A RECIPROCITY THEOREM FOR CERTAIN HYPERGEOMETRIC SERIES

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Abstract. A reciprocity theorem for certain infinite series of gamma functions found in Chapter 14 of Ramanujan's second notebook is proved and generalized.

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1. Introduction

In Entry 2 of Chapter 14 in his second notebook [4], Ramanujan states a beautiful reciprocity theorem (with no hypotheses or proof) for certain hypergeometric series. In his notebooks [4], Ramanujan recorded many "reciprocity theorems" or "modular relations" for infinite series, but we are unaware of any other reciprocity theorem for hypergeometric series, either in Ramanujan's notebooks or elsewhere in the literature, other than the theorem below.

Theorem 1.1. *Let x, y, m, and n be complex numbers such that* $\mathrm{Re}(m+n)>0$ *and* $xy\neq 0$. *Suppose that* $\Gamma(1+xz)$ *and* $\Gamma(1+yz)$*, as functions of z, have no coincident poles, and also suppose that $z=1$ is not a pole of either function. Let*

$$S(m,n;x,y):=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Gamma(1-ky/x)}{\Gamma(m-k+1)\Gamma(n+1-ky/x)\Gamma(k)(x+k)}.$$

Then

$$S(m,n;x,y)+S(n,m;y,x)=\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+m+1)\Gamma(y+n+1)}. \tag{1.1}$$

The proof of Theorem 1.1 given by the first author in [2] and [3, pp. 245–246] should have provided readers with more details in the application of Stirling's formula [1, p. 21]

$$\Gamma(z)\sim\sqrt{2\pi}z^{z-1/2}e^{-z}, \tag{1.2}$$

as $|z|\to\infty$ in $\{z: |\arg z|\leq \pi-\delta\}$, where $\delta>0$ is fixed.

The purpose of this paper is to not only provide a more complete proof of Theorem 1.1 but to also establish a considerable generalization of (1.1). Observe that the left-hand side of (1.1) may be regarded as a quasi-partial fraction decomposition of the right-hand side, where we regard the right-hand side as a function of two complex variables x and y. After proving our generalization in Section 2, we conclude our paper with a couple corollaries and remarks in Section 3.

2. A Generalization of Theorem 1.1

Theorem 2.1. *Let $x_1,\dots,x_r$ and $m_1,\dots,m_r$ be complex numbers such that* $\mathrm{Re}(m_1+\cdots+m_r)>0$ *and $x_1\cdots x_r\neq 0$, where r is a positive integer at least equal to 2. Suppose that* $\Gamma(1+x_1z)$, …,

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$\Gamma(1+x_rz)$, as functions of z, have no coincident poles and also suppose that $z=1$ is not a pole of any of these r functions. Let

$$G_{j,k}(x_1,\dots,x_r):=\prod_{\substack{\mu=1\\ \mu\neq j}}^{r}\frac{\Gamma(1-kx_\mu/x_j)}{\Gamma(m_\mu+1-kx_\mu/x_j)}$$

and

$$S_j(m_1,\dots,m_r;x_1,\dots,x_r):=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}G_{j,k}(x_1,\dots,x_r)}{\Gamma(m_j-k+1)\Gamma(k)(x_j+k)}.$$

Then

$$\sum_{j=1}^{r}S_j(m_1,\dots,m_r;x_1,\dots,x_r)=\frac{\Gamma(x_1+1)\cdots\Gamma(x_r+1)}{\Gamma(x_1+m_1+1)\cdots\Gamma(x_r+m_r+1)}. \tag{2.1}$$

Proof. Throughout the proof, all implied constants in our upper bound estimates depend upon x_j and m_j, $1\le j\le r$. Define

$$f(z):=\frac{\Gamma(1+x_1z)\cdots\Gamma(1+x_rz)}{\Gamma(m_1+1+x_1z)\cdots\Gamma(m_r+1+x_rz)}\,\frac{1}{z-1}. \tag{2.2}$$

Let C_n, $n>0$, denote a square centered at the origin with vertical sides through $\pm N_n$ and horizontal sides through $\pm iN_n$. We assume that the sequence $\{N_n\}$, tending to ∞ as $n\to\infty$, is chosen so that the squares C_n remain at a bounded distance away from the poles of $f(z)$. We apply the residue theorem to

$$\frac{1}{2\pi i}\int_{C_n}f(z)dz=\frac{1}{2\pi i}\int_{\gamma_n}\left(f(z)-f(-z)\right)dz, \tag{2.3}$$

where γ_n is that portion of C_n lying in the right half-plane. Let R_{z_0} denote the residue of $f(z)$ at a pole z_0.

First, $f(z)$ has a pole at $z=1$, which is simple by hypothesis, and it is easy to see that

$$R_1=\frac{\Gamma(1+x_1)\cdots\Gamma(1+x_r)}{\Gamma(m_1+1+x_1)\cdots\Gamma(m_r+1+x_r)}. \tag{2.4}$$

Second, since $\Gamma(z)$ has a simple pole at $z=-k$ with residue $(-1)^k/k!$, for each nonnegative integer k [1, p. 7], we see that, for $1\le j\le r$, $\Gamma(1+x_jz)$ has a simple pole at $z=-k/x_j$, $k\ge 1$, with residue

$$\frac{(-1)^{k-1}}{x_j(k-1)!}. \tag{2.5}$$

By a simple calculation with the use of (2.5), it follows that

$$R_{-k/x_j}=\frac{(-1)^kG_{j,k}(x_1,\dots,x_r)}{(k-1)!\Gamma(m_j+1-k)(x_j+k)}. \tag{2.6}$$

We next examine $f(z)$ on the contour γ_n. If Re $x_j>0$, we apply Stirling's formula (1.2) to both $\Gamma(1+x_jz)$ and $\Gamma(m_j+1+x_jz)$. However, If Re $x_j<0$, we first apply the reflection formula for the gamma function to each of these gamma functions before applying Stirling's formula. To that end,

$$\frac{\Gamma(1+x_jz)}{\Gamma(m_j+1+x_jz)}=\frac{\Gamma(-m_j-x_jz)}{\Gamma(-x_jz)}\frac{\sin\pi(-m_j-x_jz)}{\sin\pi(-x_jz)}. \tag{2.7}$$

Since

$$\frac{\sin\pi(m_j+x_jz)}{\sin\pi(x_jz)}=\cos(\pi m_j)+\sin(\pi m_j)\cot(\pi x_jz),$$

and since, by the choice of our contour C_n, $|\cot(\pi x_jz)|\ll_{x_j}1$, we find that

$$\left|\frac{\sin\pi(m_j+x_jz)}{\sin\pi(x_jz)}\right|\ll 1.$$

Hence, from (2.7) and Stirling's formula (1.2), we can deduce that

$$\frac{\Gamma(1+x_jz)}{\Gamma(m_j+1+x_jz)} = O\left(|z|^{-\text{Re } m_j}\right), \tag{2.8}$$

as $|z| \to \infty$ on γ_n. (In fact, a weaker asymptotic formula than Stirling's formula could have been employed in (2.8) and (2.10) below [1, p. 29, eqn. (1.4.3)].) If Re $x_j > 0$, then we also deduce (2.8) by direct applications of Stirling's formula.

The examination of $f(-z)$ on γ_n is similar. However, in this case, if Re $x_j < 0$, we can apply Stirling's formula directly to each quotient

$$\frac{\Gamma(1-x_jz)}{\Gamma(m_j+1-x_jz)}, \tag{2.9}$$

while if Re $x_j > 0$, we must first use the reflection formula on each of the gamma functions in (2.9) before applying Stirling's formula. In either case, we obtain the upper bound

$$\frac{\Gamma(1-x_jz)}{\Gamma(m_j+1-x_jz)} = O\left(|z|^{-\text{Re } m_j}\right). \tag{2.10}$$

Hence, using (2.8) and (2.10), $1 \leq j \leq r$, as $|z|$ tends to ∞ on γ_n, we conclude that

$$f(z) = O\left(|z|^{-\text{Re}(m_1+\cdots+m_r)-1}\right). \tag{2.11}$$

Therefore, since $\text{Re}(m_1+\cdots+m_r) > 0$, by (2.2) and (2.3),

$$\frac{1}{2\pi i}\int_{C_n} f(z)dz = o(1), \tag{2.12}$$

as n tends to infinity.

Finally, applying the residue theorem in (2.3), letting $n \to \infty$, and using (2.12), (2.4), and (2.6), we conclude that

$$\frac{\Gamma(1+x_1)\cdots\Gamma(1+x_r)}{\Gamma(m_1+1+x_1)\cdots\Gamma(m_r+1+x_r)} + \sum_{j=1}^{r}\sum_{k=1}^{\infty}\frac{(-1)^k G_{j,k}(x_1,\ldots,x_r)}{(k-1)!\Gamma(m_j+1-k)(x_j+k)} = 0,$$

from which (2.1) is immediate. □

3. Corollaries and Concluding Remarks

Observe that if we let $r = 2$, $x_1 = x$, $x_2 = y$, $m_1 = m$, and $m_2 = n$ in Theorem 2.1, we obtain Ramanujan's Theorem 1.1. If we further suppose in Theorem 1.1 that m and n are nonnegative integers, then (1.1) reduces to the quasi-partial fraction decomposition (in two variables)

$$\sum_{k=1}^{m}\frac{(-1)^{k-1}}{(1-ky/x)_n(k-1)!(m-k)!(x+k)} + \sum_{k=1}^{n}\frac{(-1)^{k-1}}{(1-kx/y)_m(k-1)!(n-k)!(y+k)} = \frac{1}{(x+1)_m(y+1)_n}, \tag{3.1}$$

where

$$(a)_0 := 1, \qquad (a)_n := a(a+1)\cdots(a+n-1), \qquad n \geq 1.$$

Of course, a similar result can be derived from Theorem 2.1 when $m_1, \ldots, m_r$ are all nonnegative integers. If $m = 0$, then (3.1) reduces to the genuine partial fraction expansion

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{(k-1)!(n-k)!(y+k)} = \frac{1}{(y+1)_n}.$$

The function $1/(z-1)$ in the definition (2.2) of $f(z)$ can be replaced by

$$R(z) := \frac{p(z)}{q(z)},$$

where $p(z)$ and $q(z)$ are polynomials such that $\mathrm{Re}(m_1+\cdots+m_r)+\deg q(z)-\deg p(z)>1$, to obtain a generalization of Theorem 2.1. In particular, if we set $R(z)\equiv 1$ and assume that $\mathrm{Re}(m+n)>1$, then

$$\frac{1}{x}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Gamma(1-ky/x)}{\Gamma(m-k+1)\Gamma(n+1-ky/x)\Gamma(k)}=-\frac{1}{y}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Gamma(1-kx/y)}{\Gamma(n-k+1)\Gamma(m+1-kx/y)\Gamma(k)}. \tag{3.2}$$

Thus, if we set $m=n$, $z=-x/y$, and

$$F(z):=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Gamma(1+kz)}{\Gamma(n-k+1)\Gamma(n+1+kz)\Gamma(k)},$$

then we can deduce from (3.2) that

$$F\left(\frac{1}{z}\right)=zF(z), \tag{3.3}$$

provided that $\mathrm{Re}\ n>\frac{1}{2}$ and $z\neq\mathbb{Q}\cap(-\infty,0]$. In particular, (3.3) holds when $\mathrm{Re}\ n>\frac{1}{2}$ and $\mathrm{Re}\ z>0$.

Since the proof of the more general theorem is similar to that of Theorem 2.1, we do not give it here.

References

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