AN INTEGRAL OF DEDEKIND ETA-FUNCTIONS IN RAMANUJAN'S LOST NOTEBOOK

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Abstract. On page 207 in his lost notebook, Ramanujan recorded a curious formula for an integral of a certain quotient of Dedekind eta-functions. The formula involves a mysterious constant C, which Ramanujan claims, with two question marks appended, is a simple multiple of a Dirichlet L-function evaluated at the argument 2. The integral formula was first proved by S. H. Son, but he did not establish Ramanujan's formula for C. We give an entirely different proof of this result which also shows that Ramanujan's questioned formula for C is correct.

1. Introduction

On pages 46 and 207 in his lost notebook [3], Ramanujan recorded eight identities for integrals of theta functions. One of them was proved by G. E. Andrews [1], while the remaining seven were established by S. H. Son [4]. One of the formulas on page 207 was particularly troublesome to prove. It involves Dedekind eta-functions in the integrand and a particular constant C, which Ramanujan claims, with two question marks appended, is a simple multiple of a value of a Dirichlet L-function evaluated at the argument 2. Although Ramanujan's formula was elegantly proved by Son [4], he could not establish Ramanujan's queried value of C. The purpose of this paper is to give a completely different proof of Ramanujan's integral formula, which yields the tenuously claimed value of C as well.

The Dedekind eta-function $\eta(z)$ is defined by

$$\eta(z) := e^{2\pi i z/24}\prod_{n=1}^{\infty}(1-e^{2\pi i n z}) =: q^{1/24}f(-q), \qquad q=e^{2\pi i z}, \qquad \operatorname{Im} z>0, \tag{1.1}$$

where we employ Ramanujan's notation $f(-q)$.

We can now state the integral formula of Ramanujan which we want to prove.

Theorem 1.1. *For* $0<q<1$,

$$q^{1/9}\frac{(1-q)(1-q^4)^4(1-q^7)^7\cdots}{(1-q^2)^2(1-q^5)^5(1-q^8)^8\cdots} = \exp\left(-C-\frac{1}{9}\int_q^1 \frac{f^9(-t)}{f^3(-t^3)}\frac{dt}{t}\right), \tag{1.2}$$

where

$$C := \frac{3\sqrt{3}}{4\pi}\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{1}{n^2}, \tag{1.3}$$

where $\left(\frac{n}{3}\right)$ *denotes the Legendre symbol.*

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David Masser has kindly informed us that C also has the representation $C = L'(-1,\chi)$, where $\chi(n) = \left(\frac{n}{3}\right)$. This can be demonstrated by differentiating the functional equation for the Dirichlet L-function $L(s,\chi)$.

2. Preliminary Results

To prove his less precise version of Theorem 1.1, Son [4, Lemma 2.5] established the following lemma using a theorem of N. J. Fine [2, p. 22].

Lemma 2.1. *For* $|q| < 1$,

$$\frac{f^9(-q)}{f^3(-q^3)} = 1 + 9\sum_{n=1}^{\infty}\left\{\frac{(3n-1)^2q^{3n-1}}{1-q^{3n-1}} - \frac{(3n-2)^2q^{3n-2}}{1-q^{3n-2}}\right\}. \tag{2.1}$$

In our proof, a different representation for $f^9(-q)/f^3(-q^3)$ arises, and we establish this in the next lemma.

Lemma 2.2. *For* $|q| < 1$,

$$\frac{f^9(-q)}{f^3(-q^3)} = 1 - 9\sum_{n=1}^{\infty}\frac{q^n - q^{2n} - 6q^{3n} - q^{4n} + q^{5n}}{(1+q^n+q^{2n})^3}. \tag{2.2}$$

Proof. Multiplying numerators and denominators by $(1-q^n)^3$ and then inverting the order of summation, we find that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{q^n-q^{2n}-6q^{3n}-q^{4n}+q^{5n}}{(1+q^n+q^{2n})^3}\\
=&\sum_{n=1}^{\infty}\frac{q^n-4q^{2n}+13q^{4n}-13q^{5n}+4q^{7n}-q^{8n}}{(1-q^{3n})^3}\\
=&\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=2}^{\infty}m(m-1)\left(q^n-4q^{2n}+13q^{4n}-13q^{5n}+4q^{7n}-q^{8n}\right)q^{3n(m-2)}\\
=&\frac{1}{2}\sum_{m=2}^{\infty}m(m-1)\left(\frac{q^{3m-5}}{1-q^{3m-5}}-4\frac{q^{3m-4}}{1-q^{3m-4}}+13\frac{q^{3m-2}}{1-q^{3m-2}}\right.\\
&\left.-13\frac{q^{3m-1}}{1-q^{3m-1}}+4\frac{q^{3m+1}}{1-q^{3m+1}}-\frac{q^{3m+2}}{1-q^{3m+2}}\right)\\
=&\frac{1}{2}\sum_{m=1}^{\infty}(m+1)m\frac{q^{3m-2}}{1-q^{3m-2}}-2\sum_{m=1}^{\infty}(m+1)m\frac{q^{3m-1}}{1-q^{3m-1}}\\
&+\frac{13}{2}\sum_{m=1}^{\infty}m(m-1)\frac{q^{3m-2}}{1-q^{3m-2}}-\frac{13}{2}\sum_{m=1}^{\infty}m(m-1)\frac{q^{3m-1}}{1-q^{3m-1}}\\
&+2\sum_{m=1}^{\infty}(m-1)(m-2)\frac{q^{3m-2}}{1-q^{3m-2}}-\frac{1}{2}\sum_{m=1}^{\infty}(m-1)(m-2)\frac{q^{3m-1}}{1-q^{3m-1}}\\
=&-\sum_{m=1}^{\infty}\left\{\frac{(3m-1)^2q^{3m-1}}{1-q^{3m-1}}-\frac{(3m-2)^2q^{3m-2}}{1-q^{3m-2}}\right\},
\end{aligned}$$

where in the last step we merely added together the coefficients of each of the two distinct q-quotients. The result now follows from Lemma 2.1. ☐

3. Proof of Theorem 1.1

Our proof will proceed in four steps. First, we show that Ramanujan's formula (1.2) implies (2.2), and conversely that (2.2) implies (1.2), except for the identification of the additive constant C. It then remains to prove that C has the prescribed value (1.3), which we do in three steps. We first show that C can be represented as the limit of a certain q-series as $q\to 1^-$. Second, we show that this limit can be represented by an integral. Lastly, we evaluate this integral to prove (1.3).

Proof. Assume throughout the proof that $0<q<1$. Taking the logarithm of both sides of (1.2) and using the Taylor expansion of $\log(1-z)$ about $z=0$, we find that

$$\frac{1}{9}\log q-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n}{3}\right)\frac{nq^{mn}}{m}=-C-\frac{1}{9}\int_q^1\frac{f^9(-t)}{f^3(-t^3)}\,\frac{dt}{t}. \tag{3.1}$$

It is easy to see that

$$\sum_{n\equiv 1\,(\mathrm{mod}\,3)} q^n = \frac{q}{1-q^3} \qquad \text{and} \qquad \sum_{n\equiv 2\,(\mathrm{mod}\,3)} q^n = \frac{q^2}{1-q^3}. \tag{3.2}$$

Differentiating (3.2), we find that

$$\sum_{n\equiv 1\,(\mathrm{mod}\,3)} nq^{n-1} = \frac{1+2q^3}{(1-q^3)^2} \qquad \text{and} \qquad \sum_{n\equiv 2\,(\mathrm{mod}\,3)} nq^{n-1} = \frac{2q+q^4}{(1-q^3)^2}. \tag{3.3}$$

Combining the two equalities of (3.3), we deduce that

$$\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) nq^n = \frac{q-q^3}{(1+q+q^2)^2}. \tag{3.4}$$

Using (3.4) in (3.1), we find that (3.1) is equivalent to

$$\sum_{m=1}^{\infty} \frac{q^m - q^{3m}}{m(1+q^m+q^{2m})^2} = \frac{1}{9}\log q + C + \frac{1}{9}\int_q^1 \frac{f^9(-t)}{f^3(-t^3)}\,\frac{dt}{t}. \tag{3.5}$$

For brevity, let L and R denote the left and right sides, respectively, of (3.5). Elementary differentiations show that

$$\begin{aligned} q\frac{dL}{dq} =& q\sum_{m=1}^{\infty} \frac{(mq^{m-1}-3mq^{3m-1})(1+q^m+q^{2m}) - 2(q^m-q^{3m})(mq^{m-1}+2mq^{2m-1})}{m(1+q^m+q^{2m})^3} \\ =& \sum_{m=1}^{\infty} \frac{q^m - q^{2m} - 6q^{3m} - q^{4m} + q^{5m}}{(1+q^m+q^{2m})^3} \end{aligned} \tag{3.6}$$

and

$$q\frac{dR}{dq} = \frac{1}{9} - \frac{1}{9}\frac{f^9(-q)}{f^3(-q^3)}. \tag{3.7}$$

Employing (3.6) and (3.7) in (3.5), we conclude that Ramanujan's formula (1.2) implies the equality

$$1 - 9\sum_{m=1}^{\infty} \frac{q^m - q^{2m} - 6q^{3m} - q^{4m} + q^{5m}}{(1+q^m+q^{2m})^3} = \frac{f^9(-q)}{f^3(-q^3)}. \tag{3.8}$$

Conversely, (3.8) implies that (1.2) holds for $0 < q < 1$ and for some constant C. However, indeed (3.8) is valid by Lemma 2.2. Thus, it remains to prove that C has the value given by (1.3), which we now do in the three steps outlined above.

First, by (3.5), it is clear that

$$C = \lim_{q\to 1^-} \sum_{m=1}^{\infty} \frac{q^m - q^{3m}}{m(1+q^m+q^{2m})^2}. \tag{3.9}$$

Secondly, we prove that

$$C = \int_{-\infty}^{\infty} \frac{\sinh u}{u(1+2\cosh u)^2} du. \tag{3.10}$$

To prove (3.10), set $q = \exp(-1/N)$, where N is a large positive integer. Then (3.9) may be written in the form

$$C = \lim_{N\to\infty} \sum_{m=1}^{\infty} \frac{e^{-m/N} - e^{-3m/N}}{m(1+e^{-m/N}+e^{-2m/N})^2} = \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{\infty} \frac{e^{-m/N} - e^{-3m/N}}{(m/N)(1+e^{-m/N}+e^{-2m/N})^2}. \tag{3.11}$$

On the far right side of (3.11), we have a Riemann sum. Taking the limit as $N \to \infty$, we deduce that

$$\begin{aligned} C =& \int_0^{\infty} \frac{e^{-u} - e^{-3u}}{u(1+e^{-u}+e^{-2u})^2} du = \int_0^{\infty} \frac{e^{u} - e^{-u}}{u(e^{u}+1+e^{-u})^2} du \\ =&2 \int_0^{\infty} \frac{\sinh u}{u(1+2\cosh u)^2} du = \int_{-\infty}^{\infty} \frac{\sinh u}{u(1+2\cosh u)^2} du, \end{aligned}$$

since the integrand is even, which establishes (3.10).

The function

$$g(z) := \frac{\sinh z}{z(1+2\cosh z)^2} \tag{3.12}$$

is meromorphic in the entire complex plane, and has double poles at the points $2\pi i n/3$, for each integer n that is not a multiple of 3. Let $\gamma_{R_m}, 1 \le m < \infty$, be a sequence of positively oriented rectangles with vertices $\pm\sqrt{R_m}$ and $\pm\sqrt{R_m} + R_m^{3/2} i$, which are chosen so that the points $R_m^{3/2} i$ remain at a bounded distance from the points $2\pi i n/3$, as m tends to ∞. For brevity, let $L_1 = L_1(m)$ and $L_2 = L_2(m)$ denote, respectively, the left and right sides, and let $L_3 = L_3(m)$ denote the top side of γ_{R_m}. Then, it is not difficult to see that, for $j = 1, 2$,

$$\left| \int_{L_j} g(z) dz \right| \ll R_m e^{-\sqrt{R_m}}, \tag{3.13}$$

as $R_m \to \infty$. It is also not difficult to see that

$$\left| \int_{L_3} g(z) dz \right| \ll \frac{1}{R_m}, \tag{3.14}$$

as $R_m \to \infty$. In summary, the inequalities (3.13) and (3.14) imply that, if $\gamma'_{R_m} = L_1 \cup L_2 \cup L_3$, then

$$\int_{\gamma'_{R_m}} g(z) dz = o(1), \tag{3.15}$$

as $R_m \to \infty$.

Letting $R(a)$ denote the residue of $g(z)$ at a pole a, we find by the residue theorem that

$$\frac{1}{2\pi i} \int_{-\sqrt{R_m}}^{\sqrt{R_m}} g(z) dz + \frac{1}{2\pi i} \int_{\gamma'_{R_m}} g(z) dz = \sum_{\substack{1 \le n < 3R_m^{3/2}/(2\pi) \\ 3 \nmid n}} R\left(\frac{2\pi i n}{3}\right). \tag{3.16}$$

Letting R_m tend to ∞ in (3.16) and using (3.15), we deduce from (3.10) that

$$C = 2\pi i \sum_{\substack{n=1 \\ 3\nmid n}}^{\infty} R\left(\frac{2\pi i n}{3}\right). \tag{3.17}$$

In order to compute the residues, we introduce simpler notation. If the positive integer n is not a multiple of 3, set $a = 2\pi i n/3$ and $\omega = e^{2\pi i/3}$. Then $e^a = \omega$, if $n \equiv 1 \pmod 3$ and $e^a = \bar{\omega}$, if $n \equiv 2 \pmod 3$. We use the Taylor expansions,

$$\frac{1}{z} = \frac{1}{a} - \frac{z-a}{a^2} + \cdots, \tag{3.18}$$

$$\sinh z = \sinh a + (z-a)\cosh a + \cdots, \tag{3.19}$$

and

$$\cosh z = \cosh a + (z-a)\sinh a + \tfrac{1}{2}(z-a)^2\cosh a + \cdots. \tag{3.20}$$

Since $1 + 2\cosh a = 0$, it follows from (3.20) that

$$1 + 2\cosh z = 2(z-a)\sinh a\left(1 + (z-a)\frac{\cosh a}{2\sinh a} + \cdots\right),$$

and so

$$\frac{1}{(1+2\cosh z)^2} = \frac{1 - (z-a)\dfrac{\cosh a}{\sinh a} + \cdots}{4(z-a)^2\sinh^2 a}. \tag{3.21}$$

Using (3.18), (3.19), and (3.21) in (3.12), we find that

$$\begin{aligned} g(z) &= \frac{\left(1 + (z-a)\dfrac{\cosh a}{\sinh a} + \cdots\right)\left(1 - \dfrac{z-a}{a} + \cdots\right)\left(1 - (z-a)\dfrac{\cosh a}{\sinh a} + \cdots\right)}{4a(z-a)^2\sinh a} \\ &= \frac{1 - \dfrac{z-a}{a} + \cdots}{4a(z-a)^2\sinh a}, \end{aligned}$$

and so

$$R(a) = -\frac{1}{4a^2\sinh a} = \frac{1}{2a^2(e^{-a} - e^{a})}.$$

We distinguish two cases. If $n \equiv 1 \pmod 3$, then $e^{-a} - e^a = \bar{\omega} - \omega = -i\sqrt{3}$, and hence

$$R(a) = \frac{i}{2a^2\sqrt{3}} = -\frac{3\sqrt{3}\,i}{8\pi^2 n^2}. \tag{3.22}$$

If $n \equiv 2 \pmod 3$, then $e^{-a} - e^a = \omega - \bar{\omega} = i\sqrt{3}$, and hence

$$R(a) = -\frac{i}{2a^2\sqrt{3}} = \frac{3\sqrt{3}\,i}{8\pi^2 n^2}. \tag{3.23}$$

Using (3.22) and (3.23) in (3.17), we conclude that

$$C = \frac{3\sqrt{3}}{4\pi}\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{1}{n^2},$$

which is (1.3). This then completes the proof of Theorem 1.1.

$\square$

References

[1] G. E. Andrews, *Ramanujan's "lost" notebook III. The Rogers–Ramanujan continued fraction*, Adv. Math. 41 (1981), 186–208.

[2] N. J. Fine, *Basic Hypergeometric Series and Applications*, American Mathematical Society, Providence, RI, 1988.

[3] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[4] S. H. Son, *Some integrals of theta functions in Ramanujan's lost notebook*, in *Fifth Conference of the Canadian Number Theory Association* (R. Gupta and K. S. Williams, eds.), CRM Proc. and Lecture Notes, Vol. 19, American Mathematical Society, Providence, RI, 1999, pp. 323–332.

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