CONGRUENCES FOR THE COEFFICIENTS OF QUOTIENTS OF EISENSTEIN SERIES[1]

BRUCE C. BERNDT[2] AND AE JA YEE[3]

1. Introduction

Ramanujan, in [6], [7, pp. 232–238], established the following famous congruences for $p(n)$, the number of partitions of n:

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

In calculating the coefficients of certain quotients of the Eisenstein series

$$P(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k} = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n, \tag{1.1}$$

$$Q(q) := 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k} = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{1.2}$$

and

$$R(q) := 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1-q^k} = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \tag{1.3}$$

studied in [1] and [2], where $|q| < 1$, we noticed that for some quotients of Eisenstein series the coefficients in certain arithmetic progressions are divisible by prime powers, usually a power of 3. In view of Ramanujan's famous congruences for $p(n)$, it seemed natural for us to systematically investigate congruences of this type for Eisenstein series. In some cases, it

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[1] 2001 Mathematics Subject Classification: 11F30

[2] Research partially supported by grant MDA904-00-1-0015 from the National Security Agency.

[3] Research partially supported by a grant from the Number Theory Foundation.

was very easy to establish our observations, but in other cases, the task was considerably more difficult.

We summarize our findings in the table below. In general, write $F(q) := \sum_{n=0}^{\infty} \alpha_n q^n$.

$F(q)$	$n \equiv 2 \pmod 3$	$n \equiv 4 \pmod 8$
$1/P(q)$	$\alpha_n \equiv 0 \pmod{3^4}$	
$1/Q(q)$	$\alpha_n \equiv 0 \pmod{3^2}$	
$1/R(q)$	$\alpha_n \equiv 0 \pmod{3^3}$	$\alpha_n \equiv 0 \pmod{7^2}$
$P(q)/Q(q)$	$\alpha_n \equiv 0 \pmod{3^3}$	
$P(q)/R(q)$	$\alpha_n \equiv 0 \pmod{3^2}$	$\alpha_n \equiv 0 \pmod 7$
$Q(q)/R(q)$	$\alpha_n \equiv 0 \pmod{3^3}$	
$P^2(q)/R(q)$	$\alpha_n \equiv 0 \pmod{3^5}$	

To prove our observations, we need to carefully examine $\sigma_k(n)$, the sum of the kth power of the divisors of the positive integer n for odd k. In Section 2, we calculate $\sigma_k(n)$ for $n \equiv 2 \pmod 3$, and state congruences and equalities for $\sigma_k(n)$ established by D. B. Lahiri in [4] and [5]. In Section 3, congruences for the coefficients of $1/Q(q)$, $1/R(q)$, $P(q)/Q(q)$, $P(q)/R(q)$, and $Q(q)/R(q)$ are proved very easily. We show the congruences for the coefficients of $1/P(q)$ and $P^2(q)/R(q)$ in Section 4 and Section 5, respectively.

2. PRELIMINARIES

In the sequel, let k be an odd positive integer, and we write $\sigma(n)$ for $\sigma_1(n)$. We examine $\sigma_k(p^r)$, where p is a prime. We consider 3 cases: (i) $p = 3x - 1$ and r is odd, (ii) $p = 3x - 1$ and r is even, and (iii) $p = 3x + 1$.

Case (i): $p = 3x - 1$ and r is odd. It follows easily from elementary considerations below that

$$\sigma_k(p^r) \equiv 0 \pmod 3. \tag{2.1}$$

Moreover,

$$\sigma_3(p^r) \equiv 0 \pmod{3^2}. \tag{2.2}$$

However, we need a more refined congruence in some of our applications. To that end,

$$\begin{aligned}\sigma_k(p^r) &= 1 + p^k + \cdots + p^{rk}\\ &= (1 + p^k)(1 + p^{2k} + \cdots + p^{(r-1)k})\\ &\equiv (1 + (3x-1)^k)(a_k + 3b_k + 3^2 c_k) \pmod{3^4},\end{aligned} \tag{2.3}$$

where

$$a_k := \sum_{j=0}^{(r-1)/2} (-1)^{2jk} = \frac{r+1}{2}, \tag{2.4}$$

$$b_k := \sum_{j=0}^{(r-1)/2} (-1)^{2jk-1} \binom{2jk}{1} x = -x\frac{r^2-1}{4}k, \tag{2.5}$$

and

$$c_k := \sum_{j=0}^{(r-1)/2} (-1)^{2jk-2} \binom{2jk}{2} x^2 = \frac{(r^2-1)r}{12}x^2k^2 - \frac{r^2-1}{8}x^2k =: uk^2 + vk. \tag{2.6}$$

Case (ii): $p = 3x - 1$ and r is even. Recall that k is odd. Then

$$\sigma_k(p^r) = 1 + p^k + \cdots + p^{rk} \equiv (A_k + 3B_k + 3^2C_k) \pmod{3^3}, \tag{2.7}$$

where

$$A_k := \sum_{j=0}^{r} (-1)^{jk} = 1, \tag{2.8}$$

$$B_k := \sum_{j=0}^{r} (-1)^{jk-1} \binom{jk}{1} x = -\frac{r}{2}xk, \tag{2.9}$$

and

$$\begin{aligned} C_k &:= \sum_{j=0}^{r} (-1)^{jk-2} \binom{jk}{2} x^2 \\ &= \left(\sum_{j=0}^{r} (-1)^j \frac{j^2}{2} x^2 \right) k^2 - \left(\sum_{j=0}^{r} (-1)^j \frac{j}{2} x^2 \right) k =: u_k k^2 + v_k k. \end{aligned} \tag{2.10}$$

Case (iii): $p = 3x + 1$. Then

$$\sigma_k(p^r) = 1 + p^k + \cdots + p^{rk} \equiv (A_k + 3B_k + 3^2C_k) \pmod{3^3}, \tag{2.11}$$

where

$$A_k := \sum_{j=0}^{r} 1 = r + 1, \tag{2.12}$$

$$B_k := \sum_{j=0}^{r} \binom{jk}{1} x = \frac{r(r+1)}{2}xk, \tag{2.13}$$

and

$$\begin{aligned}C_k &:= \sum_{j=0}^{r} \binom{jk}{2} x^2 \\ &= \frac{r(r+1)(2r+1)}{12} x^2k^2 - \frac{r(r+1)}{4} x^2 k =: u_k k^2 + v_k k. \end{aligned} \tag{2.14}$$

We need a congruence for $\sigma_k(n)$ for $n \equiv 2 \pmod 3$. There is at least one prime factor p of n such that $p \equiv 2 \pmod 3$ and the maximum power of p is odd for any $n \equiv 2 \pmod 3$. Let $n = p^r p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$, where $p \equiv 2 \pmod 3$, r is odd, and $p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m} \equiv 1 \pmod 3$. Since σ_k is multiplicative, we see by (2.1) and (2.2) that

$$\sigma_k(n) \equiv 0 \pmod 3 \qquad \text{and} \qquad \sigma_3(n) \equiv 0 \pmod{3^2}. \tag{2.15}$$

Now, we consider $\sigma_k(p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m})$. Let

$$\sigma_k(p_i^{r_i}) :\equiv A_{ki} + 3B_{ki} + 3^2 C_{ki} \pmod{3^3}, \tag{2.16}$$

for $i = 1, 2, \dots, m$, and set, as in (2.10) or (2.14), $C_{ki} = u_{ki}k^2 + v_{ki}k$. Then

$$\begin{aligned}\sigma_k(p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}) &= \sigma_k(p_1^{r_1})\sigma_k(p_2^{r_2}) \cdots \sigma_k(p_m^{r_m}) \\ &\equiv \hat{A}_k + 3\hat{B}_k + 3^2 \hat{C}_k \pmod{3^3},\end{aligned} \tag{2.17}$$

where

$$\hat{A}_k := \prod_{i=1}^{m} A_{ki}, \tag{2.18}$$

$$\hat{B}_k := \hat{A}_k \sum_{i=1}^{m} \frac{B_{ki}}{A_{ki}}, \tag{2.19}$$

and

$$\hat{C}_k := \hat{A}_k \sum_{i=1}^{m} \frac{C_{ki}}{A_{ki}} + \hat{A}_k \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{B_{ki}B_{kj}}{A_{ki}A_{kj}}. \tag{2.20}$$

Then we easily see by elementary calculations on $\sigma_k(p_i^{r_i})$ that

$$\hat{A}_k = \hat{A}_1, \tag{2.21}$$

$$\hat{B}_k = k\hat{B}_1, \tag{2.22}$$

and

$$\hat{C}_k = k^2 U + kV, \tag{2.23}$$

where

$$U = \hat{A}_1 \sum_{i=1}^{m} \frac{u_{1i}}{A_{1i}} + \hat{A}_1 \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{B_{1i}B_{1j}}{A_{1i}A_{1j}} \quad \text{and} \quad V = \hat{A}_1 \sum_{i=1}^{m} \frac{v_{1i}}{A_{1i}}. \tag{2.24}$$

Necessary for our proofs are certain identities and congruences for $\sigma_k(n)$. Before stating them, recall that Ramanujan's tau function $\tau(n)$ is defined by

$$\sum_{n=1}^{\infty} \tau(n) q^n := q \prod_{n=1}^{\infty} (1-q^n)^{24}, \qquad \text{for} \quad |q|<1.$$

Lahiri [4], [5] established many identities and congruences for $\sigma_k(n)$ and $\tau(n)$. Among them we state the identities and congruences we use in the remainder of the paper. Thus,

$$2^2 \cdot 3 \sum_{k=1}^{n-1} \sigma(k)\,\sigma(n-k) = 5\sigma_3(n) - (6n-1)\sigma(n), \tag{2.25}$$

$$\begin{aligned} 2^6 \cdot 3 \sum_{k_1+k_2=1}^{n-1} \sigma(k_1)\,\sigma(k_2)\,\sigma(n-k_1-k_2) =& 7\sigma_5(n) + (10-30n)\sigma_3(n) \\ &+ (1-12n+24n^2)\sigma(n), \end{aligned} \tag{2.26}$$

$$2^4 \cdot 3^2 \cdot 5 \cdot 7 \sum_{k=1}^{n-1} \sigma_3(k)\,\sigma_5(n-k) = 11\sigma_9(n) - 3 \cdot 7\sigma_5(n) + 2 \cdot 5\sigma_3(n), \tag{2.27}$$

$$2^3 \cdot 3^2 \cdot 7 \sum_{k=1}^{n-1} k\,\sigma(k)\,\sigma_5(n-k) = 5n\sigma_7(n) - 2 \cdot 3n^2\sigma_5(n) + n\sigma(n), \tag{2.28}$$

$$2^2 \cdot 3^2 \cdot 7 \cdot 691 \sum_{k=1}^{n-1} \sigma_5(k)\,\sigma_5(n-k) = -\,2^2 \cdot 3^3 \cdot 7\tau(n) + 5 \cdot 13\sigma_{11}(n) + 691\sigma_5(n), \tag{2.29}$$

and

$$\begin{aligned} -2^2 \cdot 3^3 \cdot 7\tau(n) + 5 \cdot 13\sigma_{11}(n) \equiv& 691\{20\sigma_7(n) - 2(21n-10)\sigma_5(n) - 105\sigma_3(n) \\ &+ 2(63n-10)\sigma(n)\} \pmod{2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 691}. \end{aligned} \tag{2.30}$$

The most thorough examination of divisor sum identities like those in (2.25)–(2.29) has been given by J. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams in [3].

By combining (2.29) and (2.30), we obtain

$$\begin{aligned}2^2\cdot 3^2\cdot 7\sum_{k=1}^{n-1}\sigma_5(k)\,\sigma_5(n-k) \equiv{}& 20\sigma_7(n)-2(21n-10)\sigma_5(n)-105\sigma_3(n)\\ &+2(63n-10)\sigma(n)+\sigma_5(n)\pmod{2^4\cdot 3^4\cdot 5^2\cdot 7}.\end{aligned}\tag{2.31}$$

We need more congruences, which are found in [4] and [5], namely,

$$n\sigma_7(n)\equiv 14n\sigma_5(n)-(24n^2-11n)\sigma_3(n)\pmod{2^5\cdot 3^2\cdot 5}\tag{2.32}$$

and

$$\begin{aligned}11\sigma_9(n)\equiv{}&50(30n-2)\sigma_7(n)-30(24n^2-28n+7)\sigma_5(n)+20(72n^3-108n^2+45n\\ &-5)\sigma_3(n)-(864n^4-1440n^3+720n^2-120n+5)\sigma(n)\pmod{2^{12}\cdot 3^4}.\end{aligned}\tag{2.33}$$

3. COEFFICIENTS OF $\frac{1}{R}$, $\frac{1}{Q}$, $\frac{P}{Q}$, $\frac{P}{R}$, AND $\frac{Q}{R}$

In this section, we show that the coefficient of q^n in $1/R(q)$ is divisible by 3^3 and 7^2 for $n\equiv 2\pmod 3$ and $n\equiv 4\pmod 8$, respectively. Since the proofs of the assertions for $1/Q(q)$, $P(q)/Q(q)$, $P(q)/R(q)$, and $Q(q)/R(q)$ are similar, we omit the proofs.

Theorem 3.1. *In each case, set* $F(q)=\sum_{n=0}^{\infty}\alpha_n q^n$, $|q|<1$. *Let* $n\equiv 2\pmod 3$.

(a) *If* $F(q)=1/R(q)$, *then* $\alpha_n\equiv 0\pmod{3^3}$;
(b) *if* $F(q)=1/Q(q)$, *then* $\alpha_n\equiv 0\pmod{3^2}$;
(c) *if* $F(q)=P(q)/Q(q)$, *then* $\alpha_n\equiv 0\pmod{3^3}$;
(d) *if* $F(q)=P(q)/R(q)$, *then* $\alpha_n\equiv 0\pmod{3^2}$;
(e) *if* $F(q)=Q(q)/R(q)$, *then* $\alpha_n\equiv 0\pmod{3^3}$.

Let $n\equiv 4\pmod 8$.

(f) *If* $F(q)=1/R(q)$, *then* $\alpha_n\equiv 0\pmod{7^2}$;
(g) *if* $F(q)=P(q)/R(q)$, *then* $\alpha_n\equiv 0\pmod 7$.

Proof of (a). For sufficiently small $|q|$, from (1.3), we consider the geometric series expansion of $1/R(q)$. Then

$$\begin{aligned}\frac{1}{R(q)} &= 1 + 504\sum_{n=1}^{\infty}\sigma_5(n)q^n + 504^2\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\sigma_5(k)\,\sigma_5(n-k)q^n + \cdots\\ &=: \sum_{n=0}^{\infty}\alpha_n q^n.\end{aligned}$$

Since $\sigma_k(n)$ is divisible by 3 when n is congruent to 2 modulo 3, as we noted in (2.15), we can easily see that

$$\alpha_n \equiv 0 \pmod{3^3}, \qquad \text{if } n \equiv 2 \pmod 3.$$

Proof of (f). To show that $\alpha_n \equiv 0 \pmod{7^2}$ when $n \equiv 4 \pmod 8$, we need to calculate $\sigma_5(8y+4)$,

$$\sigma_5(8y+4) = \sigma_5(2^2)\sigma_5(2y+1) = (1+2^5+2^{10})\sigma_5(2y+1) \equiv 0 \pmod 7.$$

That implies that $\alpha_n \equiv 0 \pmod{7^2}$ when $n \equiv 4 \pmod 8$. □

4. Coefficients of $\frac{1}{P}$

We prove the congruence for the coefficients of q^n for $1/P(q)$.

Theorem 4.1. *Set* $1/P(q) = \sum_{n=0}^{\infty}\alpha_n q^n$, $|q| < 1$. *Then*

$$\alpha_n \equiv 0 \pmod{3^4} \qquad \textit{for} \quad n \equiv 2 \pmod 3.$$

Proof. For sufficiently small $|q|$, from (1.1), we take the geometric series expansion of $1/P(q)$. Then

$$\begin{aligned}\frac{1}{P(q)} =& 1 + 24\sum_{n=1}^{\infty}\sigma(n)q^n + 24^2\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\sigma(k)\,\sigma(n-k)q^n\\ &+ 24^3\sum_{n=3}^{\infty}\sum_{k=2}^{n-1}\sum_{k_1=1}^{k-1}\sigma(k_1)\,\sigma(k-k_1)\,\sigma(n-k)q^n + \cdots.\end{aligned}$$

So, for $n \equiv 2 \pmod 3$,

$$\begin{aligned}\alpha_n \equiv& 3\cdot 8\sigma(n) + 3^2\cdot 8^2\sum_{k=1}^{n-1}\sigma(k)\,\sigma(n-k)\\ &+ 3^3\cdot 8^3\sum_{k=2}^{n-1}\sum_{k_1=1}^{k-1}\sigma(k_1)\,\sigma(k-k_1)\,\sigma(n-k) \pmod{3^4}. \qquad (4.1)\end{aligned}$$

By (2.25), (2.26), (2.15) and (4.1), we see that for $n \equiv 2 \pmod 3$,

$$\alpha_n \equiv 3 \cdot 8(10\sigma_3(n) + 3(1-4n)\sigma(n)) \pmod{3^4}.$$

So we only need to show that for $n \equiv 2 \pmod 3$,

$$8(10\sigma_3(n) + 3(1-4n)\sigma(n)) \equiv 0 \pmod{3^3}, \tag{4.2}$$

and

$$7\sigma_5(n) + \sigma(n) \equiv 0 \pmod{3^2}. \tag{4.3}$$

Since $n \equiv 2 \pmod 3$, it has at least one prime factor p that is congruent to 2 modulo 3 and whose power r in n is odd. Furthermore, the number of such prime factors must be odd since $n \equiv 2 \pmod 3$. Suppose that there are more than two prime factors of n that are congruent to 2 modulo 3 and with powers in n that are odd. Then the congruence (4.2) can be achieved easily by (2.1) and (2.2), since $\sigma_k(n)$ is multiplicative. So we can suppose that there is only one prime factor $p \equiv 2 \pmod 3$ whose power in n is odd. Let $n = p^r(3N+1)$, where $p = 3x-1$, r is odd, and N is nonnegative. By substituting $p^r(3N+1)$ for n in (4.2), we obtain

$$\begin{aligned} &8(10\sigma_3(n) + 3(1-4n)\sigma(n)) \\ &\equiv 8(10\sigma_3(p^r)\sigma_3(3N+1) + 3(1-4p^r(3N+1))\sigma(p^r)\sigma(3N+1)) \pmod{3^3}. \end{aligned} \tag{4.4}$$

We replace p by $3x-1$ and simplify it using (2.1). Then (4.4) is equivalent to

$$\begin{aligned} &8(10\sigma_3(n) + 3(1-4n)\sigma(n)) \\ &\equiv 8(10\sigma_3((3x-1)^r)\sigma_3(3N+1) + 15\sigma((3x-1)^r)\sigma(3N+1)) \pmod{3^3}. \end{aligned} \tag{4.5}$$

By (2.3) and (2.21), we see that equation (4.5) is equivalent to

$$40(2 \cdot 3^2 x \cdot a_3\hat{A}_3 + 3 \cdot 3x \cdot a_1\hat{A}_1) \equiv 40 \cdot 3^2 x(2a_1\hat{A}_1 + a_1\hat{A}_1) \equiv 0 \pmod{3^3}, \tag{4.6}$$

since $\hat{A}_3 = \hat{A}_1$. By (4.4)–(4.6), the congruence (4.2) is derived. In similar way, we can show the congruence (4.3). Thus the proof of Theorem 4.1 is complete. ☐

5. COEFFICIENTS OF $\frac{P^2}{R}$

In this section, we prove the congruence for the coefficients of q^n for $P^2(q)/R(q)$.

Theorem 5.1. *Set* $P^2(q)/R(q) = \sum_{n=0}^{\infty} \alpha_n q^n$, $|q| < 1$. *Then*

$$\alpha_n \equiv 0 \pmod{3^5}, \qquad \textit{for} \quad n \equiv 2 \pmod 3.$$

Proof. As we did in the previous sections, for sufficiently small $|q|$, from (1.1) and (1.3), we take the geometric series expansion of $P^2(q)/R(q)$,

$$\begin{aligned}
\frac{P^2(q)}{R(q)} &= \left(1 - 48\sum_{n=1}^{\infty}\sigma(n)q^n + 24^2\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\sigma(k)\,\sigma(n-k)q^n\right)\left(1 + 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right. \\
&\qquad \left. + 504^2\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\sigma_5(k)\,\sigma_5(n-k)q^n + \cdots\right) \\
&= \left(1 - 48\sum_{n=1}^{\infty}\sigma(n)q^n + 48\sum_{n=1}^{\infty}\big(5\sigma_3(n) - (6n-1)\sigma(n)\big)q^n\right)\left(1 + 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right. \\
&\qquad \left. + 504^2\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\sigma_5(k)\,\sigma_5(n-k)q^n + \cdots\right),
\end{aligned}$$

where the last step is obtained by (2.25). Then, for $n \equiv 2 \pmod 3$,

$$\begin{aligned}
\alpha_n \equiv& -48\sigma(n) + 48\big(5\sigma_3(n) - (6n-1)\sigma(n)\big) + 504\sigma_5(n) + 504^2\sum_{k=1}^{n-1}\sigma_5(k)\,\sigma_5(n-k) \\
&- 48\cdot 504\sum_{k=1}^{n-1}\sigma(k)\sigma_5(n-k) + 48\cdot 504\sum_{k=1}^{n-1}\big(5\sigma_3(k) - (6k-1)\sigma(k)\big)\sigma_5(n-k) \pmod{3^5} \\
\equiv& 48\big(5\sigma_3(n) - 6n\sigma(n)\big) + 504\sigma_5(n) + 504^2\sum_{k=1}^{n-1}\sigma_5(k)\,\sigma_5(n-k) \\
&+ 48\cdot 504\sum_{k=1}^{n-1}5\sigma_3(k)\sigma_5(n-k) - 48\cdot 504\sum_{k=1}^{n-1}6k\sigma(k)\sigma_5(n-k) \pmod{3^5}.
\end{aligned} \tag{5.1}$$

By (2.27), (2.28), and (5.1), we see that

$$\begin{aligned}
\alpha_n \equiv& 2^4\cdot 3\big(5\sigma_3(n) - 6n\sigma(n)\big) + 2^3\cdot 3^2\cdot 7\sigma_5(n) + 2^6\cdot 3^4\cdot 7^2\sum_{k=1}^{n-1}\sigma_5(k)\,\sigma_5(n-k) \\
&+ 2^3\cdot 3\big(11\sigma_9(n) - 3\cdot 7\sigma_5(n) + 2\cdot 5\sigma_3(n)\big) - 2^5\cdot 3^2\big(5n\sigma_7(n) - 2\cdot 3n^2\sigma_5(n) \\
&+ n\sigma(n)\big) \pmod{3^5}.
\end{aligned}$$

By (2.31) and simplification, we see that

$$\begin{aligned}\alpha_n \equiv& -2^6\cdot 3^2\cdot 5\cdot 7\sigma(n)+2^5\cdot 3^2\cdot 439n\sigma(n)-2^4\cdot 3\cdot 5\cdot 439\sigma_3(n)+2^4\cdot 3^3\cdot 7^2\sigma_5(n)\\ &-2^5\cdot 3^3\cdot 7^2 n\sigma_5(n)+2^6\cdot 3^3 n^2\sigma_5(n)+2^6\cdot 3^2\cdot 5\cdot 7\sigma_7(n)-2^5\cdot 3^2\cdot 5n\sigma_7(n)\\ &+2^3\cdot 3\cdot 11\sigma_9(n) \pmod{3^5}.\end{aligned}$$

By congruence (2.33), we see that

$$\begin{aligned}\alpha_n \equiv&(-2^3\cdot 3\cdot 5\cdot 13^2+2^5\cdot 3^2\cdot 449n-2^7\cdot 3^3\cdot 5\cdot n^2+2^8\cdot 3^3\cdot 5n^3-2^8\cdot 3^4n^4)\sigma(n)\\ &+(-2^4\cdot 3\cdot 5\cdot 449+2^5\cdot 3^3\cdot 5^2n-2^7\cdot 3^4\cdot 5\cdot n^2+2^8\cdot 3^3\cdot 5n^3)\sigma_3(n)\\ &+(2^8\cdot 3^2\cdot 7-2^5\cdot 3^2\cdot 7\cdot 11n-2^6\cdot 3^5n^2)\sigma_5(n)\\ &+(2^5\cdot 3\cdot 5\cdot 37+2^8\cdot 3^3\cdot 5n)\sigma_7(n) \pmod{3^5}.\end{aligned}$$

We use congruence (2.32) to obtain the equivalent congruence,

$$\begin{aligned}\alpha_n \equiv&(2^2\cdot 3\cdot 11+2^2\cdot 3^2n+2^3\cdot 3^3n^2+2\cdot 3^3n^3+2\cdot 3^4n^4)\sigma(n)+(2^2\cdot 3\cdot 11+3^4n\\ &-3^4n^2+2\cdot 3^3n^3)\sigma_3(n)+(2\cdot 3^2\cdot 5+3^2\cdot 23n)\sigma_5(n)+3\cdot 7\sigma_7(n) \pmod{3^5}.\end{aligned}$$

Since $n\equiv 2 \pmod 3$, terms with a factor of $3^4\sigma(n)$, $3^3\sigma_3(n)$, $3^4\sigma_5(n)$ and $3^4\sigma_7(n)$ cancel by (2.15). Next, setting $n=3k-1$ everywhere, expanding all powers of $3k-1$, using (2.15), we find that

$$\alpha_n \equiv (24+9n)\sigma(n)+24\sigma_3(n)+(63+18n)\sigma_5(n)+21\sigma_7(n) \pmod{3^5}.$$

Therefore, when $n\equiv 2 \pmod 3$,

$$\alpha_n \equiv 3\{(8+3n)\sigma(n)+8\sigma_3(n)+(21+6n)\sigma_5(n)+7\sigma_7(n)\} \pmod{3^5}.$$

So we only need to show that

$$(8+3n)\sigma(n)+8\sigma_3(n)+(21+6n)\sigma_5(n)+7\sigma_7(n)\equiv 0 \pmod{3^4}. \tag{5.2}$$

Since $n\equiv 2 \pmod 3$, it has at least one prime factor p that is congruent to 2 modulo 3 and whose power r in n is odd. Furthermore, the number of such prime factors must be odd since $n\equiv 2 \pmod 3$. Suppose that there are more than three prime factors of n that are congruent to 2 modulo 3 and with powers in n that are odd. Then the congruence (5.2) can be achieved easily by (2.1), since $\sigma_k(n)$ is multiplicative. So we can suppose that there are at most three prime factors congruent to 2 modulo 3 whose powers in n is odd. Let $n=p^rp_1^{r_1}\cdots p_m^{r_m}$. We consider two cases : (i) there are exactly three primes $p,p_1,p_2\equiv 2 \pmod 3$ whose powers r,r_1,r_2 in n are odd, (ii) there is only one prime $p\equiv 2 \pmod 3$ whose power r in n is odd. We use $a_k,b_k,c_k,\hat{A}_k,\hat{B}_k$, and $\hat{C}_k$ as defined in Section 2.

Case (i): $n=p^rp_1^{r_1}\cdots p_m^{r_m}$, where $p=3x-1$, r is odd, and $p_i=3x_i-1$, r_i is odd for $i=1,2$. Let $p_1^{r_1}\cdots p_m^{r_m}=3N+1$. By substituting $p^r(3N+1)$

for n in (5.2), the congruence becomes

$$(8+3p^r+3^2p^rN)\sigma(n)+8\sigma_3(n)+(21+6p^r+18p^rN)\sigma_5(n)+7\sigma_7(n)\equiv 0 \pmod{3^4},$$

which is equivalent to

$$8\sigma(n)+8\sigma_3(n)+7\sigma_7(n)\equiv 0 \pmod{3^4}, \tag{5.3}$$

since $\sigma_k(n)$ is multiplicative and $\sigma_k(p^r), \sigma_k(p_1^{r_1})$ and $\sigma_k(p_2^{r_2})$ are divisible by 3 by (2.1). Furthermore, $\sigma_3(p^r)$ is divisible by 3^2 by (2.2). So, we see that $\sigma_3(n)\equiv 0 \pmod{3^4}$. Hence, (5.3) is equivalent to

$$\begin{aligned}8\sigma(n)+7\sigma_7(n) &= 8\sigma(p^r)\sigma(3N+1)+7\sigma_7(p^r)\sigma_7(3N+1)\\ &\equiv 8\sigma(p^r)(\hat{A}_1+3\hat{B}_1+3^2\hat{C}_1)+7\sigma_7(p^r)(\hat{A}_7+3\hat{B}_7+3^2\hat{C}_7) \pmod{3^4},\end{aligned} \tag{5.4}$$

where $\hat{A}_k, \hat{B}_k$, and $\hat{C}_k$ are defined by (2.18)–(2.20). We see that $\hat{A}_k$ and $\hat{B}_k$, $k=1,7$, are zero since $p_1\equiv p_2\equiv 2 \pmod 3$ and $r_1\equiv r_2\equiv 1 \pmod 2$. By (2.3), we see that (5.4) is equivalent to

$$8\cdot 3^3xa_1\hat{C}_1+7^2\cdot 3^3xa_7\hat{C}_7\equiv 3^3xa_1(8+7^3)\hat{C}_1\equiv 0 \pmod{3^4},$$

since $a_1=a_7$ and $\hat{C}_1=7\hat{C}_7$ by (2.23).

Case (ii) : $n=p^rp_1^{r_1}\cdots p_m^{r_m}$, where $p=3x-1$ and r is odd. Let $p_1^{r_1}\cdots p_m^{r_m}=3N+1$. Then, by substituting $p^r(3N+1)$ for n, (5.2) becomes

$$\begin{aligned}&(8+3p^r+3^2p^rN)\sigma(p^r)\sigma(3N+1)+8\sigma_3(p^r)\sigma_3(3N+1)\\ &+(21+6p^r+18p^rN)\sigma_5(p^r)\sigma_5(3N+1)+7\sigma_7(p^r)\sigma_7(3N+1)\equiv 0 \pmod{3^4},\end{aligned}$$

which, by (2.17), is equivalent to

$$\begin{aligned}&(8+3p^r+3^2p^rN)\sigma(p^r)(\hat{A}_1+3\hat{B}_1+3^2\hat{C}_1)+8\sigma_3(p^r)(\hat{A}_3+3\hat{B}_3+3^2\hat{C}_3)\\ &+(21+6p^r+18p^rN)\sigma_5(p^r)(\hat{A}_5+3\hat{B}_5+3^2\hat{C}_5)+7\sigma_7(p^r)(\hat{A}_7+3\hat{B}_7+3^2\hat{C}_7)\equiv 0 \pmod{3^4}.\end{aligned} \tag{5.5}$$

By (2.21)–(2.23), formula (5.5) is equivalent to

$$\begin{aligned}&\big\{(8+3p^r)\sigma(p^r)+8\sigma_3(p^r)+(21+6p^r)\sigma_5(p^r)+7\sigma_7(p^r)\big\}\hat{A}_1\\ &+\big\{(8+3p^r)\sigma(p^r)+3\cdot 8\sigma_3(p^r)+5(21+6p^r)\sigma_5(p^r)+7^2\sigma_7(p^r)\big\}3\hat{B}_1\\ &+\big\{(8+3p^r)\sigma(p^r)+3^2\cdot 8\sigma_3(p^r)+5^2(21+6p^r)\sigma_5(p^r)+7^3\sigma_7(p^r)\big\}3^2U\\ &+\big\{(8+3p^r)\sigma(p^r)+3\cdot 8\sigma_3(p^r)+5(21+6p^r)\sigma_5(p^r)+7^2\sigma_7(p^r)\big\}3^2V\\ &+3^2p^rN\big\{\sigma(p^r)+2\sigma_5(p^r)\big\}\hat{A}_1\equiv 0 \pmod{3^4}.\end{aligned} \tag{5.6}$$

To show (5.6), we examine carefully each expression in curly brackets in (5.6). Since $p = 3x - 1$, we see that $p^r \equiv -1 + 3rx \pmod{3^2}$. By (2.3) we see that

$$\begin{aligned}
&(8+3p^r)\sigma(p^r) + 8\sigma_3(p^r) + (21+6p^r)\sigma_5(p^r) + 7\sigma_7(p^r)\\
&\equiv (5+3^2rx)(3x)(a_1+3b_1+3^2c_1) + 8\cdot 3^2x(1-3x+3x^2)(a_3+3b_3+3^2c_3)\\
&\quad + (15+2\cdot 3^2rx)(15x)(1-6x+18x^2)(a_5+3b_5+3^2c_5)\\
&\quad + 7^2\cdot 3x(1-9x+45x^2)(a_7+3b_7+3^2c_7) \pmod{3^4}. \qquad (5.7)
\end{aligned}$$

By (2.4)–(2.6), after reducing some coefficients modulo 3^4, we see that (5.7) is equivalent to

$$\begin{aligned}
&27x(2+x^2)\frac{r+1}{2} + 27x^2(2r+1)\frac{r+1}{2} - 27x^2\frac{r^2-1}{4} + (5+7^4)3^3xu + (5+7^3)3^3xv\\
&\equiv 27x(2+x^2)\frac{r+1}{2} + 27x^2\frac{(r+1)(3r+3)}{4}\\
&\equiv 0 \pmod{3^4}, \qquad (5.8)
\end{aligned}$$

since $x(2+x^2) \equiv 0 \pmod 3$.

We next examine the coefficient of $3\hat{B}_1$ in (5.6). By the congruence $p^r \equiv -1 + 3rx \pmod{3^2}$, (2.3), (2.4), and (2.5), we see that

$$\begin{aligned}
&(8+3p^r)\sigma(p^r) + 3\cdot 8\sigma_3(p^r) + 5(21+6p^r)\sigma_5(p^r) + 7^2\sigma_7(p^r)\\
&\equiv (5+3^2rx)(3x)(a_1+3b_1+3^2c_1) + 3^3\cdot 8x(1-3x+3x^2)(a_3+3b_3+3^2c_3)\\
&\quad + 5^2(15+2\cdot 3^2rx)(3x)(1-6x+18x^2)(a_5+3b_5+3^2c_5)\\
&\quad + 7^3\cdot 3x(1-9x+45x^2)(a_7+3b_7+3^2c_7)\\
&\equiv 15xa_1 + 3^2\cdot 5xb_1 + 3^2\cdot 5^3xa_5 + 3\cdot 7^3xa_7 + 3^2\cdot 7^3xb_7\\
&\equiv 9xa_1 \pmod{3^3}. \qquad (5.9)
\end{aligned}$$

Using the congruence $p^r \equiv -1 + 3rx \pmod{3^2}$, (2.3), and (2.4), we find that the coefficient of 3^2U in (5.6) is

$$\begin{aligned}
&(8+3p^r)\sigma(p^r) + 3^2\cdot 8\sigma_3(p^r) + 5^2(21+6p^r)\sigma_5(p^r) + 7^3\sigma_7(p^r)\\
&\equiv (5+3^2rx)(3x)(a_1+3b_1+3^2c_1) + 3^4\cdot 8x(1-3x+3x^2)(a_3+3b_3+3^2c_3)\\
&\quad + 5^3(15+2\cdot 3^2rx)(3x)(1-6x+18x^2)(a_5+3b_5+3^2c_5)\\
&\quad + 7^4\cdot 3x(1-9x+45x^2)(a_7+3b_7+3^2c_7)\\
&\equiv 15xa_1 + 3\cdot 7^4xa_7\\
&\equiv 0 \pmod{3^2}. \qquad (5.10)
\end{aligned}$$

By (5.9), we see that the coefficient of 3^2V in (5.6) is

$$(8+3p^r)\sigma(p^r)+3^2\cdot 8\sigma_3(p^r)+5^2(21+6p^r)\sigma_5(p^r)+7^3\sigma_7(p^r)\equiv 0 \pmod{3^2}. \tag{5.11}$$

We examine the coefficient of the last term in (5.6). By (2.3), we see that

$$3^2p^rN(\sigma(p^r)+2\sigma_5(p^r))\equiv 9p^rN(3xa_1+30xa_5)\equiv 27xa_1N \pmod{3^4}. \tag{5.12}$$

By combining (5.8)–(5.12), we see that (5.6) is equivalent to

$$3^3xa_1\hat{B}_1+3^3xa_1\hat{A}_1N\equiv 0 \pmod{3^4}. \tag{5.13}$$

By (2.8), (2.9), (2.12), (2.13), and (2.19), we see that (5.13) is equivalent to

$$3^3xa_1\hat{A}_1\left(3\sum_{j=1}^{m_1}\frac{-r_j}{2}x_j+3\sum_{j=m_1+1}^{m}\frac{r_j}{2}x_j\right)\equiv 0 \pmod{3^4},$$

where m_1 is the number of $p_i\equiv 2 \pmod 3$ in $3N+1$, and $\hat{A}_1r_j/2$ is an integer since $\hat{A}_1=\prod_{j=m_1+1}^{m}(r_j+1)$.

This then completes the proof of (5.6) and hence also of (5.2) in Case (ii). The proof of Theorem 5.1 is thus complete. □

Acknowledgment. The authors are grateful to Alexandru Zaharescu for informing them that they had originally missed the congruence modulo 7 in the table.

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Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA
E-mail address: `berndt@math.uiuc.edu`

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA
E-mail address: `yee@math.uiuc.edu`