^8} \frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2} \\
=& \frac{16}{\pi Q^2(e^{-2\pi})} \sum_{\substack{c,d\\ c^2+d^2 \text{ even}}} \frac{1}{(ci+d)^7} \left(-\frac{27(ci-d)}{8} \right) \frac{1}{1-(q/\underline{q})^2} \\
&+ \frac{36}{Q^2(e^{-2\pi})} \sum_{\substack{c,d\\ c^2+d^2 \text{ even}}} \frac{1}{(ci+d)^8} \frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2} \\
=& -\frac{54}{\pi Q^2(e^{-2\pi})} \sum_{\substack{c,d\\ c^2+d^2 \text{ even}}} \frac{(ci-d)(ci+d)}{(ci+d)^8} \frac{1}{1-(q/\underline{q})^2} \\
&+ \frac{36}{Q^2(e^{-2\pi})} \sum_{\substack{c,d\\ c^2+d^2 \text{ even}}} \frac{1}{(ci+d)^8} \frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2} \\
=& \frac{54}{\pi Q^2(e^{-2\pi})} \sum_{\substack{c,d\\ c^2+d^2 \text{ even}}} \frac{\mu}{(ci+d)^8} \frac{1}{1-(q/\underline{q})^2} \\
&+ \frac{36}{Q^2(e^{-2\pi})} \sum_{\substack{c,d\\ c^2+d^2 \text{ even}}} \frac{1}{(ci+d)^8} \frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}.
\end{aligned}
$$

Hence, as in the proof in [9, Thm. 3], [16, p. 319] or [5, Thm. 4.1], we separate the terms for positive and negative c and observe that if c is replaced by $-c$ in $\underline{q}$, then $\underline{q}$ is replaced by $\overline{\underline{q}}$. From above, we then deduce that

$$\begin{aligned}
\frac{1}{B^2(q^2)} =&\frac{54}{\pi Q^2(e^{-2\pi})}\left(\frac{1}{2^4}\frac{1}{1+e^{\pi}q^2}\right.\\
&+\left.\sum_{\substack{c,d\\ \mu>2}}\left\{\frac{\mu}{(ci+d)^8}\frac{1}{1-(q/\underline{q})^2}+\frac{\mu}{(-ci+d)^8}\frac{1}{1-(q/\overline{\underline{q}})^2}\right\}\right)\\
&+\frac{36}{Q^2(e^{-2\pi})}\left(\frac{1}{2^4}\frac{-e^{\pi}q^2}{1+e^{\pi}q^2}\right.\\
&+\left.\sum_{\substack{c,d\\ \mu>2}}\left\{\frac{1}{(ci+d)^8}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}+\frac{1}{(-ci+d)^8}\frac{(q/\overline{\underline{q}})^2}{(1-(q/\overline{\underline{q}})^2)^2}\right\}\right)\\
=&\frac{54}{\pi Q^2(e^{-2\pi})}\left(\frac{1}{2^4}\sum_{n=0}^{\infty}(-1)^n e^{\pi n}q^{2n}\right.\\
&+\left.\sum_{\substack{c,d\\ \mu>2}}\left\{\frac{\mu}{(ci+d)^8}\sum_{n=0}^{\infty}\underline{q}^{-2n}q^{2n}+\frac{\mu}{(-ci+d)^8}\sum_{n=0}^{\infty}\overline{\underline{q}}^{-2n}q^{2n}\right\}\right)\\
&+\frac{36}{Q^2(e^{-2\pi})}\left(\frac{1}{2^4}\sum_{n=0}^{\infty}(-1)^n n e^{\pi n}q^{2n}\right.\\
&+\left.\sum_{\substack{c,d\\ \mu>2}}\left\{\frac{1}{(ci+d)^8}\sum_{n=0}^{\infty}n\underline{q}^{-2n}q^{2n}+\frac{1}{(-ci+d)^8}\sum_{n=0}^{\infty}n\overline{\underline{q}}^{-2n}q^{2n}\right\}\right).
\end{aligned}$$

Equating coefficients of $q^{2n}, n\geq 0$, on both sides and proceeding as in the proof in [9, Thm. 3], [16, p. 319] or [5, Thm. 4.1], we find that

$$\begin{aligned}
b'_n =&\frac{54}{\pi Q^2(e^{-2\pi})}\left(\frac{(-1)^n}{2^4}e^{\pi n}+\sum_{\substack{(\mu_e)\\ \mu_e>2}}\sum_{c,d}\frac{\mu_e e^{2\pi n/\mu_e}}{\mu_e^4}2\cos\left(\frac{2\pi n}{\mu_e}(ac+bd)+8\tan^{-1}\frac{c}{d}\right)\right)\\
&+\frac{36}{Q^2(e^{-2\pi})}\left(\frac{(-1)^n}{2^4}ne^{\pi n}+\sum_{\substack{(\mu_e)\\ \mu_e>2}}\sum_{c,d}\frac{ne^{2\pi n/\mu_e}}{\mu_e^4}\cos\left(\frac{2\pi n}{\mu_e}(ac+bd)+8\tan^{-1}\frac{c}{d}\right)\right)\\
=&\frac{27}{\pi}\sum_{(\mu_e)}\mu_e T_{\mu_e}(n)+18n\sum_{(\mu_e)}\mu_e T_{\mu_e}(n)\\
=&18\sum_{(\mu_e)}\left(\frac{3\mu_e}{2\pi}+n\right)T_{\mu_e}(n),
\end{aligned}$$

where the sums on μ_e and c, d are as given in the statement of Theorem 5.4. □

Using *Mathematica*, we calculated $b'_n, 1 \leq n \leq 10$, and the first two terms in (5.33). As with b_n, the accuracy is remarkable.

n	b'_n	$18\left((n+\frac{3}{\pi})T_2(n)+(n+\frac{15}{\pi})T_{10}(n)\right)$
1	-48	-48.001187
2	$1,680$	$1,679.997897$
3	$-52,032$	$-52,031.997988$
4	$1,508,496$	$1,508,496.002778$
5	$-41,952,672$	$-41,952,671.998915$
6	$1,133,840,832$	$1,133,840,831.996875$
7	$-30,010,418,304$	$-30,010,418,304.008563$
8	$781,761,426,576$	$781,761,426,576.003783$
9	$-20,110,673,188,848$	$-20,110,673,188,847.986981$
10	$512,123,093,263,584$	$512,123,093,263,584.006307$

Ramanujan's theorem on the coefficients of $1/B^2(q)$ is also closely related to one further power series expansion for Eisenstein series, which was stated by Ramanujan in one of his letters to Hardy from Matlock House in 1918 and first proved by Berndt and Bialek [5, Thm. 5.1].

Theorem 5.5. *Define the coefficients* η_n *by*

$$\frac{P(q^2)}{R(q^2)} =: \sum_{n=0}^{\infty} \eta_n q^{2n}, \qquad |q| < q_0. \tag{5.38}$$

Then, for $n \geq 0$,

$$\eta_n = \frac{3}{\pi} \sum_{(\mu)} \mu T_\mu(n), \tag{5.39}$$

where μ *runs over all integers of the form* (3.4)*, and where* $T_\mu(n)$ *is defined by* (5.30)–(5.32).

Now observe from (5.29) and (5.38) that

$$\sum_{n=0}^{\infty} c_n q^n := q\left(\frac{1}{R(q)}\right)' + \frac{P(q)}{2R(q)} = \sum_{n=0}^{\infty} np_n q^n + \frac{1}{2}\sum_{n=0}^{\infty} \eta_n q^n, \tag{5.40}$$

and so by (5.29) and (5.39),

$$c_n = \sum_{\mu} \left(n + \frac{3\mu}{2\pi}\right) T_\mu(n) =: c_{n,e} + c_{n,o}, \tag{5.41}$$

where $c_{n,e}$ and $c_{n,o}$ are the subseries over the even and odd values of μ, respectively. Equality (5.41) should be compared with (5.33); in particular, note that $b'_n = 18c_{n,e}$.

Moreover, by (1.6),

$$q\left(\frac{1}{R(q)}\right)' + \frac{P(q)}{2R(q)} = \frac{-P(q)R(q)+Q^2(q)}{2R^2(q)} + \frac{P(q)}{2R(q)} = \frac{1}{2}\left(\frac{Q(q)}{R(q)}\right)^2 =: \frac{1}{2}\sum_{n=0}^{\infty} d_n q^n. \tag{5.42}$$

Defining $d_{n,e}$ and $d_{n,o}$ as we did above for $c_{n,e}$ and $c_{n,o}$, we see by (5.33) and (5.40)–(5.42) that

$$d_{n,e} = 2c_{n,e}, \qquad d_{n,o} = 2c_{n,o}, \qquad \text{and} \qquad b'_n = 9d_{n,e}.$$

By using the formula for the coefficients of $Q(q)/R(q)$ given in Theorem 3.1, we can obtain a relation between these coefficients and the coefficients d_n above.

Although we have stated Theorem 5.1 only for modular forms on $\Gamma_0(2)$ with a single double pole on a fundamental region, there is an obvious analogue for modular forms on the full modular group. In fact, as a check on our work, we applied this analogue to $Q^2(q^2)/R^2(q^2)$ to show that $d_n = 2c_n$, where c_n is given by (5.41). Also, Lehner's theorem [10] can also now be obviously extended for forms with double poles.

Some of the results in Sections 2–4 appeared in the second author's doctoral dissertation [7]. The authors thank Marvin Knopp for informing them of Lehner's paper [10] and especially Dennis Hejhal for supplying several useful comments, corrections, and references.

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