

THEOREMS ON PARTITIONS FROM A PAGE IN RAMANUJAN'S LOST NOTEBOOK

BRUCE C. BERNDT¹, AE JA YEE², AND JINHEE YI

Abstract. On page 189 in his lost notebook, Ramanujan recorded five assertions about partitions. Two are famous identities of Ramanujan immediately yielding the congruences $p(5n + 4) \equiv 0 \pmod{5}$ and $p(7n + 5) \equiv 0 \pmod{7}$ for the partition function $p(n)$. Two of the identities, also originally due to Ramanujan, were rediscovered by M. Newman, who used the theory of modular forms to prove them. The fifth claim is false, but Ramanujan corrected it in his unpublished manuscript on the partition and τ -functions. The purpose of this paper is to give completely elementary proofs of all four claims. In particular, although Ramanujan's elementary proof for his identity implying the congruence $p(7n + 5) \equiv 0 \pmod{7}$ is sketched in his unpublished manuscript on the partition and τ -functions, it has never been given in detail. This proof depends on some elementary identities mostly found in his notebooks; new proofs of these identities are given here.

1. INTRODUCTION

Let $p(n)$ denote, as usual, the number of unrestricted partitions of the positive integer n . In [7], [8, p. 213], Ramanujan offered the beautiful identities

$$\sum_{n=0}^{\infty} p(5n + 4)q^n = 5 \frac{(q^5; q^5)_{\infty}^5}{(q; q)_{\infty}^6} \quad (1.1)$$

and

$$\sum_{n=0}^{\infty} p(7n + 5)q^n = 7 \frac{(q^7; q^7)_{\infty}^3}{(q; q)_{\infty}^4} + 49q \frac{(q^7; q^7)_{\infty}^7}{(q; q)_{\infty}^8}, \quad (1.2)$$

where, as usual,

$$(a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k), \quad |q| < 1.$$

References to several proofs of (1.1) and (1.2) can be found in the latest edition of [8, pp. 372–373]. Ramanujan gave a brief proof of (1.1) in [7]. He did not prove (1.2) in [7], but he did give a sketch of his proof of (1.2) in his unpublished manuscript on the partition and τ -functions [10, pp. 242–243], [3, Sect. 24]. Note that (1.1) and (1.2) immediately yield the congruences $p(5n + 4) \equiv 0 \pmod{5}$ and $p(7n + 5) \equiv 0 \pmod{7}$, respectively.

¹Research partially supported by grant MDA904-00-1-0015 from the National Security Agency.

²Research partially supported by a grant from the Number Theory Foundation.

³2000 AMS Classification Numbers: Primary, 11P83; Secondary, 11F20.

The two identities (1.1) and (1.2) are stated on page 189 of Ramanujan's lost notebook in the pagination of [10]. Also given by Ramanujan are two further identities. Define $q_m(n), n \geq 0$, by

$$(q; q)_\infty^m =: \sum_{n=0}^{\infty} q_m(n)q^n. \quad (1.3)$$

Then

$$\sum_{n=0}^{\infty} q_5(5n)q^n = \frac{(q; q)_\infty^6}{(q^5; q^5)_\infty}. \quad (1.4)$$

and

$$\sum_{n=0}^{\infty} q_7(7n)q^n = \frac{(q; q)_\infty^8}{(q^7; q^7)_\infty} + 49q(q; q)_\infty^4 (q^7; q^7)_\infty^3. \quad (1.5)$$

For completeness, in Section 2, we begin with essentially Ramanujan's proof of (1.1). We then prove (1.4).

In Section 3, we amplify Ramanujan's sketch in [10] and give a complete proof of (1.2). We also prove (1.5) in Section 3. Both proofs depend on some theta function identities which Ramanujan stated without proof. Thus, in Section 3 we also give proofs of these required identities.

One of the latter identities is found in Entry 18(i) of Chapter 19 in Ramanujan's second notebook [9], [2, p. 305, eq. (18.2)]. However, the proof given in [2] is very complicated, and the proof given here is much shorter. Two related identities are also given by Ramanujan in the same section of [9]. In Section 4, we give much easier proofs of these identities than those given in [2, pp. 306–312].

At the bottom of page 189 in [10], Ramanujan offers an elegant assertion on the divisibility of a certain difference of partition functions. Although his claim is true in some cases, it is unfortunately false in general. In the last section, Section 5, of this paper we briefly discuss this claim.

Throughout the paper, J, J_1, J_2, \dots represent power series with integral coefficients and integral powers, not necessarily the same with each occurrence. We have adhered to Ramanujan's notation, whereas in contemporary notation we would use congruence signs instead.

We emphasize that the theory of modular forms can be utilized to provide proofs of all identities in this paper. However, we think that it is instructive to construct proofs as Ramanujan would possibly have given them.

2. THE IDENTITIES FOR MODULUS 5

Theorem 2.1. *If $p(n)$ denotes the ordinary partition function, then*

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \frac{(q^5; q^5)_\infty^5}{(q; q)_\infty^6}. \quad (2.1)$$

Proof. Recall that the Rogers–Ramanujan continued fraction $R(q)$ is defined by

$$R(q) := \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \dots}}}}, \quad |q| < 1.$$

This continued fraction satisfies two beautiful and famous identities [7], [8, p. 212], [2, p. 267, eqs. (11.5), (11.6)]

$$R(q^5) - q - \frac{q^2}{R(q^5)} = \frac{(q; q)_\infty}{(q^{25}; q^{25})_\infty} \quad (2.2)$$

and

$$R^5(q^5) - 11q^5 - \frac{q^{10}}{R^5(q^5)} = \frac{(q^5; q^5)_\infty^6}{(q^{25}; q^{25})_\infty^6}. \quad (2.3)$$

(It should be remarked here that the introduction of the continued fraction $R(q)$ is not strictly necessary. In the representations (2.2) and (2.3) we only need to know that the function $R(q)$ can be represented as a power series in q with integral coefficients.) Using the generating function for $p(n)$, (2.2) and (2.3), and “long division,” we find that

$$\begin{aligned} \sum_{n=0}^{\infty} p(n)q^n &= \frac{1}{(q; q)_\infty} = \frac{(q^{25}; q^{25})_\infty^5}{(q^5; q^5)_\infty^6} \frac{(q^5; q^5)_\infty^6}{(q^{25}; q^{25})_\infty^6} \frac{(q^{25}; q^{25})_\infty}{(q; q)_\infty} \\ &= \frac{(q^{25}; q^{25})_\infty^5}{(q^5; q^5)_\infty^6} \frac{R^5(q^5) - 11q^5 - q^{10}/R^5(q^5)}{R(q^5) - q - q^2/R(q^5)} \\ &= \frac{(q^{25}; q^{25})_\infty^5}{(q^5; q^5)_\infty^6} \{R^4 + qR^3 + 2q^2R^2 + 3q^3R + 5q^4 \\ &\quad - 3q^5R^{-1} + 2q^6R^{-2} - q^7R^{-3} + q^8R^{-4}\}, \end{aligned} \quad (2.4)$$

where $R := R(q^5)$. Choosing only those terms on each side of (2.4) where the powers of q are of the form $5n + 4$, we find that

$$\sum_{\substack{n=0 \\ n \equiv 4 \pmod{5}}}^{\infty} p(n)q^n = 5q^4 \frac{(q^{25}; q^{25})_\infty^5}{(q^5; q^5)_\infty^6}$$

or

$$\sum_{n=0}^{\infty} p(5n + 4)q^{5n} = 5 \frac{(q^{25}; q^{25})_\infty^5}{(q^5; q^5)_\infty^6}. \quad (2.5)$$

Replacing q^5 by q in (2.5), we complete the proof of (2.1). \square

Theorem 2.2. *If $q_5(n), n \geq 0$, is defined by (1.3), then*

$$\sum_{n=0}^{\infty} q_5(5n)q^n = \frac{(q; q)_\infty^6}{(q^5; q^5)_\infty}. \quad (2.6)$$

Proof. By (2.2) and (2.3),

$$\begin{aligned} \sum_{n=0}^{\infty} q_5(n)q^n &= (q; q)_\infty^5 = \frac{(q^5; q^5)_\infty^6}{(q^{25}; q^{25})_\infty} \frac{(q; q)_\infty^5}{(q^{25}; q^{25})_\infty^5} \frac{(q^{25}; q^{25})_\infty^6}{(q^5; q^5)_\infty^6} \\ &= \frac{(q^5; q^5)_\infty^6}{(q^{25}; q^{25})_\infty} \frac{(R(q^5) - q - q^2/R(q^5))^5}{R^5(q^5) - 11q^5 - q^{10}/R^5(q^5)}. \end{aligned} \quad (2.7)$$

Now, the terms where the exponents of q are multiples of 5 in

$$\left(R(q^5) - q - \frac{q^2}{R(q^5)} \right)^5$$

are given by

$$\begin{aligned} R^5(q^5) - \frac{q^{10}}{R^5(q^5)} - q^5 - \binom{5}{2, 1, 2} q^5 + \binom{5}{1, 3, 1} q^5 \\ = R^5(q^5) - \frac{q^{10}}{R^5(q^5)} - q^5 - 30q^5 + 20q^5 = R^5(q^5) - \frac{q^{10}}{R^5(q^5)} - 11q^5. \end{aligned} \quad (2.8)$$

Thus, choosing only those terms from (2.7) where the powers of q are multiples of 5, we find upon using (2.8) that

$$\sum_{n=0}^{\infty} q_5(5n)q^{5n} = \frac{(q^5; q^5)_{\infty}^6}{(q^{25}; q^{25})_{\infty}} \frac{R^5(q^5) - q^{10}/R^5(q^5) - 11q^5}{R^5(q^5) - q^{10}/R^5(q^5) - 11q^5} = \frac{(q^5; q^5)_{\infty}^6}{(q^{25}; q^{25})_{\infty}}. \quad (2.9)$$

Replacing q^5 by q , we complete the proof of (2.6). \square

Recall that $q_m(n)$ is defined by (1.3). Note that $q_m(n)$ denotes the number of m -colored partitions of n into an even number of distinct parts minus the number of m -colored partitions of n into an odd number of distinct parts.

Corollary 2.3. *We have*

$$q_5(5n) \equiv \begin{cases} (-1)^j \pmod{5}, & \text{if } n = j(3j - 1)/2, \quad -\infty < j < \infty, \\ 0 \pmod{5}, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.2 and the binomial theorem,

$$\sum_{n=0}^{\infty} q_5(5n)q^n = \frac{(q; q)_{\infty}^6}{(q^5; q^5)_{\infty}} \equiv \frac{(q; q)_{\infty}^6}{(q; q)_{\infty}^5} = (q; q)_{\infty} \pmod{5}.$$

The result now follows from Euler's pentagonal number theorem. \square

To illustrate Corollary 2.3, we find, from *Mathematica*, that

$$\begin{aligned} \frac{(q; q)_{\infty}^6}{(q^5; q^5)_{\infty}} = & 1 - 6q + 9q^2 + 10q^3 - 30q^4 + q^5 + 5q^6 + 51q^7 + 10q^8 - 100q^9 + 20q^{10} - 55q^{11} \\ & + 109q^{12} + 110q^{13} - 130q^{14} - q^{15} - 110q^{16} + 160q^{17} + 10q^{18} - 230q^{19} + 100q^{20} + \dots \end{aligned}$$

3. THE IDENTITIES FOR MODULUS 7

Our primary goal in this section is to give a completely elementary proof along the lines outlined by Ramanujan in [10], [3, Sect. 24] of his famous theorem below, Theorem 3.1, as well as a proof of the new related theorem, Theorem 3.2, or (1.5) in the Introduction.

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7 \frac{(q^7; q^7)_{\infty}^3}{(q; q)_{\infty}^4} + 49q \frac{(q^7; q^7)_{\infty}^7}{(q; q)_{\infty}^8}. \quad (3.1)$$

Proof. Recall Euler's pentagonal number theorem [2, p. 36, Entry 22(iii)],

$$(q; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}, \quad |q| < 1. \quad (3.2)$$

Using (3.2) in both the numerator and denominator and then separating the indices of summation in the numerator into residue classes modulo 7, we readily find that

$$\frac{(q^{1/7}; q^{1/7})_{\infty}}{(q^7; q^7)_{\infty}} = J_1 + q^{1/7} J_2 - q^{2/7} + q^{5/7} J_3, \quad (3.3)$$

where $J_1, J_2,$ and J_3 are power series in q with integral coefficients. Now recall Jacobi's identity [2, p. 39, Entry 24(ii)],

$$(q; q)_{\infty}^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}, \quad |q| < 1. \quad (3.4)$$

Cubing both sides of (3.3) and substituting (3.4) into the left side, we find that

$$\begin{aligned} & \frac{\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/14}}{\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{7n(n+1)/2}} \\ &= (J_1^3 + 3J_2^2 J_3 q - 6J_1 J_3 q) + q^{1/7} (3J_1^2 J_2 - 6J_2 J_3 q + J_3^2 q^2) + 3q^{2/7} (J_1 J_2^2 - J_1^2 + J_3 q) \\ & \quad + q^{3/7} (J_2^3 - 6J_1 J_2 + 3J_1 J_3^2 q) + 3q^{4/7} (J_1 - J_2^2 + J_2 J_3^2 q) \\ & \quad + 3q^{5/7} (J_2 + J_1^2 J_3 - J_3^2 q) + q^{6/7} (6J_1 J_2 J_3 - 1). \end{aligned} \quad (3.5)$$

On the other hand, by separating the indices of summation in the numerator on the left side of (3.5) into residue classes modulo 7, we easily find that

$$\frac{\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/14}}{\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{7n(n+1)/2}} = G_1 + q^{1/7} G_2 + q^{3/7} G_3 - 7q^{6/7}, \quad (3.6)$$

where $G_1, G_2,$ and G_3 are power series in q with integral coefficients. By comparing coefficients in (3.5) and (3.6), we conclude that

$$\begin{cases} J_1 J_2^2 - J_1^2 + J_3 q &= 0, \\ J_1 - J_2^2 + J_2 J_3^2 q &= 0, \\ J_2 + J_1^2 J_3 - J_3^2 q &= 0, \\ 6J_1 J_2 J_3 - 1 &= -7. \end{cases} \quad (3.7)$$

Now write (3.3) in the form

$$\frac{(\omega q^{1/7}; \omega q^{1/7})_{\infty}}{(q^7; q^7)_{\infty}} = J_1 + \omega q^{1/7} J_2 - \omega^2 q^{2/7} + \omega^5 q^{5/7} J_3, \quad (3.8)$$

where $\omega^7 = 1$. Multiplying (3.8) over all seventh roots of unity, we find that

$$\frac{(q; q)_\infty^8}{(q^7; q^7)_\infty^8} = \prod_{i=0}^6 (J_1 + \omega^i q^{1/7} J_2 - \omega^{2i} q^{2/7} + \omega^{5i} q^{5/7} J_3). \quad (3.9)$$

Using the generating function for $p(n)$, (3.3), and (3.9), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} p(n)q^n &= \frac{1}{(q; q)_\infty} = \frac{(q^{49}; q^{49})_\infty^7}{(q^7; q^7)_\infty^8} \frac{(q^7; q^7)_\infty^8}{(q^{49}; q^{49})_\infty^8} \frac{(q^{49}; q^{49})_\infty}{(q; q)_\infty} \\ &= \frac{(q^{49}; q^{49})_\infty^7}{(q^7; q^7)_\infty^8} \frac{\prod_{i=0}^6 (J_1 + \omega^i q J_2 - \omega^{2i} q^2 + \omega^{5i} q^5 J_3)}{J_1 + q J_2 - q^2 + q^5 J_3} \\ &= \frac{(q^{49}; q^{49})_\infty^7}{(q^7; q^7)_\infty^8} \left\{ \prod_{i=1}^6 (J_1 + \omega^i q J_2 - \omega^{2i} q^2 + \omega^{5i} q^5 J_3) \right\}. \end{aligned} \quad (3.10)$$

We only need to compute the terms in $\prod_{i=1}^6 (J_1 + \omega^i q J_2 - \omega^{2i} q^2 + \omega^{5i} q^5 J_3)$ where the powers of q are of the form $7n + 5$ to complete the proof. In order to do this, we need to prove the identities,

$$J_1^7 + J_2^7 q + J_3^7 q^5 = \frac{(q; q)_\infty^8}{(q^7; q^7)_\infty^8} + 14q \frac{(q; q)_\infty^4}{(q^7; q^7)_\infty^4} + 57q^2, \quad (3.11)$$

$$J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 = -\frac{(q; q)_\infty^4}{(q^7; q^7)_\infty^4} - 8q, \quad (3.12)$$

$$J_1^2 J_2^3 + J_2^2 J_3^3 q + J_3^2 J_1^3 q^2 = -\frac{(q; q)_\infty^4}{(q^7; q^7)_\infty^4} - 5q. \quad (3.13)$$

Since $J_2^2 = J_1 + J_2 J_3^2 q$, $J_1^2 = J_1 J_2^2 + J_3 q$, $J_3^2 q = J_2 + J_1^2 J_3$, and $J_1 J_2 J_3 = -1$ by (3.7), we find that

$$\begin{aligned} J_1^2 J_2^3 + J_2^2 J_3^3 q + J_3^2 J_1^3 q^2 &= J_1^3 J_2 + J_1^2 J_2^2 J_3^2 q + J_1^2 J_2^2 J_3^2 q + J_3^3 J_1 q^2 + J_2^3 J_3 q + J_1^2 J_2^2 J_3^2 q \\ &= J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 + 3q, \end{aligned} \quad (3.14)$$

$$\begin{aligned} J_1 J_2^5 + J_3 J_1^5 + J_2 J_3^5 q^3 &= J_1 J_2 (J_1 + J_2 J_3^2 q)^2 + J_3 J_1 (J_1 J_2^2 + J_3 q)^2 + J_2 J_3 (J_2 + J_1^2 J_3)^2 q \\ &= J_1^3 J_2 + 2J_1^2 J_2^2 J_3^2 q + J_1 J_2^2 J_3^4 q^2 + J_3 J_1^3 J_2^4 + 2J_1^2 J_2^2 J_3^2 q + J_3^3 J_1 q^2 \\ &\quad + J_2^3 J_3 q + 2J_1^2 J_2^2 J_3^2 q + J_2 J_3^3 J_1^4 q \\ &= J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 - (J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2) + 6q \\ &= 3q, \end{aligned} \quad (3.15)$$

where (3.15) is obtained from (3.14). (Observe from (3.14) that it suffices to prove only (3.12) or (3.13).) By squaring the left side of (3.12) and using (3.7), (3.15), and

(3.14), we find that

$$\begin{aligned}
(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2)^2 &= J_1^6 J_2^2 + J_2^6 J_3^2 q^2 + J_3^6 J_1^2 q^4 \\
&\quad + 2(J_1^3 J_2^4 J_3 q + J_1 J_2^3 J_3^4 q^3 + J_1^4 J_2 J_3^3 q^2) \\
&= J_1^7 + J_1^6 J_2 J_3^2 q + J_2^7 q + J_1^2 J_2^6 J_3 q + J_3^7 q^5 + J_1 J_2^2 J_3^6 q^4 \\
&\quad - 2(J_1^2 J_2^3 q + J_3^2 J_1^3 q^2 + J_2^2 J_3^3 q^3) \\
&= J_1^7 + J_2^7 q + J_3^7 q^5 - (J_1 J_2^5 q + J_3 J_1^5 q + J_2 J_3^5 q^4) \\
&\quad - 2(J_1^2 J_2^3 q + J_2^2 J_3^3 q^3 + J_1^3 J_3^2 q^2) \\
&= J_1^7 + J_2^7 q + J_3^7 q^5 - 2q(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2) - 9q^2.
\end{aligned}$$

Thus,

$$(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 + q)^2 = (J_1^7 + J_2^7 q + J_3^7 q^5) - 8q^2. \quad (3.16)$$

Expanding the right side of (3.9) and using (3.7), (3.15), and (3.14), we obtain

$$\begin{aligned}
\frac{(q; q)_\infty^8}{(q^7; q^7)_\infty^8} &= J_1^7 + J_2^7 q + J_3^7 q^5 + 7(J_1 J_2^5 q + J_3 J_1^5 q + J_2 J_3^5 q^4) + 7(J_1^4 J_2^2 J_3 q + J_1 J_2^4 J_3^2 q^2 \\
&\quad + J_2 J_3^4 J_1^2 q^3) + 7(J_1^3 J_2 q + J_2^3 J_3 q^2 + J_3^3 J_1 q^3) \\
&\quad + 14(J_1^2 J_2^3 q + J_3^2 J_1^3 q^2 + J_2^2 J_3^3 q^3) + 7J_1^2 J_2^2 J_3^2 q^2 + 14J_1 J_2 J_3 q^2 - q^2 \\
&= J_1^7 + J_2^7 q + J_3^7 q^5 + 21q^2 - 7q(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2) + 7q(J_1^3 J_2 + J_2^3 J_3 q \\
&\quad + J_3^3 J_1 q^2) + 14q(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 + 3q) + 7q^2 - 14q^2 - q^2 \\
&= J_1^7 + J_2^7 q + J_3^7 q^5 + 14q(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2) + 55q^2. \quad (3.17)
\end{aligned}$$

Combining (3.16) and (3.17), we find that

$$\begin{aligned}
\frac{(q; q)_\infty^8}{(q^7; q^7)_\infty^8} &= (J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 + q)^2 + 8q^2 + 14(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2)q + 55q^2 \\
&= (J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 + 8q)^2.
\end{aligned}$$

By (3.3), we see that for q sufficiently small and positive, $J_2 < 0$. Thus, taking the square root of both sides above, we find that

$$J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 = -\frac{(q; q)_\infty^4}{(q^7; q^7)_\infty^4} - 8q, \quad (3.18)$$

which proves (3.12). We now see that (3.11) follows from (3.17) and (3.18), and (3.13) follows from (3.14) and (3.18).

Returning to (3.10), we are now ready to compute the terms in $\prod_{i=1}^6 (J_1 + \omega^i q J_2 - \omega^{2i} q^2 + \omega^{5i} q^5 J_3)$ where the powers of q are of the form $7n + 5$. Using the computer algebra system MAPLE, (3.12), (3.13), and (3.15), we find that the desired terms with

powers of the form q^{7n+5} are equal to

$$\begin{aligned}
& - (J_1 J_2^5 + J_3 J_1^5 + 3J_1^3 J_2 + 4J_1^2 J_2^3) q^5 - (3J_2^3 J_3 + 4J_3^2 J_1^3 - 8) q^{12} \\
& - (4J_2^2 J_3^3 + 3J_3^3 J_1) q^{19} - J_2 J_3^5 q^{26} \\
= & - 3(J_1^3 J_2 + J_2^3 J_3 q^7 + J_3^3 J_1 q^{14}) q^5 - 4(J_1^2 J_2^3 + J_3^2 J_1^3 q^7 + J_2^2 J_3^3 q^{14}) q^5 \\
& - (J_1 J_2^5 + J_3 J_1^5 + J_2 J_3^5 q^{21}) q^5 + 8q^{12} \\
= & 7 \frac{(q^7; q^7)_\infty^4}{(q^{49}; q^{49})_\infty^4} q^5 + 49q^{12}. \tag{3.19}
\end{aligned}$$

Choosing only those terms on each side of (3.10) where the powers of q are of the form $7n + 5$ and using the calculation from (3.19), we find that

$$\sum_{\substack{n=0 \\ n \equiv 5 \pmod{7}}}^{\infty} p(n) q^n = q^5 \frac{(q^{49}; q^{49})_\infty^7}{(q^7; q^7)_\infty^8} \left(7 \frac{(q^7; q^7)_\infty^4}{(q^{49}; q^{49})_\infty^4} + 49q^7 \right),$$

or

$$\sum_{n=0}^{\infty} p(7n+5) q^{7n} = 7 \frac{(q^{49}; q^{49})_\infty^3}{(q^7; q^7)_\infty^4} + 49q^7 \frac{(q^{49}; q^{49})_\infty^7}{(q^7; q^7)_\infty^8}. \tag{3.20}$$

Replacing q^7 by q in (3.20), we complete the proof of (3.1). \square

By comparing (3.3) with Entry 17(v) in Chapter 19 of Ramanujan's second notebook [9], [2, p. 303], we see that

$$J_1 = \frac{f(-q^2, -q^5)}{f(-q, -q^6)}, \quad J_2 = -\frac{f(-q^3, -q^4)}{f(-q^2, -q^5)}, \quad \text{and} \quad J_3 = \frac{f(-q, -q^6)}{f(-q^3, -q^4)},$$

where

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

In the notation of Section 18 of Chapter 19 in [9], [2, p. 306],

$$\alpha = u^{1/7} = q^{-2/7} J_1, \quad \beta = -v^{1/7} = q^{-1/7} J_2, \quad \text{and} \quad \gamma = w^{1/7} = q^{3/7} J_3. \tag{3.21}$$

Thus, the identity (3.11) is equivalent to an identity in Entry 18 in Chapter 19 of Ramanujan's second notebook [9], [2, p. 305, eq. (18.2)]. The proof of (3.11) given here is much simpler than that given in [2, pp. 306–312].

Theorem 3.2. *If $q_7(n)$, $n \geq 0$, is defined by (1.3), then*

$$\sum_{n=0}^{\infty} q_7(7n) q^n = \frac{(q; q)_\infty^8}{(q^7; q^7)_\infty} + 49q(q; q)_\infty^4 (q^7; q^7)_\infty^3. \tag{3.22}$$

Proof. By (3.3), with q replaced by q^7 ,

$$\begin{aligned}
\sum_{n=0}^{\infty} q_7(n) q^n &= (q; q)_\infty^7 = \frac{(q^7; q^7)_\infty^8}{(q^{49}; q^{49})_\infty} \frac{(q; q)_\infty^7}{(q^{49}; q^{49})_\infty^7} \frac{(q^{49}; q^{49})_\infty^8}{(q^7; q^7)_\infty^8} \\
&= \frac{(q^7; q^7)_\infty^8}{(q^{49}; q^{49})_\infty} (J_1 + qJ_2 - q^2 + q^5 J_3)^7 \frac{(q^{49}; q^{49})_\infty^8}{(q^7; q^7)_\infty^8}. \tag{3.23}
\end{aligned}$$

Using (3.7) and employing (3.11)–(3.13) with q replaced by q^7 , we find that the terms where the exponents of q are multiples of 7 in

$$(J_1 + qJ_2 - q^2 + q^5J_3)^7$$

are given by

$$\begin{aligned} & \sum_{\substack{u,v,w \geq 0 \\ 7|(u+2v+5w)}} \binom{7}{u,v,w} (-1)^v J_1^{7-u-v-w} J_2^u J_3^w q^{u+2v+5w} \\ &= J_1^7 + J_2^7 q^7 + J_3^7 q^{35} - q^{14} + (105J_1^4 J_2^2 J_3 - 42J_1 J_2^5 + 210J_1^2 J_2^3 - 42J_1^5 J_3 - 140J_1^3 J_2) q^7 \\ & \quad + (105J_1 J_2^4 J_3^2 - 630J_1^2 J_2^2 J_3^2 + 210J_1 J_2 J_3 - 140J_2^3 J_3 + 210J_1^3 J_3^2) q^{14} \\ & \quad + (-140J_1 J_3^3 + 105J_1^2 J_2 J_3^4 + 210J_2^2 J_3^3) q^{21} - 42J_2 J_3^5 q^{28} \\ &= J_1^7 + J_2^7 q^7 + J_3^7 q^{35} - 245(J_1^3 J_2 + J_2^3 J_3 q^7 + J_3^3 J_1 q^{14}) q^7 - 42(J_1 J_2^5 + J_3 J_1^5 + J_2 J_3^5 q^{21}) q^7 \\ & \quad + 210(J_1^2 J_2^3 + J_3^2 J_1 q^7 + J_2^2 J_3^3 q^{14}) q^7 - 841 q^{14} \\ &= J_1^7 + J_2^7 q^7 + J_3^7 q^{35} - 35(J_1^3 J_2 + J_2^3 J_3 q^7 + J_3^3 J_1 q^{14}) q^7 + 630 q^{14} - 126 q^{14} - 841 q^{14} \\ &= \frac{(q^7; q^7)_\infty^8}{(q^{49}; q^{49})_\infty^8} + 14q^7 \frac{(q^7; q^7)_\infty^4}{(q^{49}; q^{49})_\infty^4} + 57q^{14} - 35q^7 \left(-\frac{(q^7; q^7)_\infty^4}{(q^{49}; q^{49})_\infty^4} - 8q^7 \right) - 337q^{14} \\ &= \frac{(q^7; q^7)_\infty^8}{(q^{49}; q^{49})_\infty^8} + 49q^7 \frac{(q^7; q^7)_\infty^4}{(q^{49}; q^{49})_\infty^4}. \end{aligned} \tag{3.24}$$

Thus, choosing only those terms from (3.23) where the powers of q are multiples of 7, we find upon using (3.24) that

$$\begin{aligned} \sum_{n=0}^{\infty} q_7(7n) q^{7n} &= \frac{(q^7; q^7)_\infty^8}{(q^{49}; q^{49})_\infty^8} \left(\frac{(q^7; q^7)_\infty^8}{(q^{49}; q^{49})_\infty^8} + 49q^7 \frac{(q^7; q^7)_\infty^4}{(q^{49}; q^{49})_\infty^4} \right) \frac{(q^{49}; q^{49})_\infty^8}{(q^7; q^7)_\infty^8} \\ &= \frac{(q^7; q^7)_\infty^8}{(q^{49}; q^{49})_\infty^8} + 49q^7 (q^7; q^7)_\infty^4 (q^{49}; q^{49})_\infty^3. \end{aligned} \tag{3.25}$$

Replacing q^7 by q , we complete the proof of (3.22). \square

Corollary 3.3. *We have*

$$q_7(7n) \equiv \begin{cases} (-1)^j \pmod{7}, & \text{if } n = j(3j-1)/2, \quad -\infty < j < \infty, \\ 0 \pmod{7}, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 3.2 and the binomial theorem,

$$\sum_{n=0}^{\infty} q_7(7n) q^n = \frac{(q; q)_\infty^8}{(q^7; q^7)_\infty^8} + 49q (q; q)_\infty^4 (q^7; q^7)_\infty^3 \equiv \frac{(q; q)_\infty^8}{(q; q)_\infty^7} = (q; q)_\infty \pmod{7}.$$

The result now follows from Euler's pentagonal number theorem. \square

To illustrate Corollary 3.3, we find, from *Mathematica*, that

$$\begin{aligned} \frac{(q; q)_\infty^8}{(q^7; q^7)_\infty^8} &= 1 - 8q + 20q^2 - 70q^4 + 64q^5 + 56q^6 + q^7 - 133q^8 - 140q^9 + 308q^{10} - 70q^{11} \\ & \quad + 174q^{12} + 56q^{13} - 518q^{14} - 141q^{15} - 63q^{16} + 868q^{17} - 140q^{18} + 238q^{19} + 294q^{20} + \dots \end{aligned}$$

Apparently, proofs of Theorems 2.2 and 3.2 were first given by M. Newman [5] using modular forms, although he credits D. H. Lehmer with first discovering the identities. Of course, they were unaware that these identities are in the lost notebook. A more complicated proof of Theorem 2.2 was given by K. G. Ramanathan [6].

4. RELATED IDENTITIES FOR MODULUS 7

Our goal in this section is to use results from the preceding section to give much simpler proofs of two further identities from Section 18 of Chapter 19 in Ramanujan's second notebook than those given in [2, pp. 306–312]. It will be convenient to use the notation (3.21).

Theorem 4.1. *Let u, v , and w be defined by (3.21). Then*

$$uv - uw + vw = 289 + 126 \frac{(q; q)_\infty^4}{q(q^7; q^7)_\infty^4} + 19 \frac{(q; q)_\infty^8}{q^2(q^7; q^7)_\infty^8} + \frac{(q; q)_\infty^{12}}{q^3(q^7; q^7)_\infty^{12}}. \quad (4.1)$$

If

$$\mu := \frac{(q; q)_\infty^4}{q(q^7; q^7)_\infty^4} \quad \text{and} \quad \nu := \frac{(q^{1/7}; q^{1/7})_\infty}{q^{2/7}(q^7; q^7)_\infty}, \quad (4.2)$$

then

$$2\mu = 7(\nu^3 + 5\nu^2 + 7\nu) + (\nu^2 + 7\nu + 7)(4\nu^3 + 21\nu^2 + 28\nu)^{1/2}. \quad (4.3)$$

Proof. Recall the definitions of α, β , and γ from (3.21). It will be convenient to define

$$\begin{cases} A := \alpha^3\beta + \beta^3\gamma + \gamma^3\alpha, \\ B := \alpha^2\beta^3 + \beta^2\gamma^3 + \gamma^2\alpha^3, \\ C := \alpha\beta^5 + \beta\gamma^5 + \gamma\alpha^5, \\ D := \alpha^7 + \beta^7 + \gamma^7, \\ E := \alpha^6\beta^2 + \beta^6\gamma^2 + \gamma^6\alpha^2. \end{cases} \quad (4.4)$$

By (3.21), the equalities (3.7) now take the shapes

$$\begin{cases} \gamma - \alpha^2 + \alpha\beta^2 & = 0, \\ \alpha - \beta^2 + \beta\gamma^2 & = 0, \\ \beta - \gamma^2 + \gamma\alpha^2 & = 0, \\ \alpha\beta\gamma & = -1. \end{cases} \quad (4.5)$$

From (4.4) and (4.5), it is easy to see that

$$B = A + 3, \quad C = B - A = 3, \quad \text{and} \quad D = E + C = E + 3. \quad (4.6)$$

Furthermore, by (4.5),

$$\begin{aligned} \alpha^7\beta^7 + \beta^7\gamma^7 + \gamma^7\alpha^7 &= (\alpha^8\beta^5 + \beta^8\gamma^5 + \gamma^8\alpha^5) + (\alpha^5\beta^4 + \beta^5\gamma^4 + \gamma^5\alpha^4), \\ \alpha^8\beta^5 + \beta^8\gamma^5 + \gamma^8\alpha^5 &= (\alpha^9\beta^3 + \beta^9\gamma^3 + \gamma^9\alpha^3) + E, \\ \alpha^9\beta^3 + \beta^9\gamma^3 + \gamma^9\alpha^3 &= (\alpha^{10}\beta + \beta^{10}\gamma + \gamma^{10}\alpha) + D, \\ \alpha^5\beta^4 + \beta^5\gamma^4 + \gamma^5\alpha^4 &= E + A. \end{aligned} \quad (4.7)$$

Using successively the four preceding equalities, as well as (4.6), we readily find that

$$T := \alpha^7\beta^7 + \beta^7\gamma^7 + \gamma^7\alpha^7 = (\alpha^{10}\beta + \beta^{10}\gamma + \gamma^{10}\alpha) + A + 3D - 6. \quad (4.8)$$

By (3.21), (3.12), and (3.17),

$$\begin{aligned} \frac{(q; q)_\infty^4}{q(q^7; q^7)_\infty^4} \frac{(q; q)_\infty^8}{q^2(q^7; q^7)_\infty^8} &= -(A + 8)(D + 14A + 55) \\ &= -AD - 14A^2 - 167A - 8D - 440. \end{aligned} \quad (4.9)$$

By direct calculations with the use of (4.6) and the equality $\alpha\beta\gamma = -1$ from (4.5), we easily deduce that

$$A^2 = D - 2A - 9$$

and

$$AD = (\alpha^{10}\beta + \beta^{10}\gamma + \gamma^{10}\alpha) + (\alpha^3\beta^8 + \beta^3\gamma^8 + \gamma^3\alpha^8) - E.$$

Using the foregoing two identities and (4.6) in (4.9), we find that

$$\frac{(q; q)_\infty^{12}}{q^3(q^7; q^7)_\infty^{12}} = -(\alpha^{10}\beta + \beta^{10}\gamma + \gamma^{10}\alpha) - (\alpha^3\beta^8 + \beta^3\gamma^8 + \gamma^3\alpha^8) - 139A - 21D - 317. \quad (4.10)$$

Appealing to (4.5) once again, (4.7), and (4.6), we find that

$$\begin{aligned} \alpha^4\beta^6 + \beta^4\gamma^6 + \gamma^4\alpha^6 &= B + (\alpha^5\beta^4 + \beta^5\gamma^4 + \gamma^5\alpha^4) \\ &= B + (E + A) = (A + 3) + (D - 3) + A = 2A + D, \end{aligned} \quad (4.11)$$

and by (4.5), (4.11), and (4.6), we deduce that

$$\begin{aligned} \alpha^3\beta^8 + \beta^3\gamma^8 + \gamma^3\alpha^8 &= (\alpha^4\beta^6 + \beta^4\gamma^6 + \gamma^4\alpha^6) + C \\ &= 2A + D + C = 2A + D + 3. \end{aligned} \quad (4.12)$$

Hence, by (4.10), (4.8), (4.12), (3.17), and (3.12),

$$\begin{aligned} \frac{(q; q)_\infty^{12}}{q^3(q^7; q^7)_\infty^{12}} &= -T + A + 3D - 6 - 2A - D - 3 - 139A - 21D - 317 \\ &= -T - 19D - 140A - 326 \\ &= -T - 19 \left(\frac{(q; q)_\infty^8}{q^2(q^7; q^7)_\infty^8} - 14A - 55 \right) - 140A - 326 \\ &= -T - 19 \frac{(q; q)_\infty^8}{q^2(q^7; q^7)_\infty^8} + 126A + 719 \\ &= -T - 19 \frac{(q; q)_\infty^8}{q^2(q^7; q^7)_\infty^8} - 126 \frac{(q; q)_\infty^4}{q(q^7; q^7)_\infty^4} - 289. \end{aligned}$$

Noting that, by (3.21), $T = -uv + uw - vw$, we see that (4.1) has now been established.

We next prove (4.3). Let $x = \alpha + \beta + \gamma$ and $y = \alpha\beta + \beta\gamma + \gamma\alpha$. Then, by (4.5),

$$\begin{aligned} A &= \alpha^3\beta + \beta^3\gamma + \gamma^3\alpha \\ &= (\alpha^2 + \beta^2 + \gamma^2)(\alpha\beta + \beta\gamma + \gamma\alpha) - (\alpha\beta^3 + \beta\gamma^3 + \gamma\alpha^3) + (\alpha + \beta + \gamma) \\ &= (x^2 - 2y)y - (\alpha\beta^3 + \beta\gamma^3 + \gamma\alpha^3) + x. \end{aligned} \quad (4.13)$$

But, from (4.5),

$$\alpha\beta^3 + \beta\gamma^3 + \gamma\alpha^3 = (\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha) - (\alpha\beta + \beta\gamma + \gamma\alpha) = (\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha) - y, \quad (4.14)$$

and

$$\begin{aligned} \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha &= (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - (\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2) + 3 \\ &= xy - (\alpha^2 + \beta^2 + \gamma^2) + (\alpha + \beta + \gamma) + 3 \\ &= xy - (x^2 - 2y) + x + 3. \end{aligned} \quad (4.15)$$

So, using (4.15) in (4.14) and then (4.14) in (4.13), and recalling that $\nu = \alpha + \beta + \gamma - 1 = x - 1$, by (3.3), we deduce that

$$A = -2y^2 + (x^2 - x - 1)y + x^2 - 3 = -2y^2 + (\nu^2 + \nu - 1)y + \nu^2 + 2\nu - 2. \quad (4.16)$$

Returning once more to (4.5), we find that

$$(\alpha\beta + \beta\gamma + \gamma\alpha) - (\alpha^3 + \beta^3 + \gamma^3) + (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) = 0. \quad (4.17)$$

Easy calculations show that

$$\alpha^3 + \beta^3 + \gamma^3 = x^3 - 3xy - 3 \quad \text{and} \quad \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = y^2 + 2x.$$

Using the two preceding equalities in (4.17), we find that

$$y^2 + (3x + 1)y - x^3 + 2x + 3 = 0,$$

and since $x = \nu + 1$, the latter equation can be written as

$$y^2 + (3\nu + 4)y - \nu^3 - 3\nu^2 - \nu + 4 = 0. \quad (4.18)$$

Solving (4.18) for y , observing from (3.21) that $y \sim -q^{-3/7}$ as $q \rightarrow 0^+$, and accordingly choosing the correct sign, we find that

$$2y = -(3\nu + 4) - K, \quad (4.19)$$

where $K = (4\nu^3 + 21\nu^2 + 28\nu)^{1/2}$. Hence, by (3.12), (4.16), and (4.18),

$$\mu = -A - 8 = 2y^2 - (\nu^2 + \nu - 1)y - \nu^2 - 2\nu - 6 = 2\nu^3 + 5\nu^2 - 14 - (\nu^2 + 7\nu + 7)y. \quad (4.20)$$

Finally, using (4.19) above, we conclude that

$$\begin{aligned} 2\mu &= 4\nu^3 + 10\nu^2 - 28 + (\nu^2 + 7\nu + 7)\{(3\nu + 4) + K\} \\ &= 7\nu^3 + 35\nu^2 + 49\nu + (\nu^2 + 7\nu + 7)K, \end{aligned}$$

which completes the proof of (4.3). \square

5. A BEAUTIFUL, BUT FALSE, CLAIM OF RAMANUJAN

At the bottom of page 189 in his lost notebook [10], Ramanujan wrote

“ n is the least positive integer such that $24n - 1$ is divisible by a positive integer k . Then

$$p(n + vk) - p(n)q(v) \quad (5.1)$$

is divisible by k for all positive integral values of v , where

$$(x; x)_{\infty}^{(24n-1)/k} = \sum_{\lambda=0}^{\infty} q(\lambda)x^{\lambda}.”$$

Of course, $q(v)$ depends on n (and k). Ramanujan then gives the examples

$$p(4), p(9), p(14), \dots \equiv 0 \pmod{5},$$

$$p(5), p(12), p(19), \dots \equiv 0 \pmod{7},$$

$$p(6), p(17), p(28), \dots \equiv 0 \pmod{11},$$

$$p(24) + 1, p(47) + 1, p(70), p(93), p(116) - 1, p(139), p(162) - 1, p(185), \dots \equiv 0 \pmod{23}.$$

All four sets of congruences would follow from Ramanujan's claim, if it were true. Although it is well known that the first three examples are indeed true, the fourth is false. For example, $p(24) + 1 = 1576$ is not divisible by 23.

However, Ramanujan himself modified his assertion in his unpublished manuscript on the partition and τ -functions [10, pp. 157–162], [3, Sects. 15, 16]. In particular, Ramanujan wrote “From this we can always deduce in every particular case that

$$\begin{aligned} & \sum_{n=1}^{\infty} p\left(n\varpi + \varpi \left[\frac{\varpi}{24}\right] - \frac{\varpi^2 - 1}{24}\right) q^{n+[\varpi/24]}(q^{\varpi}; q^{\varpi})_{\infty} \\ & = (Q^3 - R^2)^{1+[\varpi/24]} \sum k_{\ell,m} Q^{\ell} R^m + \varpi J \end{aligned} \quad (5.2)$$

where $k_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$4\ell + 6m = \varpi - 13 \quad (5.3)$$

and $[t]$ denotes as usual the greatest integer in t .” It is clear that Ramanujan did not have a proof of his claim (5.2); however, for all examples that he determined, (5.2) holds. Here, Q and R are the Eisenstein series defined for $|q| < 1$ by

$$Q(q) := 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1 - q^k}$$

and

$$R(q) := 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1 - q^k}.$$

Thus, Ramanujan's original claim about (5.1) must be modified by multiplying the appropriate power of $(q; q)_{\infty}$ by a polynomial in Q and R . It was first pointed out to us by Heng Huat Chan that the summation condition (5.3) is incorrect and that it should be replaced by

$$4\ell + 6m = \varpi - 13 - 12 \left[\frac{\varpi}{24}\right]. \quad (5.4)$$

Moreover, Chan supplied us with a manuscript, based on correspondence with J.-P. Serre and work of K. S. Chua [4] and himself, in which (5.2) is proved with the condition (5.4) in place of (5.3). S. Ahlgren and M. Boylan [1] have also established a corrected version of (5.2). In fact, they have generalized Ramanujan's (corrected) claim by replacing the prime ϖ by any prime power ϖ^j , $\varpi \geq 5$, $j > 0$.

Ramanujan worked out examples up to $\varpi = 23$, and so consequently could not observe that (5.3) needed to be replaced by (5.4). In particular, he writes, “In these cases we can easily prove that

$$\sum_{n=1}^{\infty} p(17n - 12)q^n (q^{17}; q^{17})_{\infty} = 7 \sum_{n=1}^{\infty} \tau_2(n)q^n + 17J,$$

where

$$\sum_{n=1}^{\infty} \tau_2(n)q^n = Qq(q; q)_{\infty}^{24};$$

$$\sum_{n=1}^{\infty} p(19n - 15)q^n (q^{19}; q^{19})_{\infty} = 5 \sum_{n=1}^{\infty} \tau_3(n)q^n + 19J,$$

where

$$\sum_{n=1}^{\infty} \tau_3(n)q^n = Rq(q; q)_{\infty}^{24n}.$$

Thus, as predicted by (5.2), in the cases $\varpi = 17, 19$, respectively, Ramanujan’s original claim must be modified by multiplying the appropriate power of $(q; q)_{\infty}$ by Q and R , respectively.

We are grateful to Heng Huat Chan for informing us that the results on page 189 of the lost notebook were briefly discussed by Ramanathan [6, pp. 154–155], and for supplying us with a proof of (5.2). We also thank Scott Ahlgren for several informative conversations.

REFERENCES

- [1] S. Ahlgren and M. Boylan, *Arithmetic properties of the partition function*, Invent. Math., to appear.
- [2] B. C. Berndt, *Ramanujan’s Notebooks*, Part III, Springer–Verlag, New York, 1991.
- [3] B. C. Berndt and K. Ono, *Ramanujan’s unpublished manuscript on the partition and tau functions with proofs and commentary*, Sémin. Lotharingien de Combinatoire **42** (1999), 63 pp; The Andrews Festschrift, D. Foata and G.–N. Han, eds., Springer–Verlag, Berlin, 2001, pp. 39–110.
- [4] K. S. Chua, *Explicit congruences of the partition function modulo every prime*, Archiv Math., to appear.
- [5] M. Newman, *Remarks on some modular identities*, Trans. Amer. Math. Soc. **73** (1952), 313–320.
- [6] K. G. Ramanathan, *Ramanujan and the congruence properties of partitions*, Proc. Indian Acad. Sci. (Math. Sci.) **89** (1980), 133–157.
- [7] S. Ramanujan, *Some properties of $p(n)$, the number of partitions of n* , Proc. Cambridge Philos. Soc. **19** (1919), 207–210.
- [8] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1960; reprinted by the American Mathematical Society, Providence, RI, 2000.
- [9] S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.
- [10] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN STREET, URBANA, IL 61801, USA

E-mail address: berndt@math.uiuc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN STREET, URBANA, IL 61801, USA

E-mail address: yee@math.uiuc.edu

DEPARTMENT OF MATHEMATICS, MONMOUTH COLLEGE, 700 EAST BROADWAY, MONMOUTH, IL 61462 USA

E-mail address: jinheeyi@monm.edu