COMBINATORIAL PROOFS OF IDENTITIES IN RAMANUJAN'S LOST NOTEBOOK ASSOCIATED WITH THE ROGERS–FINE IDENTITY AND FALSE THETA FUNCTIONS

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Abstract. Ramanujan's lost notebook contains several identities arising from the Rogers–Fine identity and/or Rogers' false theta functions. Combinatorial proofs for many of these identities are given.

1. Introduction

Ramanujan's lost notebook [8] contains several hundred q-series identities. At least 27 of them arise from or are closely related to the Rogers–Fine identity

$$\sum_{n=0}^{\infty} \frac{(\alpha;q)_n}{(\beta;q)_n}\tau^n = \sum_{n=0}^{\infty} \frac{(\alpha;q)_n(\alpha\tau q/\beta;q)_n\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta;q)_n(\tau;q)_{n+1}}, \tag{1.1}$$

and in most of these identities one of Rogers' false theta functions appears. As customary, here and in the sequel, we employ the notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad n \geq 1,$$

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \quad |q| < 1.$$

The identity (1.1) was first proved by L. J. Rogers [9] in 1917. It was independently discovered by N. J. Fine [5, p. 15] several years before he published it, and he made several applications of it in his monograph [5]. G. E. Andrews [1, Sect. 4] proved (1.1) combinatorially and gave it the appellation, the Rogers–Fine identity. Subsequently, it was learned that G. W. Starcher [10, p. 803] had also discovered and proved most of (1.1). Each of the three original proofs is essentially the same; the idea is to study a defining functional equation. While Ramanujan appears not to have stated this result explicitly, he did consider a closely related more general result, namely, Entry 7 of Chapter 16 in his second notebook [7], [4, p. 16]. In fact, (1.1) follows from the last equation in [4, p. 16] by setting $a = \alpha\tau q/\beta$, $c = \alpha\tau$, and $d = \alpha\tau q$. It is difficult to say

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exactly when Ramanujan discovered his generalization of (1.1), but it was likely in the period 1910–1913.

False theta functions are series that would be instances of classical theta series, except for an alteration of the signs of some of the series terms. These series were first introduced generally by Rogers [9], but several special cases were considered by Ramanujan in his notebooks [7] and lost notebook [8].

In their first volume [3, Chap. 9] on Ramanujan's lost notebook, Andrews and Berndt establish 27 identities arising either directly or indirectly from the Rogers–Fine identity (1.1). In most of them, false theta functions appear. As is to be expected, the proofs vary somewhat in difficulty. However, none of the proofs are long or especially difficult. The purpose of this paper is to provide combinatorial proofs for the majority of these 27 identities. As readers will see, some of the proofs are fairly easy, depending eventually, for example, on Franklin's involution. On the other hand, in many cases it is not easy to ascertain what is happening combinatorially, and so some proofs are considerably more difficult. We emphasize that, although Andrews [1, Sect. 4] proved (1.1) combinatorially, further insights are needed for most of the proofs in this paper.

The appearance of a false theta function in most of the identities is quite interesting combinatorially. In these identities, an enormous amount of cancellation of partitions takes place on the left side of the identity leaving only a sparse set of nonzero quadratic exponents in the false theta function series on the right side. Of course, such occurrences of enormous cancellation are not rare in partition theory, with Euler's pentagonal number theorem being a primary example.

In most of our proofs we employ Ferrers graphs of partitions. The Ferrers graph of a partition of n is a graphical representation of the partition whose kth row has as many boxes as the kth part of the partition. In this paper, we consider a variation of Ferrers graphs by putting a nonnegative number $f(i,j)$ into the box in row i and column j. Such a Ferrers graph represents a partition of n, where n equals the sum of all numbers $f(i,j)$ in the boxes. For example, the figure below is the Ferrers graph of a partition of 23.

0	1	2	2	3	4
1	2	2	2		
1	2				
1					

We conclude the Introduction with a few remarks about the organization of Ramanujan's claims and our proofs. We have chosen a different criterion from that used in [3]. In the next section, we examine identities involving a refinement of the generating function for partitions and a refinement of the generating function for partitions into distinct parts, namely the function ϕ defined by Ramanujan below. In one of our proofs we establish an apparently new bijection between partitions into distinct parts and partitions into odd parts. In Section 3, we give proofs of identities mostly depending on Franklin's involution. Those proofs not falling under the purviews of the former classifications are found in Section 4 and are generally more difficult. Page numbers attached to entries indicate their locations in the lost notebook.

2. Identities Arising from Rearranging Parts

In this section we examine identities related to refinements of the generating functions for partitions and partitions into distinct parts.

Entry 2.1 (p. 32). *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^n}{(aq;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{2n^2+2n}}{(a^2q^2;q^4)_{n+1}}. \tag{2.1}$$

Proof. Replace aq by a and q^2 by q in (2.1). Then multiply both sides by $-a$ and replace a by aq to deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^n (aq)^n}{(aq;q)_n} = \sum_{n=1}^{\infty} \frac{(-1)^n a^{2n-1} q^{n^2}}{(a^2q^2;q^2)_n} = \sum_{n=1}^{\infty} \frac{(-1)^n a^{2n-1} q^{n^2}}{(aq;q)_n(-aq;q)_n}. \tag{2.2}$$

Consider the left side of (2.2), which generates partitions of positive integers with weight 1 if there are an even number of parts and weight -1 if there are an odd number of parts. Furthermore, the power of a represents the number of parts plus the largest part minus 1. Consider the right side of (2.2). Take a Durfee square of side n and append to it on the right the partitions with parts in nonincreasing order generated by $1/(aq)_n$, and append to the bottom of the square the partitions generated by $1/(-aq)_n$. We then easily see that the right side of (2.2) counts the number of partitions whose Durfee squares are of size n^2, the power of a gives the number of parts plus the largest part minus 1, and the power of -1 is the number of parts in the partition. Thus (2.2) has been shown as desired. □

Entry 2.2 (p. 36). *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n^2}}{(a^2q^2;q^2)_n} = 1 - a\sum_{n=1}^{\infty} \frac{a^n q^n}{(-aq;q)_n}. \tag{2.3}$$

Proof. Write the sum on the left side of (2.3) in the form

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n^2}}{(a^2q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n^2}}{(aq;q)_n(-aq;q)_n}. \tag{2.4}$$

By taking Durfee squares, we easily see that the right side of (2.4) counts the number of partitions whose Durfee squares are of size n^2, the power of a gives the number of parts plus the largest part, and the power of -1 is the number of parts in the partition. On the other hand, the right side of (2.3), namely,

$$1 + \sum_{n=1}^{\infty} \frac{-a^{n+1} q^n}{(-aq;q)_n} \tag{2.5}$$

counts the same partitions now arranged according to the largest part. The minus sign in the numerator of (2.5) accounts for the part n, and so again the power of -1

indicates the number of parts. Furthermore, the power of a gives us the number of parts in the partition. Thus, the two sides of (2.3) count the same partitions, and the proof is complete. $\square$

Entry 2.3 (p. 30)**.** *We have*

$$aq(-aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-a;q)_{n+1}} = \sum_{n=0}^{\infty} \frac{a^{n+1} q^{(n+1)^2}}{(q^2;q^2)_n(1+aq^{2n})}.$$

Proof. We divide both sides by aq. Then

$$(-aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-a;q)_{n+1}} = \sum_{n=0}^{\infty} \frac{a^n q^{n^2+2n}}{(q^2;q^2)_n(1+aq^{2n})}.$$

We begin with the right side of the identity above. Examine the summand

$$\frac{a^n q^{n^2+2n}}{(q^2;q^2)_n(1+aq^{2n})},$$

which generates partitions into n odd parts greater than $2k+1$ for a positive integer k, where k equals the power of aq^{2n} arising from the terms of $1/(1+aq^{2n})$. The generating function for such partitions can be rewritten as

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2+2n}}{(q^2;q^2)_n(1+aq^{2n})} = \sum_{k=0}^{\infty}(-a)^k(-aq^{2k+3};q^2)_\infty = (-aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-a)^n}{(-aq;q^2)_{n+1}},$$

where the last identity follows from cancellation. Hence we only need to show that

$$\sum_{n=0}^{\infty} \frac{(-a)^n}{(-aq;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-a)^n q^{n(n+1)/2}}{(1+a)(-aq;q)_n}.$$

Multiply both sides by $-a$ and replace a by $-aq$ to find that

$$\sum_{n=1}^{\infty} \frac{(aq)^n}{(aq^2;q^2)_n} = \sum_{n=1}^{\infty} \frac{a^n q^{n(n+1)/2}}{(aq;q)_n}, \tag{2.6}$$

which implies that partitions into odd parts and partitions into distinct parts are equinumerous. To show this, we establish a bijection preserving the statistic tracked by a. We consider partitions whose Ferrers graphs have boxes each containing a 1 in the first column and boxes each containing a 2 in the other columns. For example, the

1	2	2	2	2
1	2	2	2	2
1	2	2	2	
1	2			
1				

following figure represents the partition of 29. Such partitions are generated by the left side of (2.6).

To obtain the right side of (2.6), we take the largest Durfee square arising from the columns of 2's. Let d be the side of the Durfee square; in the example above, $d = 3$. If the partition has part d below the Durfee square, then we take the $(d+1)\times(d+1)$ square, which is the largest one fitting into the Ferrers graph, where now the column of 1's is included. Otherwise, we take the $d \times (d+1)$ rectangle, which is the largest one fitting into the Ferrers graph as well. In the first case, the square contributes $(d+1)(2d+1)$ to the partition. We now subdivide each box containing a 2 into two boxes each containing a 1, rearrange parts in the square so that the square becomes the partition with parts 1 through $2d+1$, which we arrange in a triangle of parts $1, 2, \ldots, 2d+1$. Since the columns to the right of the rectangle are even and less than or equal to $2d$ and the rows below the rectangle are odd and less than or equal to $2d+1$, by putting those columns and rows below the triangle, we can obtain a partition into distinct parts. We can similarly show that partitions obtained in the the other case can be reformulated as partitions into $2d$ distinct parts. During both processes, the exponent of a is preserved. □

Remark. The bijection between partitions into odd parts and distinct parts described above is likely new.

In the remainder of this section, we examine the first three entries from page 41 of Ramanujan's lost notebook, which are related to the generating function for partitions into distinct parts. Define

$$\phi(a) := \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(bq;q)_n}. \tag{2.7}$$

Note that $\phi(1)$ is the generating function for partitions into distinct parts when $b = 1$.

Entry 2.4 (p. 41)**.** *If $\phi(a)$ is defined by (2.7), then*

$$\phi(a) = (b + aq)\phi(aq) + 1 - b. \tag{2.8}$$

Proof. As we noted above, $\phi(a)$ generates partitions into distinct parts. In the definition of $\phi(a)$, the power of a denotes the number of distinct parts, and the sum of the powers of a and b denotes the largest part. We now divide the partitions into two sets; one is the set of partitions having a part 1, and the other is the set of partitions not having a part 1. Consider now

$$aq\phi(aq) := \sum_{n=0}^{\infty} \frac{a^{n+1} q^{(n+1)(n+2)/2}}{(bq;q)_n}.$$

The sum above generates partitions into distinct parts. But note that the smallest part is 1, since each summand generates partitions into exactly $n+1$ parts by the numerator, whereas the denominator $(bq;q)_n$ does not have an effect on the last part. The power of a is equal to the number of parts and the sum of the powers of a and b is equal to the largest part.

Examine

$$b\phi(aq) := \sum_{n=0}^{\infty} \frac{b a^n q^{n(n+3)/2}}{(bq;q)_n}. \tag{2.9}$$

In each summand, the exponent of q in the numerator is the sum of integers 2 through $n+1$. Thus, we obtain a partition into distinct parts, but now there are no 1's. The power of a in (2.9) still denotes the number of parts, and the sum of the powers of a and b on the right side of (2.9) is equal to the largest part. But observe that the empty partition corresponding to the term 1 is absent, and so we must add it. On the other hand, the term with $n=0$ in (2.9) is equal to b. Thus, we must subtract it.

We have now accounted for all partitions into distinct parts on the right side of (2.8), and so the proof of Entry 2.4 is complete. □

Entry 2.5 (p. 41). *If $\phi(a)$ is defined by (2.7), then*

$$\phi(a)=\sum_{n=0}^{\infty}\frac{(-aq/b;q)_n a^n b^n q^{n(3n+1)/2}(1+aq^{2n+1})}{(bq;q)_n}.$$

Proof. We rewrite the identity above as

$$\phi(a)=\sum_{n=0}^{\infty}\frac{(-aq/b;q)_n a^n b^n q^{n(3n+1)/2}}{(bq;q)_n}+\sum_{n=1}^{\infty}\frac{(-aq/b;q)_{n-1} a^n b^{n-1} q^{n(3n-1)/2}}{(bq;q)_{n-1}}. \tag{2.10}$$

Let us divide the partitions counted by $\phi(a)$ into two sets; one is the set of partitions such that the Durfee squares have side n and the Ferrers diagrams to the right of the Durfee squares contain the Ferrers diagram $(n, n-1, \dots, 1)$, and the other is the set of partitions such that the Durfee squares have side n and the Ferrers diagrams to the right of the Durfee squares contain the Ferrers diagram $(n-1, n-2, \dots, 1)$. In the first case, the parts below the Durfee square correspond to the distinct parts generated by $(-aq/b;q)_n$ arranged in decreasing order, and the parts to the right of the Durfee square and also to the right of the triangle $(n, n-1, \dots, 1)$ correspond to the partitions generated by $1/(bq;q)_n$. Carefully observe that the parts of each partition are distinct. The partitions have largest part $2n$ plus the largest part generated by $1/(bq;q)_n$, i.e., the sum of the powers of a and b keeps track of the largest part, since no contributions are made by the terms arising from $(-aq/b;q)_n$, as the powers of a and b cancel each other. Note that the power of a corresponds to the number of distinct parts of the partition. Thus the first infinite series on the right side of (2.10) is the generating function for partitions counted in the first case.

Similarly, we can show that the second infinite series on the right side of (2.10) is the generating function for partitions counted in the other case. □

Entry 2.6 (p. 41). *If $\phi(a)$ is defined by (2.7), then*

$$\phi(a)=1+\sum_{n=1}^{\infty}\frac{(-aq/b;q)_{n-1} a^n b^{n-1} q^{n(3n-1)/2}(1+aq^{2n})}{(bq;q)_n}.$$

Proof. We rewrite the identity above as

$$\phi(a)=1+\sum_{n=1}^{\infty}\frac{(-aq/b;q)_{n-1} a^n b^{n-1} q^{n(3n-1)/2}}{(bq;q)_n}+\sum_{n=1}^{\infty}\frac{(-aq/b;q)_{n-1} a^{n+1} b^{n-1} q^{n(3n+3)/2}}{(bq;q)_n}. \tag{2.11}$$

We divide the partitions counted by $\phi(a)$ into two sets; one is the set of partitions such that the Durfee squares have side n and the Ferrers diagrams to the right of the Durfee squares have no column of length n, and the other is the set of partitions such that the Durfee squares have side n, the Ferrers diagrams to the right of the Durfee squares have a column of length n, and the Ferrers diagrams below the Durfee squares have a row of length n

Using an argument similar to that in the proof of Entry 2.5, on the right side of (2.11), we again obtain the generating function for partitions into distinct parts, where the power of a denotes the number of distinct parts and the sum of the powers of a and b give the largest part. The result then follows as in Entry 2.4. □

3. Identities Arising from Franklin's Involution

The Franklin involution for Euler's generalized pentagonal number theorem [2, pp. 10–11] is a weight reversing bijection, which explains cancellation among most of the partitions into distinct parts. In this section, we examine identities that can be proved by this involution.

Entry 3.1 (p. 37). *For any complex number a,*

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)/2}}{(-aq;q)_n} = \sum_{n=0}^{\infty} a^{3n} q^{n(3n+1)/2}(1-a^2q^{2n+1}). \tag{3.1}$$

Proof. Replace $-a$ by a in (3.1). Then,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)/2}}{(aq;q)_n} &= \sum_{n=0}^{\infty} (-1)^n a^{3n} q^{n(3n+1)/2}(1-a^2q^{2n+1}) \\ &= \sum_{n=0}^{\infty} (-1)^n a^{3n} q^{n(3n+1)/2} + \sum_{n=1}^{\infty} (-1)^n a^{3n-1} q^{n(3n-1)/2}. \end{aligned} \tag{3.2}$$

This entry then immediately follows from the Franklin involution. Note that the power of a on both sides of (3.2) gives the number of parts plus the largest part. □

We state the following trivial special case of Entry 3.1, because we use it in the proof of Entry 3.3.

Entry 3.2 (p. 37). *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}). \tag{3.3}$$

Proof. Set $a = 1$ in Entry 3.1. □

Entry 3.3 (p. 37). *We have*

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}). \tag{3.4}$$

Proof. Consider a triangle with sides of length $2n$ in the upper left hand corner of a Ferrers diagram. The number of boxes in this triangle is $n(2n+1)$. Append to the right of this triangle a partition of less than or equal to $2n+1$ parts arranged in weakly decreasing order generated by $1/(-q;q)_{2n+1}$. Note that if there are exactly $2n+1$ parts, then the last part will lie below the triangle. But in any case, we obtain a partition into either $2n$ or $2n+1$ distinct parts with a weight of 1 or -1 according to whether the largest part is even or odd, respectively.

Having accounted for the partitions on the left side of (3.4), we examine the left side of (3.3) and observe that by a similar argument, but now beginning with a triangle with sides of size n in the upper left hand corner, we generate again partitions into distinct parts weighted by the sign of the largest part. Comparing the right sides of (3.3) and (3.4), we complete the proof. □

Entry 3.4 (p. 37). *We have*

$$2-\sum_{n=0}^{\infty}\frac{q^{n(2n-1)}}{(-q;q)_{2n}}=\sum_{n=0}^{\infty}q^{n(3n+1)/2}(1-q^{2n+1}). \tag{3.5}$$

Proof. Entry 3.4 is equivalent to the identity

$$1+\sum_{n=1}^{\infty}\frac{-q^{n(2n-1)}}{(-q;q)_{2n}}=\sum_{n=0}^{\infty}q^{n(3n+1)/2}(1-q^{2n+1}).$$

Form a triangle with sides of length $2n-1$ and append to it on the right the partitions generated by $1/(-q;q)_{2n}$. In fact, the last row of these partitions may be one row below the bottom box of the Ferrers triangle. We know that $-q^{n(2n-1)}/(-q;q)_{2n}$ generates partitions into either $2n-1$ or $2n$ distinct parts with weight 1 or -1 if the largest part is odd or even, respectively. Thus, the left side of Entry 3.4 generates the same partitions as the left side of Entry 3.2 does. □

Entry 3.5 (p. 36). *We have*

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(-q^2;q^2)_n}=\sum_{n=0}^{\infty}q^{n(3n+1)/2}(1-q^{2n+1}). \tag{3.6}$$

Proof. We consider partitions generated by the left side above. To represent such partitions, we use Ferrrers graphs whose first column consists of boxes of 1's and the other columns consist of boxes of 2's, as we did in the proof of Entry 2.3.

Taking the largest $n\times(n+1)$ rectangle that fits inside the Ferrers graph, we divide the Ferrers graph into three areas. The columns to the right of the rectangle form a partition into even parts and rows below the rectangle form a partition into odd parts. The weight of the partition is $(-1)^{r+c}$, where r and c are the numbers of rows and columns in the Ferrers graph of the partition. We can therefore rewrite the left side of (3.6) as

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(-q^2;q^2)_n}=\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{(-q;q^2)_{n+1}(-q^2;q^2)_n}=\sum_{n=0}^{\infty}q^{n(3n+1)/2}-\sum_{n=1}^{\infty}q^{n(3n-1)/2},$$

where the last equality is obtained using Entry 3.3. □

Entry 3.6 (p. 41). *We have*

$$1+2\sum_{n=1}^{\infty}\frac{q^{n^2+2n}}{(q^2;q^2)_{n-1}(1-q^{4n})}=(-q;q^2)_\infty\sum_{n=0}^{\infty}q^{n(3n+1)/2}(1-q^{2n+1}).$$

Proof. By Entry 2.3 and the Franklin involution,

$$\begin{aligned}1+2\sum_{n=1}^{\infty}\frac{q^{n^2+2n}}{(q^2;q^2)_n(1+q^{2n})} &= 2\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^2;q^2)_n(1+q^{2n})}\\ &= (-q;q^2)_\infty\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n}\\ &= (-q;q^2)_\infty\sum_{n=0}^{\infty}q^{n(3n+1)/2}(1-q^{2n+1}).\end{aligned}$$

□

In the remainder of this section, we consider a variation of Ferrers graphs by putting either 0 or 1 or 2 into boxes. We put 0 in the box at the upper left corner, 1's into the boxes either in the first row or column, 2's in each box except those in the first row and column. Such a Ferrers graph represents a partition of n, where n equals the sum of all numbers in the boxes. For example, the figure below is the Ferrers graph of a partition of 16. Throughout the remainder of this section, we consider only partitions

0	1	1	1	1	1
1	2	2	2		
1	2				
1					

with such Ferrers graphs.

Entry 3.7 (p. 37)**.** *For any complex number a,*

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}q^{n(n+1)}}{(-aq;q^2)_{n+1}} = \sum_{n=0}^{\infty}a^{3n}q^{3n^2+2n}(1-aq^{2n+1}). \tag{3.7}$$

Proof. We rewrite the identity as

$$\sum_{n=0}^{\infty}\frac{(-1)^n (aq)^{2n}q^{n(n-1)}}{(-aq;q^2)_{n+1}} = \sum_{n=0}^{\infty}((aq)^{3n}q^{n(3n-1)}-(aq)^{3n+1}q^{n(3n+1)}).$$

The left side of the identity above generates partitions described above with distinct rows and weight $(-1)^{c-1}$, where c denotes the number of columns of the Ferrers graph. Moreover, a keeps track of each box with a 1 in it. By applying the Franklin involution, we obtain the right side. □

Entry 3.8 (p. 39)**.** *If*

$$g(a) := \sum_{n=0}^{\infty}a^{3n}q^{n(3n+1)/2}(1-a^2q^{2n+1}), \tag{3.8}$$

then

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{4n+3} q^{2(n+1)^2}}{(-a^2q^3;q^4)_{n+1}} = \frac{1}{2}\left(g(a) - g(-a)\right). \tag{3.9}$$

Proof. By the definition (3.8) of $g(a)$, we find that (3.9) can be written as

$$\begin{aligned}\frac{1}{2}(g(a) - g(-a)) &= \frac{1}{2}\sum_{n=0}^{\infty}(1-(-1)^n)\, a^{3n} q^{n(3n+1)/2}(1-a^2q^{2n+1}) \\ &= a^3q^2\sum_{n=0}^{\infty}(a^{6n}q^{6n^2+7n} - a^{6n+2}q^{6n^2+11n+3}) \\ &= a^3q^2\sum_{n=0}^{\infty}\frac{(-1)^n a^{4n}q^{4n}q^{2n(n-1)}}{(-a^2q^3;q^4)_{n+1}}.\end{aligned}$$

Dividing both sides of the last equality by a^3q^2, and replacing a^2q and q^2 by a and q, respectively, we obtain

$$\sum_{n=0}^{\infty}((aq)^{3n}q^{n(3n-1)} - (aq)^{3n+1}q^{n(3n+1)}) = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}q^{2n}q^{n(n-1)}}{(-aq;q^2)_{n+1}},$$

which is equivalent to Entry 3.7. ☐

Entry 3.9 (p. 29)**.** *We have*

$$\sum_{n=0}^{\infty}(q;q^2)_n q^n = \sum_{n=0}^{\infty} q^{3n^2+2n}(1-q^{2n+1}).$$

Proof. The left side of the identity above generates partitions into distinct parts. By taking out the first k boxes in the kth row from the bottom, we rewrite the generating function as

$$\sum_{n=0}^{\infty}(q;q^2)_n q^n = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} q^{3n^2+2n}(1-q^{2n+1}),$$

where the second equality is obtained using Entry 3.7. ☐

Entry 3.10 (p. 30)**.** *We have*

$$\sum_{n=0}^{\infty}\frac{(-q^2)^{n(n+1)/2}}{(-q;-q^2)_n} = \sum_{n=0}^{\infty}(-1)^{n(n+1)/2}q^{3n^2+2n}(1-(-1)^n q^{2n+1}).$$

Proof. We rewrite the identity as

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n}(-q^2)^{n(n-1)/2}}{(-q;-q^2)_n} = \sum_{n=0}^{\infty}(-1)^{n(n+1)/2}q^{3n^2+2n}(1-(-1)^n q^{2n+1}).$$

The left side generates partitions into distinct parts with weight $(-1)^b$, where b denotes the number of boxes not in the first column.

The only difference between the identity above and the identity in the proof of Entry 3.7 is that all boxes in each column except the first column contribute a weight.

By applying the Franklin involution as we did in Entry 3.7, we arrive at the right side. ☐

Entry 3.11 (p. 37). *We have*

$$\sum_{n=0}^{\infty} \frac{q^n}{(-q;q^2)_{n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{6n^2+4n}(1+q^{4n+2}).$$

Proof. The left side of the identity above generates partitions with weight $(-1)^{r-1}$, where r denotes the number of the rows in the Ferrers graph. Consider the Durfee square of the Ferrers graph. Let s_1 be the least positive integer appearing as a part an odd number of times below the Durfee square and s_2 be the smallest part of the partition to the right of the Durfee square. If neither s_1 nor s_2 exist, set both equal to ∞. If $s_1 > s_2$, then move s_2 to the partition below the Durfee square. If $s_1 \leq s_2$, then move one part s_1 to the right of the Durfee square. The process is a weight reversing involution. We can cancel such partitions. Thus only partitions with parts that appear an even number of times below and with no part to the right of the Durfee square remain. We rearrange such partitions. Let $d+1$ be the side of the Durfee square. Remove the first row of boxes each containing a 1 and add 1 to each of the first d boxes with 1 in column 1. Rearrange the d^2+d boxes of 2's in the Durfee square as $2d$, $2d-2, \ldots, 2$ boxes in rows. Merge these boxes in pairs in each row, so that each box has a 4. Merge every pair of rows below the Durfee square, so that boxes in column 1 have a 2 and the other boxes have a 4. Then we see that such partitions are generated by

$$\sum_{d=0}^{\infty} \frac{(-1)^d q^{2d(d+1)}}{(-q^2;q^4)_{d+1}}.$$

The only difference between the identity above and the identity in the proof of Entry 3.7 is that aq is replaced by q^2 and q is replaced by q^2. Thus we obtain the right hand side by applying the Franklin involution as we did in the proof of Entry 3.7. ☐

4. Identities More Difficult to Prove

Entry 4.1 (p. 29). *We have*

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n(-aq)^n}{(-aq^2;q^2)_n} = \sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2}. \tag{4.1}$$

Proof. We employ the concept of modular partitions, as described in the Introduction.

On the left side of (4.1), $(aq;q^2)_n/(aq^2;q^2)_n$ generates modular partitions $\lambda^{(1)}$, where the parts are less than or equal to n and the parts ending with 1 are distinct, and aq^n generates a partition $\lambda^{(2)}$ of only one part n. We form a new partition λ, whose Ferrers graph has boxes of either 1 or 2, by putting the Ferrers graph of $\lambda^{(1)}$ immediately below that of $\lambda^{(2)}$. For example, when $n=3$, let $\lambda^{(1)} = 6+6+3+2+1$ and $\lambda^{(2)} = 1+1+1$ be given. Then we obtain λ with the Ferrers graph below. It is easily seen that λ is generated by the left side of (4.1). Note that the exponent of a represents the sum of the size of the top row of λ and the number of rows below the top row, and λ has its

1	1	1
2	2	2
2	2	2
2	1	
2		
1		

sign defined by $(-1)^o$, where o is the number of boxes with 1 in the rows below the top row.

We define a sign reversing involution as follows. Let s_1 and s_2 be the last column and last row of the Ferrers graph of λ, respectively. We divide the proof into three cases: when $s_1 < s_2$, $s_2 < s_1$, and $s_1 = s_2$. Here, for example, $s_1 < s_2$ means that the sum of the elements in the boxes of s_1 is less than the sum of the elements in the boxes of s_2.

Case 1: $s_1 < s_2$. If the box in the last square of s_1 contains a 2, then put s_1 immediately below s_2 with the entries arranged in weakly decreasing order. If both the first and last boxes of s_1 have 1, then remove the first box and change 1 in the last box to 2. Move s_1 immediately below s_2, so that boxes of s_1 are in weakly decreasing order. If s_1 has only one box of 1, then the box produces an additional negative sign after the move. If s_1 has one or two boxes of 1, then the move results in losing a negative sign. In summary, each move changes the sign.

Case 2: $s_1 > s_2$. If s_2 has no box with a 1, then add an additional box with 1 in front of the first box and change 2 in the last box to 1. Move s_2 immediately to the right of s_1, so that the first box has 1 and the other boxes are in weakly decreasing order. This move changes the sign as well.

Case 3: $s_1 = s_2$. We separate two cases: when $s_1 = s_2$ are even and when $s_1 = s_2$ are odd. If $s_1 = s_2$ are odd, move s_2 to right next to s_1, so that the box with 1 goes to the top. If $s_1 = s_2$ are even, then s_1 must have two boxes of 1. Remove the first box of s_1, change 1 to 2, and move s_1 to immediately below s_2, so that the box of 1 is rightmost. The move changes the sign.

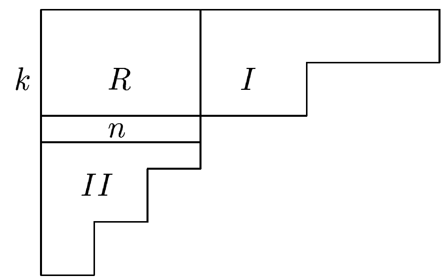
In each case, we see that the sign of a partition changes under the map. Thus the map is a sign reversing involution, which results in cancellations among such partitions.

On the other hand, there are certain partitions for which none of the moves described in Cases 1–3 is possible. These are the partitions whose Ferrers graphs are $(j+1)\times j$ rectangles or $(j+1)\times(j+1)$ rectangles for some $j \geq 1$. Furthermore, these graphs contain boxes of 1's at the top row and boxes of 2's in the other rows and so have no image under the maps described above. These are partitions of $2r^2+r = 2r(2r+1)/2$ or $2r^2+3r+1 = (2r+1)(2r+2)/2$ elements. These are counted on the right side of (4.1). On both sides the power of a equals the largest part plus the number of parts minus 1. $\square$

Entry 4.2 (p. 12). *For any a,*

$$\sum_{n=0}^{\infty} \frac{(-aq^{n+1};q)_n q^n}{(q;q)_n} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} a^n q^{3n(n+1)/2}. \tag{4.2}$$

Proof. For a nonnegative integer n, the left side generates partitions where parts are less than or equal to $2n$, parts greater than n and less than or equal to $2n$ are distinct, and n appears as a part at least once. The exponent of a is the number of parts greater than n. Let such a partition π be given, and let k be the number of parts of π greater than n. Then π has a Ferrers graph described below, where R is a $(k+1)\times n$ rectangle, the area I consists of k distinct parts, and the area II consists of parts less than or equal to n.



To obtain the right side, we need to rearrange parts. We first divide R into a $(k+1)\times(n-k)$ rectangle R_1 and a $(k+1)\times k$ rectangle R_2. We examine the parts in I. Since these k parts are distinct, we can subtract $k, k-1, \dots, 1$ from the largest part through the smallest of the k parts. We paste these parts $k, k-1, \dots, 1$ to R_2, so that we get parts $2k, 2k-1, \dots, k+1, k$, which form the partition of $3k(k+1)/2$, which appears on the right side of (4.2). The remaining k parts after the subtraction form a partition π^* which has parts less than or equal to $(n-k)$. We take the conjugates of R_1 and π^*, and paste the conjugates together so that we get a partition π^{**} with $n-k$ parts greater than or equal to $(k+1)$, and less than or equal to $2k+1$. In II, we take parts greater than k, and add the parts to π^{**} using the Sylvester map for

$$\frac{1}{(q;q)_{n-k}} = \frac{1}{(q^{k+1};q)_{n-k}} \begin{bmatrix} n+1 \\ n-k \end{bmatrix}_q .$$

Hence we get a partition into parts greater than k. Combining the resulting partition with the parts in II which are less than or equal to k, we form a partition, which is counted by $1/(q;q)_\infty$ on the right side of (4.2). □

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