ON THE GENERALIZED ROGERS–RAMANUJAN CONTINUED FRACTION

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Abstract. On page 26 in his lost notebook, Ramanujan states an asymptotic formula for the generalized Rogers–Ramanujan continued fraction. This formula is proved and made slightly more precise. A second primary goal is to prove another continued fraction representation for the Rogers–Ramanujan continued fraction conjectured by R. Blecksmith and J. Brillhart. Two further entries in the lost notebook are examined. One of them is an identity bearing a superficial resemblance to the generating function for the generalized Rogers–Ramanujan continued fraction. Thus, our third main goal is to establish, with the help of an idea of F. Franklin, a partition bijection to prove this identity.

1. Introduction

We begin by offering the standard notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad n \geq 1, \quad \text{and} \quad (a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \quad |q|<1.$$

The generalized Rogers–Ramanujan continued fraction in the title is defined for $|q|<1$ and for any complex number a by

$$R(a,q) := \frac{1}{1} \, \frac{}{+} \, \frac{aq}{1} \, \frac{}{+} \, \frac{aq^2}{1} \, \frac{}{+} \, \frac{aq^3}{1} \, \frac{}{+} \cdots. \tag{1.1}$$

The Rogers–Ramanujan continued fraction is the special case

$$R(1,q) =: R(q) = \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{1.2}$$

where the representation for $R(q)$ in (1.2) is due to L. J. Rogers [12] in 1894 and Ramanujan [8], [9, 214–215] in about 1912.

On page 26 in his lost notebook (in the pagination of [11]), Ramanujan offers the asymptotic expansion, as $x \to 0+$,

$$R(a,e^{-x}) = \frac{-1+\sqrt{1+4a}}{2a} \exp\left(\frac{ax}{1+4a} - \frac{a(1-a)x^2}{2(1+4a)^{5/2}} + \frac{a(1-a)(1-14a)x^3}{6(1+4a)^4} - \cdots\right). \tag{1.3}$$

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In Section 2, we prove (1.3) while calculating additional terms in the asymptotic expansion (1.3). We notice that each term in the expansion from the first onward has a factor of a, which is to be expected, and each term from the second onward has a factor of $1-a$. We prove indeed that these factors do appear generally.

It is natural to ask if similar asymptotic expansions can be obtained for other q-continued fractions. We establish a similar asymptotic formula for Ramanujan's cubic continued fraction.

In Section 3 we establish a beautiful formula for the Rogers–Ramanujan continued fraction conjectured by R. Blecksmith and J. Brillhart [6]. If $R(q)$ is defined by (1.2), then Blecksmith and Brillhart conjectured that

$$R(q) = 1 - \frac{q}{1} + \frac{q}{1} - \frac{q}{1} + \frac{q}{1} - \frac{q^2}{1} + \frac{q^2}{1} - \frac{q^2}{1} + \frac{q^2}{1} - \cdots. \tag{1.4}$$

In Section 4, we discuss two entries on page 200 in Ramanujan's lost notebook [11]. On this page, Ramanujan offers an identity bearing a superficial resemblance to a standard generating function for $R(a,q)$. We provide two proofs. The first derives from a familiar transformation for q-series. The second proof is more interesting. We show that each side of the identity is a generating function for certain types of partitions. We then establish the identity by deriving a bijection between the two sets of partitions.

Below the identity described above, Ramanujan offers two close cousins of the Rogers–Ramanujan continued fraction, which he links together. We emphasize that no theorem about these continued fractions is claimed by Ramanujan. We have been unable to relate the continued fractions with the identity above them, nor with any other result of Ramanujan. Thus, it remains a mystery as to why Ramanujan recorded them here.

2. Proofs of the Asymptotic Formulas

Theorem 2.1. *As $x \to 0+$,*

$$R(a,e^{-x}) = \frac{-1+\sqrt{1+4a}}{2a} \exp\left(\frac{ax}{1+4a} - \frac{a(1-a)x^2}{2(1+4a)^{5/2}} + \frac{a(1-a)(1-14a)x^3}{6(1+4a)^4} - \cdots\right). \tag{2.1}$$

Moreover, each term of the asymptotic expansion beginning with the second has a factor of $a(1-a)$.

Proof. For brevity, set $R(a,e^{-x}) = r(a,x)$. From the definition (1.1), we observe that $r(a,x)$ satisfies the functional equation

$$r(a,x) = \frac{1}{1+ae^{-x}r(ae^{-x},x)}. \tag{2.2}$$

We use a method of successive approximations. Accordingly, we first set $x=0$, so that (2.2) takes the form

$$r(a,0) = \frac{1}{1+ar(a,0)}. \tag{2.3}$$

Solving this quadratic equation for $r(a,0)$, we find that

$$r(a,0)=\frac{-1\pm\sqrt{1+4a}}{2a}.$$

Since $r(a,0)>0$, the plus sign must be taken above. Thus, our first approximation is

$$r(a,x)\approx\frac{-1+\sqrt{1+4a}}{2a}=:c_0(a):=c_0. \tag{2.4}$$

For our second approximation, set

$$r(a,x)=c_0(a)e^{c_1(a)x}=c_0e^{c_1x}. \tag{2.5}$$

Then from (2.2),

$$r(a,x)+ae^{-x}r(a,x)r(ae^{-x},x)-1=0. \tag{2.6}$$

Using (2.4) and (2.5) in (2.6), we find that

$$c_0(a)e^{c_1(a)x}+ae^{-x}c_0(a)e^{c_1(a)x}\left(\frac{-1+\sqrt{1+4ae^{-x}}}{2ae^{-x}}\right)e^{c_1(ae^{-x})x}-1\approx 0. \tag{2.7}$$

Now,

$$\frac{1}{2}\left(-1+\sqrt{1+4ae^{-x}}\right)=\frac{1}{2}\left(-1+\sqrt{1+4a}-\frac{2ax}{\sqrt{1+4a}}+\cdots\right)=ac_0-\frac{ax}{\sqrt{1+4a}}+\cdots$$

and

$$e^{c_1(ae^{-x})x}=1+c_1(ae^{-x})x+\cdots=1+c_1(a)x+O(x^2),$$

as $x\to 0$. Using the two expansions above in (2.7) and displaying only the terms up to the first power of x, which are needed to obtain the next approximation, we set

$$c_0(1+c_1x+\cdots)+c_0(1+2c_1x+\cdots)\left(ac_0-\frac{ax}{\sqrt{1+4a}}+\cdots\right)-1=0. \tag{2.8}$$

If we equate constant coefficients in (2.8), we arrive at

$$c_0+ac_0^2-1=0,$$

which again yields (2.4). If we equate coefficients of x in (2.8), we find that

$$c_1+2ac_0c_1-\frac{a}{\sqrt{1+4a}}=0.$$

Solving for c_1 and employing (2.4), we conclude that

$$c_1=\frac{a}{1+4a}, \tag{2.9}$$

which is in agreement with what Ramanujan claims in (2.1).

For the third approximation, set

$$r(a,x)=c_0(a)e^{c_1(a)x+c_2(a)x^2}$$

and use this approximation in (2.6). We repeat the procedure detailed above to calculate $c_2(a)$. In fact, at this point, we turn to *Maple* to effect the calculations. After several iterations of (2.6), we deduce the asymptotic expansion

$$
\begin{aligned}
r(a,x) =& \frac{-1+\sqrt{1+4a}}{2a}\exp\Bigg(\frac{ax}{1+4a}-\frac{a(1-a)x^2}{2(1+4a)^{5/2}}+\frac{a(1-a)(1-14a)x^3}{6(1+4a)^4}\\
&-\frac{a(1-a)(1-66a+378a^2-20a^3)x^4}{24(1+4a)^{11/2}}\\
&+\frac{a(1-a)(1-230a+4860a^2-17000a^3+1984a^4)x^5}{120(1+4a)^7}\\
&-\frac{a(1-a)(1-726a+40530a^2-455740a^3+1155960a^4-211776a^5+976a^6)x^6}{720(1+4a)^{17/2}}\\
&+O(x^7)\Bigg). \qquad (2.10)
\end{aligned}
$$

This establishes (2.1), gives further evidence that the coefficient of $x^n, n \geq 2$, has $a(1-a)$ as a factor, and indicates that finding a general formula for the coefficient of x^n is a daunting task.

We now prove the claims about the factors a and $1-a$. The assertion about a is trivial to prove. Inducting on n, suppose that $c_j(0)=0, 1 \leq j \leq n-1$. Then from (2.6),

$$\exp\left(c_n(0)x^n + O(x^{n+1})\right) = 1.$$

It follows that $c_n(0)=0$.

The assertion about the factor $1-a$ is deeper, but follows from work of Berndt and J. Sohn [5]. In [5, Thm. 4.1], a very general asymptotic formula for q-products is proved. In fact, a slightly stronger result is proved in [5], which we now state for only the product representation for the Rogers–Ramanujan continued fraction given in (1.2) [5, Thm. 4.1, Cor. 4.2]. For every positive number $N>0$, as $x \to 0+$,

$$R(1,e^{-x}) = \frac{(e^{-x};e^{-5x})_\infty(e^{-4x};e^{-5x})_\infty}{(e^{-2x};e^{-5x})_\infty(e^{-3x};e^{-5x})_\infty} = \frac{\sqrt{5}-1}{2}\exp\left(\frac{1}{5}x + O(x^N)\right). \qquad (2.11)$$

Comparing (2.11) with (2.1), we conclude that $c_n(1)=0$ for every $n \geq 2$, since $N>0$ can be made arbitrarily large. □

The ideas used to prove Theorem 2.1 can be applied to the continued fraction

$$C(a,e^{-x}) := \frac{1}{1+}\,\frac{ae^{-x}+a^2e^{-2x}}{1}\,\frac{}{+}\,\frac{ae^{-2x}+a^2e^{-4x}}{1}\,\frac{}{+}\,\frac{ae^{-3x}+a^2e^{-6x}}{1}\,\frac{}{+\cdots}, \qquad (2.12)$$

where a is any complex number and $x>0$. The continued fraction (2.12) generalizes Ramanujan's cubic continued fraction [7]

$$C(q) := \frac{(q;q^6)_\infty(q^5;q^6)_\infty}{(q^3;q^6)^2_\infty} = \frac{1}{1+}\,\frac{q+q^2}{1}\,\frac{}{+}\,\frac{q^2+q^4}{1}\,\frac{}{+}\,\frac{q^3+q^6}{1}\,\frac{}{+\cdots}, \qquad |q|<1. \qquad (2.13)$$

Theorem 2.2. *As $x \to 0+$,*

$$C(a, e^{-x}) = \frac{1}{a+1} \exp\left(\frac{ax}{1+2a} - \frac{a(1-a)x^2}{2(1+2a)^3} + \frac{a(1-a)(1-12a-4a^2)x^3}{6(1+2a)^5} - \cdots \right). \tag{2.14}$$

Moreover, each term of the asymptotic expansion beginning with the second has a factor of $a(1-a)$.

Proof. As the details are very similar to those for the proof of Theorem 2.1, we give only a brief sketch. Observe from the definition (2.12) that $C(a, e^{-x})$ satisfies the functional equation

$$C(a, e^{-x}) = \frac{1}{1 + (ae^{-x} + a^2e^{-2x})\, C(ae^{-x}, e^{-x})}.$$

The asymptotic formula (2.14) now follows as before. Showing that the factor a persists in the coefficients of $x^n, n \geq 1$, is easy, as above. To show that the factor $1-a$ appears in the coefficient of $x^n, n \geq 2$, requires a modification in the argument of [5]. First note that, by elementary manipulation,

$$\frac{(q;q^6)_\infty (q^5;q^6)_\infty}{(q^3;q^6)_\infty^2} = \frac{(q;q)_\infty (q^6;q^6)_\infty^3}{(q^2;q^2)_\infty (q^3;q^3)_\infty^3}.$$

To examine the asymptotic behavior of $C(1, e^{-x})$ as $x \to 0+$, it therefore suffices to determine the asymptotic behavior of $(e^{-x}; e^{-x})_\infty$ as $x \to 0+$. Theorem 4.1 of [5] is inapplicable, but the method employed there with Dirichlet L-functions playing the leading role can be adopted to the Riemann zeta-function. We would then find that, as $x \to 0+$,

$$C(1, e^{-x}) = \frac{1}{2} \exp\left(\frac{x}{3} + O(x^N) \right),$$

for every $N > 0$, which confirms the presence of $1-a$ in all terms of (2.14) beginning with the second. □

Recall the generating function for $R(a,q)$ given by [2, p. 30, Cor.]

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)}}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n}} = \frac{1}{1+}\, \frac{aq}{1+}\, \frac{aq^2}{1+}\, \frac{aq^3}{1+} \cdots. \tag{2.15}$$

It would seem that another approach to (2.1) might be achieved by determining the asymptotic behavior of both the numerator and denominator of (2.15) as $x \to 0+$, with $q = e^{-x}$. However, if we apply a theorem of Ramanujan [3, p. 269, Entry 7], we only obtain the easy result,

$$R(a, e^{-x}) \sim \frac{\sqrt{5}-1}{2},$$

as $x \to 0+$.

3. The Conjecture of Blecksmith and Brillhart

Our goal in this section is to prove the beautiful formula (1.4).

Theorem 3.1. *For* $|q| < 1$,

$$R(q) = 1 - \frac{q}{1} + \frac{q}{1} - \frac{q}{1} + \frac{q}{1} - \frac{q^2}{1} + \frac{q^2}{1} - \frac{q^2}{1} + \frac{q^2}{1} - \cdots. \tag{3.1}$$

Proof. Set

$$X(q,a) := 1 - \frac{q}{1} + \frac{q}{1} - \frac{q}{1} + \frac{q}{a}. \tag{3.2}$$

Then

$$X(q,a) = \frac{a + q - aq + aq^2}{a + q + q^2} \qquad \text{and} \qquad 1 - X(q,a) = \frac{q(a + q - aq)}{a + q + q^2}. \tag{3.3}$$

Hence,

$$\begin{aligned}\frac{1 - X(q,a)}{X(q,a)} &= \frac{q(a+q-aq)}{a+q-aq+aq^2}\\ &= \cfrac{q}{1 + \cfrac{aq^2}{a+q-aq}}\\ &= \frac{q}{1} + \cfrac{q^2}{1 + \cfrac{q(1-a)}{a}}\\ &= \frac{q}{1} + \frac{q^2}{1} + \frac{q(1-a)}{a}.\end{aligned} \tag{3.4}$$

Let $F(q)$ denote the right side of (3.1), and define A by

$$F(q) := 1 - \frac{q}{1} + \frac{q}{1} - \frac{q}{1} + \frac{q}{A}.$$

Then, from (3.2) and (3.3),

$$\begin{aligned}F(q) &= \frac{A + q - Aq + Aq^2}{A + q + q^2}\\ &= \cfrac{1}{1 + \cfrac{Aq + q^2 - Aq^2}{A + q - Aq + Aq^2}}\\ &= \frac{1}{1} + \frac{q(A + q - Aq)}{A + q - Aq + Aq^2}\\ &= \frac{1}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q(1-A)}{A},\end{aligned} \tag{3.5}$$

by (3.4). Replacing A on the far right side of (3.5) by $X(q^2, a)$ and using (3.4), we find that

$$\begin{aligned} F(q) &= \frac{1}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, q \left(\frac{q^2}{1} \, \underset{+}{} \, \frac{q^4}{1} \, \underset{+}{} \, \frac{q^2(1-a)}{a} \right) \\ &= \frac{1}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \, \frac{q^4}{1} \, \underset{+}{} \, \frac{q^2(1-a)}{a}. \end{aligned} \tag{3.6}$$

Next, we replace a by $X(q^3, a)$ in (3.6) and continue iterating in this manner to complete the proof of Theorem 3.1. ☐

4. Two Entries on Page 200 of Ramanujan's Lost Notebook

On page 200 of his lost notebook (in the pagination of [11]), Ramanujan recorded the following identity.

Theorem 4.1. *For each complex number a and $|q| < 1$,*

$$\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)}}{(q;q)_n^2} = (aq;q)_\infty \sum_{n=0}^{\infty} \frac{a^n q^n}{(q;q)_n^2}. \tag{4.1}$$

We provide two proofs. The first is easy, since the result follows from a corollary of a well-known transformation for basic hypergeometric series. The second is more difficult, but also more interesting, since the shape of (4.1) suggests that it has a connection with the theory of partitions.

Proof. In Entry 8 of Chapter 16 in his second notebook [10], Ramanujan recorded an identity arising from a basic hypergeometric series transformation. For $|a|, |q| < 1$,

$$\frac{(a;q)_\infty}{(b;q)_\infty} \sum_{n=0}^{\infty} \frac{(c;q)_n (b/a;q)_n}{(d;q)_n (q;q)_n} a^n = \sum_{n=0}^{\infty} \frac{(-1)^n (b/a;q)_n (d/c;q)_n}{(b;q)_n (d;q)_n (q;q)_n} a^n c^n q^{n(n-1)/2}. \tag{4.2}$$

A proof of (4.2) may be found in [2, p. 17]. In (4.2), let $d = q$, replace a by aq, and let both b and c tend to 0. The claim (4.1) readily follows. ☐

Proof. By replacing aq by a and dividing by $(aq;q)_\infty$ on both sides of (4.1), we arrive at

$$\frac{1}{(a;q)_\infty} \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n^2} = \sum_{n=0}^{\infty} \frac{a^n}{(q;q)_n^2}. \tag{4.3}$$

We prove (4.3). Recall that a generating function for partitions $p(n)$ is [1, p. 21, eq. (2.2.9)]

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2}. \tag{4.4}$$

For $a = 1$, the only difference between the right sides of (4.3) and (4.4) is the numerator q^{n^2}; the coefficient of q^N in $1/(q;q)_n^2$ counts the number of partitions of $N + n^2$ with the Durfee square of side n. Let $A(n,N)$ be the set of partitions of $N+n^2$ with the Durfee square of side n. Then

$$\sum_{n=0}^{\infty} \frac{a^n}{(q;q)_n^2} = \sum_{n=0}^{\infty}\sum_{N=0}^{\infty} |A(n,N)|\, a^n q^N. \tag{4.5}$$

On the other hand, the left side of (4.3) is the product of generating functions for two sets of certain partitions: one is for partitions with nonnegative parts and the other is for partitions with the Durfee square of side n. Thus we consider pairs of partitions. Let $B(n,N)$ be the set of pairs of partitions (μ,ν) such that $|\mu|+|\nu|=N$, μ has at most $n-d$ nonnegative parts, and ν has the Durfee square of side d, $d\le n$. Then we see that

$$\frac{1}{(a;q)_\infty}\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n^2} = \sum_{n=0}^{\infty}\sum_{N=0}^{\infty} |B(n,N)|\, a^n q^N. \tag{4.6}$$

To show (4.3), we will establish a bijection between $A(n,N)$ and $B(n,N)$ by constructing a partition λ in $A(n,N)$ for a given pair (μ,ν) in $B(n,N)$. In the proof, we assume that parts are in decreasing order. We consider an $n\times n$ square, and then attach μ and ν to the right of and below the square, respectively. If the largest part of ν is less than or equal to n, then we obtain the desired partition λ with the Durfee square of side n. Otherwise, we need to apply a bijection of F. Franklin [13, pp. 18–19] to ν in order to obtain a partition with parts less than or equal to n.

To explain the bijection of Franklin, we define a map $f_{k,s}$ from a partition $\delta = (\delta_1,\delta_2,\dots,\delta_m)$ to a partition $\rho=(\rho_1,\rho_2,\dots,\rho_m)$ as follows. If $\delta_1-\delta_{k+1}>s$, define $f_{k,s}(\delta)=\rho$, where for, $1\le i\le m$,

$$\rho_i = \begin{cases} \delta_{i+1}-1, & \text{for} \quad i<k,\\ \delta_1-s-1, & \text{for} \quad i=k,\\ \delta_i, & \text{for} \quad i>k. \end{cases}$$

Otherwise, $f_{k,s}(\delta)=\delta$.

Let $\sigma=(\sigma_1,\sigma_2,\dots,\sigma_l)$ be the partition to the right of the Durfee square of side d of ν, and let μ' be the conjugate of μ. For convention, $\sigma_{l+1}=0$. Let r_1 be the smallest j such that $f^{j}_{1,n-d}(\sigma)=f^{j+1}_{1,n-d}(\sigma)$. Then we add $n-d+1$ nodes r_1 times to μ' as parts, and denote $f^{r_1}_{1,n-d}(\sigma)$ by σ to avoid a proliferation of notation. Next, we consider the second excess of σ. Let r_2 be the smallest j such that $f^{j}_{2,n-d}(\sigma)=f^{j+1}_{2,n-d}(\sigma)$. Then we add $n-d+2$ nodes r_2 times to μ' as parts, and denote $f^{r_2}_{2,n-d}(\sigma)$ by σ. We repeat this process with $f_{k,n-d}$ and σ where $k=3,\dots,l$.

In this way, we can finally produce a partition with parts less than or equal to $n-d$ since the process terminates when $\sigma_1-\sigma_{l+1}\le n-d$. Furthermore, we add to μ' at each step the part $n-d+k$ which is less than or equal to n since the old σ has at most d parts, i.e., $l\le d$. Thus the new pair σ and μ' are the desired partitions; σ has at most d parts with the largest parts less than or equal to $n-d$, and μ' has parts less than or equal to n, i.e., μ has at most n parts. Therefore, we obtain a partition λ in $A(d,N)$ with the pair (μ,ν) in $B(d,N)$. Since the steps are invertible, the map is a bijection. □

Below (4.1) on page 200 in [11], Ramanujan wrote

$$a + \frac{q^4}{a\,+}\,\frac{q^8}{a\,+}\cdots \qquad \& \qquad \frac{q}{1\,-}\,\frac{aq}{1\,+}\,\frac{q^2}{1\,-}\,\frac{aq^3}{1\,+}\cdots. \tag{4.7}$$

We emphasize that no assertion about these two continued fractions is claimed by Ramanujan. The former continued fraction can be written as

$$a\frac{1}{R(1/a^2, q^4)},$$

but the latter continued fraction cannot be represented in terms of the generalized Rogers–Ramanujan continued fraction. The appearance of the ampersand sign & between the continued fractions most likely indicates that they have been linked together by Ramanujan in some theorem. Their appearance below (4.1) suggests that they are related to it. However, we have been unable to make such a connection. Note that there is a superficial resemblance with the series on the left side of (4.1) and the series in the numerator of the generating function of the generalized Rogers–Ramanujan continued fraction given by (2.15). In his third notebook [10], Ramanujan examined the limits of both the even indexed and odd indexed partial quotients of the Rogers–Ramanujan continued fraction when $q > 1$. Quite remarkably, these limits involve exactly the same continued fractions in (4.7), but with, of course, $a = 1$. See [4, p. 30, Entry 11] for a statement and proof of Ramanujan's result. Thus, it is natural to conjecture that Ramanujan had established a generalization of Entry 11 for the generalized Rogers–Ramanujan continued fraction. One can begin to prove a generalization of Entry 11 by using the same ideas. However, we are unable to identify the quotients of q-series that arise in place of those appearing on page 32 of [4]. Moreover, computer algebra does not reveal any connection of these q-series with the continued fractions of (4.7). Thus, it would seem that our conjecture about why Ramanujan recorded the continued fractions in (4.7) is groundless. But there is a connection with another result of Ramanujan, namely, a claim in his second notebook, recorded as Entry 13 in [4, p. 36]. The continued fractions of (4.7) are precisely those appearing in Entry 13, and Ramanujan claims that they are "close" to each other. We refer readers to [4, pp. 36–40] for the meaning of "closeness." Thus, maybe Ramanujan had Entry 13 in mind, but we have the nagging suspicion that Ramanujan had some other motivation for recording these two continued fractions, and that we have been unable to discern his reasoning.

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