

# PARTITION-THEORETIC INTERPRETATIONS OF CERTAIN MODULAR EQUATIONS OF SCHRÖTER, RUSSELL, AND RAMANUJAN

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## 1. INTRODUCTION

In [4], H. M. Farkas and I. Kra established three theta constant identities and interpreted them in terms of partition identities. Perhaps the most elegant of their partition theorems is the following result [4, p. 202, Theorem 3].

**Theorem 1.1.** *Let  $S$  denote the set consisting of one copy of the positive integers and one additional copy of those positive integers that are multiples of 7. Then for each positive integer  $k$ , the number of partitions of  $2k$  into even elements of  $S$  is equal to the number of partitions of  $2k + 1$  into odd elements of  $S$ .*

Subsequently, M. D. Hirschhorn [8] and S. O. Warnaar [19] gave different proofs of Theorem 1.1 by giving elegant derivations of the equivalent  $q$ -series identity that yields Theorem 1.1 immediately. It was observed by the referee of [8] that the identity proved by Farkas and Kra [4], Warnaar [19], and Hirschhorn [8] was, in fact, equivalent to a modular equation of degree 7 found in Ramanujan's notebooks [13, Chapter 19, Entry 19(i)], [2, p. 314], but actually first proved by C. Guetzlaff [6] in 1834. Ramanujan discovered five modular equations of this type, which are also often called modular equations of Russell-type, after the English mathematician R. Russell, who discovered many modular equations of this sort [14], [15]. All five modular equations were previously and uniformly derived by H. Schröter and can be found in his paper [18]. We have carefully examined Russell's papers [14], [15], which contain many new modular equations, and we have searched a large segment of the literature on modular equations, and these five modular equations found independently by Schröter, Russell, and Ramanujan appear to be the only known modular equations of this simple type. The other modular equations of this type found by either Russell or Ramanujan are more complicated and do not seem to have elegant partition theoretic interpretations. For a history of modular equations up to 1928, consult a paper by M. Hanna [7].

The purpose of this paper is to establish the five partition identities implied by the five aforementioned modular equations. In closing this introduction, we remark that Warnaar [19] considerably generalized the aforementioned  $q$ -series identity associated with the partition identity connected with 7's., and gave a combinatorial proof of his more general identity. Although indeed Warnaar's proof provides a combinatorial

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proof of the original theorem of Farkas and Kra as a special case, his argument does not provide a simple combinatorial proof that one hopes would exist for this elegant theorem. Thus, in the words of Hirschhorn [8], 7 still remains “mysterious.” In a similar vein, the numbers 3, 5, 11, and 23, corresponding respectively to the partition identities arising from the modular equations of degrees 3, 5, 11, and 23 which follow, also appear to be manifestly “mysterious.” It appears to be extremely difficult to construct simple bijective proofs of each of these five partition identities. Indeed, such proofs would be of enormous interest. Observe that adding 1 to each of the five numbers above yields the divisors of 24, except 1, 2, and 3, and so it might be worthwhile in seeking bijective proofs to simultaneously consider all five cases, which appear to be the only cases where primes other than two arise. In the work of Farkas and Kra [4], Warnaar [19], and N. D. Baruah [1], a few further partition identities related to modular equations have been derived, but these seem to be related to modular equations of degree 2 or of composite degrees.

## 2. MODULAR EQUATIONS

We provide some definitions in preparation for defining a modular equation, as Ramanujan would have understood it.

The complete elliptic integral of the first kind associated with the *modulus*  $k$ ,  $0 < k < 1$ , is defined by

$$K := K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The complementary modulus  $k'$  is defined by  $k' := \sqrt{1 - k^2}$ ; set  $K' := K(k')$ . If  $q = \exp(-\pi K'/K)$ , then one of the central theorems in the theory of elliptic functions asserts that [2, p. 101, Entry 6]

$$\varphi^2(q) = \frac{2}{\pi} K(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \quad (2.1)$$

where  $\varphi$  denotes the classical theta function defined by

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2},$$

${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$  denotes an ordinary hypergeometric function, and where the last equality in (2.1) follows from expanding the integrand in a binomial series and integrating termwise. It is (2.1) upon which all of Ramanujan’s modular equations ultimately rests.

Let  $K$ ,  $K'$ ,  $L$ , and  $L'$  denote complete elliptic integrals of the first kind associated with the moduli  $k$ ,  $k'$ ,  $\ell$ , and  $\ell' := \sqrt{1 - \ell^2}$ , respectively, where  $0 < k, \ell < 1$ . Suppose that

$$n \frac{K'}{K} = \frac{L'}{L} \quad (2.2)$$

for some positive integer  $n$ . A relation between  $k$  and  $\ell$  induced by (2.2) is called a *modular equation of degree  $n$* . In fact, modular equations are always algebraic equations.

After Ramanujan, set  $\alpha = k^2$  and  $\beta = \ell^2$ . In the sequel, we shall frequently say that  $\beta$  has degree  $n$  over  $\alpha$ .

After Ramanujan, define

$$\chi(q) := (-q; q^2)_\infty, \quad (2.3)$$

where, as usual,

$$(a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \cdots, \quad |q| < 1.$$

In his notebooks [13, Chapter 17, Entry 12(v)–(vii)], [2, p. 124], Ramanujan recorded the following evaluations of  $\chi$  at different arguments in terms of  $\alpha$  and  $q$ .

**Lemma 2.1.** *We have*

$$\begin{aligned} \chi(q) &= \frac{2^{1/6} q^{1/24}}{\{\alpha(1 - \alpha)\}^{1/24}}, \\ \chi(-q) &= \frac{2^{1/6} q^{1/24} (1 - \alpha)^{1/12}}{\alpha^{1/24}}, \\ \chi(-q^2) &= \frac{2^{1/3} q^{1/12} (1 - \alpha)^{1/24}}{\alpha^{1/12}}. \end{aligned}$$

*Suppose that  $\beta$  has degree  $n$  over  $\alpha$ . If we replace  $q$  by  $q^n$  above, then the same evaluations hold with  $\alpha$  replaced by  $\beta$ .*

We begin each of the following five sections by citing the requisite modular equation of degree  $n$ . We then use Lemma 2.1 to transcribe it into an equivalent  $q$ -series identity. Finally, we give the partition-theoretic interpretation of the  $q$ -series identity. In our transcriptions, we make use of Euler's famous identity

$$\frac{1}{(q; q^2)_\infty} = (-q; q)_\infty, \quad (2.4)$$

i.e., the number of partitions of the positive integer  $n$  into odd parts is identical to the number of partitions of  $n$  into distinct parts.

### 3. PARTITIONS WITH MULTIPLES OF 3

The following modular equation of degree 3 was first proved by A. M. Legendre [12] and only slightly later by C. G. J. Jacobi in his epic work [9, p. 68], [10, p. 124]. It can also be found in A. Cayley's book [3, p. 196]. This modular equation was later rediscovered by Ramanujan [13, Chapter 19, Entry 5(ii)], [2, p. 230] and by Schröter [18], and is also equivalent to one of the theorems of Farkas and Kra [4].

**Theorem 3.1.** *If  $\beta$  has degree 3 over  $\alpha$ , then*

$$(\alpha\beta)^{1/4} + \{(1 - \alpha)(1 - \beta)\}^{1/4} = 1. \quad (3.1)$$

We now translate (3.1) into a  $q$ -series identity by using Lemma 2.1. Multiply both sides of (3.1) by

$$\frac{2^{2/3} q^{1/3}}{\{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/12}}$$

to deduce that

$$4q \frac{\alpha^{1/6}}{2^{2/3} q^{1/6} (1-\alpha)^{1/12}} \frac{\beta^{1/6}}{2^{2/3} q^{3/6} (1-\beta)^{1/12}} + \frac{2^{1/3} q^{1/12} (1-\alpha)^{1/6}}{\alpha^{1/12}} \frac{2^{1/3} q^{3/12} (1-\beta)^{1/6}}{\beta^{1/12}} = \frac{2^{1/3} q^{1/12}}{\{\alpha(1-\alpha)\}^{1/12}} \frac{2^{1/3} q^{3/12}}{\{\beta(1-\beta)\}^{1/12}}. \quad (3.2)$$

Using Lemma 2.1, we can rewrite (3.2) in the form

$$\frac{4q}{\chi^2(-q^2)\chi^2(-q^6)} + \chi^2(-q)\chi^2(-q^3) = \chi^2(q)\chi^2(q^3). \quad (3.3)$$

Lastly, using the definition of  $\chi$  in (2.3), rearranging, and employing Euler's theorem (2.4), we see that (3.3) can be written in the equivalent form

$$(-q; q^2)_\infty^2 (-q^3; q^6)_\infty^2 - (q; q^2)_\infty^2 (q^3; q^6)_\infty^2 = 4q(-q^2; q^2)_\infty^2 (-q^6; q^6)_\infty^2. \quad (3.4)$$

We now easily see that (3.4) has the following interpretation in terms of partitions.

**Theorem 3.2.** *Let  $\mathcal{S}$  denote the set of partitions into 4 distinct colors with two colors, say orange and blue, each appearing at most once, and the remaining two colors, say red and green, appearing at most once and only in multiples of 3. Let  $A(N)$  denote the number of partitions of  $2N + 1$  into odd elements of  $\mathcal{S}$ . Let  $B(N)$  denote the number of partitions of  $2N$  into even elements of  $\mathcal{S}$ . Then, for  $N \geq 1$ ,*

$$A(N) = 2B(N). \quad (3.5)$$

**Example 3.3.** *Let  $N = 2$ . Then  $B(2) = 3$  and  $A(2) = 6$ , and*

$$4_o = 4_b = 2_o + 2_b,$$

$$5_o = 5_b = 3_o + 1_o + 1_b = 3_b + 1_o + 1_b = 3_r + 1_o + 1_b = 3_g + 1_o + 1_b.$$

**Example 3.4.** *Let  $N = 3$ . Then  $B(3) = 8$ ,  $A(3) = 16$ , and*

$$6_r = 6_g = 6_b = 6_o = 4_o + 2_o = 4_o + 2_b = 4_b + 2_o = 4_b + 2_b,$$

$$7_o = 7_b = 5_o + 1_o + 1_b = 5_b + 1_o + 1_b = 3_r + 3_g + 1_b,$$

11 further partitions of the type  $3 + 3 + 1$ .

#### 4. PARTITIONS WITH MULTIPLES OF 5

The following modular equation of degree 5 is originally due to Jacobi [9], [10]. Schröter [18] rediscovered it, and Ramanujan recorded it as Entry 13(i) in Chapter 19 of his second notebook [13], [2, p. 280].

**Theorem 4.1.** *If  $\beta$  has degree 5 over  $\alpha$ , then*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \quad (4.1)$$

Multiply both sides of (4.1) by

$$\frac{2^{4/3}q}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}}$$

to write (4.1) in the form

$$16q^3 \frac{\alpha^{1/3}}{2^{4/3}q^{1/3}(1-\alpha)^{1/6}} \frac{\beta^{1/3}}{2^{4/3}q^{5/3}(1-\beta)^{1/6}} + \frac{2^{2/3}q^{1/6}(1-\alpha)^{1/3}}{\alpha^{1/6}} \frac{2^{2/3}q^{5/6}(1-\beta)^{1/3}}{\beta^{1/6}} + 8q = \frac{2^{2/3}q^{1/6}}{\{\alpha(1-\alpha)\}^{1/6}} \frac{2^{2/3}q^{5/6}}{\{\beta(1-\beta)\}^{1/6}}. \quad (4.2)$$

Applying Lemma 2.1, we can rewrite (4.2) as

$$\frac{16q^3}{\chi^4(-q^2)\chi^4(-q^{10})} + \chi^4(-q)\chi^4(-q^5) + 8q = \chi^4(q)\chi^4(q^5). \quad (4.3)$$

Using the definition of  $\chi$  in (2.3), invoking Euler's identity (2.4), and rearranging, we deduce from (4.3) that

$$(-q; q^2)_\infty^4 (-q^5; q^{10})_\infty^4 - (q; q^2)_\infty^4 (q^5; q^{10})_\infty^4 = 8q + 16q^3 (-q^2; q^2)_\infty^4 (-q^{10}; q^{10})_\infty^4. \quad (4.4)$$

It is now readily seen that (4.4) has the following partition-theoretic interpretation.

**Theorem 4.2.** *Let  $\mathcal{S}$  denote the set of partitions into 8 distinct colors with four colors, say orange, blue, Orange, and Blue, each appearing at most once, and the remaining four colors, say red, green, Red, and Green, appearing at most once and only in multiples of 5. Let  $A(N)$  denote the number of partitions of  $2N + 1$  into odd elements of  $\mathcal{S}$ . Let  $B(N)$  denote the number of partitions of  $2N - 2$  into even elements of  $\mathcal{S}$ , with the convention that  $B(1) = 1$ . Then,  $A(0) = 4$ , and for  $N \geq 1$ ,*

$$A(N) = 8B(N). \quad (4.5)$$

**Example 4.3.** *Let  $N = 2$ . Then  $B(2) = 4$  and  $A(2) = 32$ , and*

$$\begin{aligned} 2_o &= 2_b = 2_O = 2_B, \\ 5_o &= 5_b = 5_O = 5_B = 5_r = 5_g = 5_R = 5_G \\ &= 3_o + 1_o + 1_b, 23 \text{ further partitions of the type } 3+1+1 \text{ into colors.} \end{aligned}$$

**Example 4.4.** *Let  $N = 3$ . Then  $B(3) = 10$  and  $A(3) = 80$ . In particular,*

$$\begin{aligned} 4_o &= 4_b = 4_O = 4_B = 2_o + 2_b, 5 \text{ further representations of the form } 2 + 2, \\ 7_o &= 7_b = 7_O = 7_B \\ &= 5_o + 1_o + 1_b, 47 \text{ further representations of the form } 5 + 1 + 1 \\ &= 3_o + 3_b + 1_o, 23 \text{ further representations of the form } 3 + 3 + 1 \\ &= 3_o + 1_o + 1_b + 1_O + 1_B, 3 \text{ further representations of the form } 3 + 1 + 1 + 1 + 1. \end{aligned}$$

## 5. PARTITIONS WITH MULTIPLES OF 7

As mentioned in the Introduction, the modular equation of degree 7 in the next theorem was discovered by Guetzlaff [6] in 1834. H. Schröter [16], in his doctoral dissertation in 1854, and E. Fiedler [5], in 1885, also proved Theorem 5.1. It can furthermore be found in Ramanujan's second notebook [13, Chapter 19, Entry 19(i)], [2, p. 314].

**Theorem 5.1.** *If  $\beta$  has degree 7 over  $\alpha$ , then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \quad (5.1)$$

Multiply both sides of (5.1) by

$$\frac{2^{1/3}q^{1/3}}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}}$$

to write (5.1) in the equivalent form

$$2q \frac{\alpha^{1/12}}{2^{1/3}q^{1/12}(1-\alpha)^{1/24}} \frac{\beta^{1/12}}{2^{1/3}q^{7/12}(1-\beta)^{1/24}} + \frac{2^{1/6}q^{1/24}(1-\alpha)^{1/12}}{\alpha^{1/24}} \frac{2^{1/6}q^{7/24}(1-\beta)^{1/12}}{\beta^{1/24}} = \frac{2^{1/6}q^{1/24}}{\{\alpha(1-\alpha)\}^{1/24}} \frac{2^{1/6}q^{7/24}}{\{\beta(1-\beta)\}^{1/24}}. \quad (5.2)$$

Applying Lemma 2.1 to (5.2), we see that

$$\frac{2q}{\chi(-q^2)\chi(-q^{14})} + \chi(-q)\chi(-q^7) = \chi(q)\chi(q^7). \quad (5.3)$$

Using the definition (2.3) and Euler's identity (2.4), we readily find that (5.3) can be recast in the form

$$(-q; q^2)_\infty (-q^7; q^{14})_\infty - (q; q^2)_\infty (q^7; q^{14})_\infty = 2q(-q^2; q^2)_\infty (-q^{14}; q^{14})_\infty. \quad (5.4)$$

Interpreting (5.4) in terms of partitions, we deduce the following theorem.

**Theorem 5.2.** *Let  $\mathcal{S}$  denote the set of partitions into 2 distinct colors with one color, say orange, appearing at most once, and the remaining color, say blue, appearing at most once and only in multiples of 7. Let  $A(N)$  denote the number of partitions of  $2N+1$  into odd elements of  $\mathcal{S}$ . Let  $B(N)$  denote the number of partitions of  $2N$  into even elements of  $\mathcal{S}$ . Then, for  $N \geq 0$ ,*

$$A(N) = B(N). \quad (5.5)$$

A combinatorial proof of a considerable generalization of Theorem 5.2 has been given by Warnaar [19].

**Example 5.3.** *Let  $N = 7$ . Then  $A(7) = B(7) = 6$ , and*

$$\begin{aligned} 14_o &= 14_b = 12_o + 2_o = 10_o + 4_o = 8_o + 6_o = 8_o + 4_o + 2_o, \\ 15_o &= 11_o + 3_o + 1_o = 9_o + 5_o + 1_o = 7_o + 7_b + 1_o = 7_o + 5_o + 3_o = 7_b + 5_o + 3_o. \end{aligned}$$

**Example 5.4.** *Let  $N = 8$ . Then  $A(8) = B(8) = 7$ , and*

$$\begin{aligned} 16_o &= 14_o + 2_o = 14_b + 2_o = 12_o + 4_o = 10_o + 6_o = 10_o + 4_o + 2_o = 8_o + 6_o + 2_o, \\ 17_o &= 13_o + 3_o + 1_o = 11_o + 5_o + 1_o = 9_o + 7_o + 1_o \\ &= 9_o + 7_b + 1_o = 9_o + 5_o + 3_o = 7_o + 7_b + 3_o. \end{aligned}$$

## 6. PARTITIONS WITH MULTIPLES OF 11

The modular equation in Theorem 6.1 is due to Schröter [17], [18], and was rediscovered by Ramanujan, who recorded it as Entry 7(i) in Chapter 20 of his second notebook [13], [2, p. 363].

**Theorem 6.1.** *If  $\beta$  has degree 11 over  $\alpha$ , then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1. \quad (6.1)$$

Multiply both sides of (6.1) by

$$\frac{2^{2/3}q}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}}$$

to deduce that

$$4q^3 \frac{\alpha^{1/6}}{2^{2/3}q^{1/6}(1-\alpha)^{1/12}} \frac{\beta^{1/6}}{2^{2/3}q^{11/6}(1-\beta)^{1/12}} + \frac{2^{1/3}q^{1/12}(1-\alpha)^{1/6}}{\alpha^{1/12}} \frac{2^{1/3}q^{11/12}(1-\beta)^{1/6}}{\beta^{1/12}} + 4q = \frac{2^{1/3}q^{1/12}}{\{\alpha(1-\alpha)\}^{1/12}} \frac{2^{1/3}q^{11/12}}{\{\beta(1-\beta)\}^{1/12}}. \quad (6.2)$$

By Lemma 2.1, the identity (6.2) can be transformed into the equivalent identity

$$\frac{4q^3}{\chi^2(-q^2)\chi^2(-q^{22})} + \chi^2(-q)\chi^2(-q^{11}) + 4q = \chi^2(q)\chi^2(q^{11}). \quad (6.3)$$

Invoking (2.3) and (2.4) in (6.3) and rearranging, we conclude that

$$(-q; q^2)_\infty^2 (-q^{11}; q^{22})_\infty^2 - (q; q^2)_\infty^2 (q^{11}; q^{22})_\infty^2 = 4q + 4q^3 (-q^2; q^2)_\infty^2 (-q^{22}; q^{22})_\infty^2. \quad (6.4)$$

Transcribing (6.4) into a theorem about partitions, we deduce the following theorem.

**Theorem 6.2.** *Let  $\mathcal{S}$  denote the set of partitions into 4 distinct colors with two colors, say orange and blue, each appearing at most once, and the remaining two colors, say red and green, each appearing at most once and only in multiples of 11. Let  $A(N)$  denote the number of partitions of  $2N + 1$  into odd elements of  $\mathcal{S}$ . Let  $B(N)$  denote the number of partitions of  $2N - 2$  into even elements of  $\mathcal{S}$ , with the convention that  $B(1) = 1$ . Then,  $A(0) = 2$ , and, for  $N \geq 1$ ,*

$$A(N) = 2B(N). \quad (6.5)$$

**Example 6.3.** *Let  $N = 5$ . Then  $B(5) = 9$  and  $A(5) = 18$ , and*

$$\begin{aligned} 8_o &= 8_b = 6_o + 2_o = 6_o + 2_b = 6_b + 2_o = 6_b + 2_b = 4_o + 4_b \\ &= 4_o + 2_o + 2_b = 4_b + 2_o + 2_b, \\ 11_o &= 11_b = 11_r = 11_g = 9_o + 1_o + 1_b = 9_b + 1_o + 1_b = 7_o + 3_o + 1_o = 7_o + 3_o + 1_b \\ &= 7_o + 3_b + 1_o = 7_o + 3_b + 1_b = 7_b + 3_o + 1_o = 7_b + 3_o + 1_b = 7_b + 3_b + 1_o \\ &= 7_b + 3_b + 1_b = 5_o + 5_b + 1_o = 5_o + 5_b + 1_b = 5_o + 3_o + 3_b = 5_b + 3_o + 3_b. \end{aligned}$$

**Example 6.4.** Let  $N = 6$ . Then  $B(6) = 14$  and  $A(6) = 28$ , and

$$\begin{aligned}
10_o &= 10_b = 8_o + 2_b = 8_o + 2_o = 8_b + 2_b = 8_b + 2_o \\
&= 6_b + 4_o = 6_b + 4_b = 6_o + 4_b = 6_o + 4_o = 4_b + 4_o + 2_b = 4_o + 4_b + 2_o \\
&= 6_b + 2_b + 2_o = 6_o + 2_b + 2_o, \\
13_o &= 13_b = 11_o + 1_o + 1_b = 11_b + 1_o + 1_b = 11_r + 1_o + 1_b = 11_g + 1_o + 1_b \\
&= 9_o + 3_o + 1_o, 7 \text{ further representations of the form } 9 + 3 + 1 \\
&= 7_o + 3_o + 3_b = 7_b + 3_o + 3_b \\
&= 7_o + 5_o + 1_o, 7 \text{ further representations of the form } 7 + 5 + 1 \\
&= 5_o + 5_b + 3_o = 5_o + 5_b + 3_b = 5_o + 3_o + 3_b + 1_o + 1_b = 5_b + 3_o + 3_b + 1_o + 1_b.
\end{aligned}$$

## 7. PARTITIONS WITH MULTIPLES OF 23

The next theorem is also due to Schröter [17], [18]. Ramanujan rediscovered this modular equation and recorded it as Entry 15(i) in Chapter 20 of his second notebook [13], [2, p. 411]. According to F. Klein [11], A. Hurwitz also discovered this modular equation, but Klein does not provide a reference for Hurwitz's work.

**Theorem 7.1.** *If  $\beta$  is of degree 23 over  $\alpha$ , then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1. \quad (7.1)$$

We multiply both sides of (7.1) by

$$\frac{2^{1/3}q}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}}$$

and deduce that

$$\begin{aligned}
2q^3 \frac{\alpha^{1/12}}{2^{1/3}q^{1/12}(1-\alpha)^{1/24}} \frac{\beta^{1/12}}{2^{1/3}q^{23/12}(1-\beta)^{1/24}} + \frac{2^{1/6}q^{1/24}(1-\alpha)^{1/12}}{\alpha^{1/24}} \frac{2^{1/6}q^{23/24}(1-\beta)^{1/12}}{\beta^{1/24}} \\
+ 2q = \frac{2^{1/6}q^{1/24}}{\{\alpha(1-\alpha)\}^{1/24}} \frac{2^{1/6}q^{23/24}}{\{\beta(1-\beta)\}^{1/24}}. \quad (7.2)
\end{aligned}$$

Using Lemma 2.1, we write (7.2) in the shape

$$\frac{2q^3}{\chi(-q^2)\chi(-q^{46})} + \chi(-q)\chi(-q^{23}) + 2q = \chi(q)\chi(q^{23}). \quad (7.3)$$

Using (2.3) and (2.4), we transform (7.3) into  $q$ -products, and rearrange to deduce that

$$(-q; q^2)_\infty (-q^{23}; q^{46})_\infty - (q; q^2)_\infty (q^{23}; q^{46})_\infty = 2q + 2q^3 (-q^2; q^2)_\infty (-q^{46}; q^{46})_\infty. \quad (7.4)$$

We now record the partition-theoretic interpretation of (7.4).

**Theorem 7.2.** *Let  $\mathcal{S}$  denote the set of partitions into 2 distinct colors with one color, say orange, appearing at most once, and the remaining color, say blue, appearing at most once and only in multiples of 23. Let  $A(N)$  denote the number of partitions of  $2N + 1$  into odd elements of  $\mathcal{S}$ . Let  $B(N)$  denote the number of partitions of  $2N - 2$  into even elements of  $\mathcal{S}$ , with the convention that  $B(1) = 1$ . Then, for  $N \geq 1$ ,*

$$A(N) = B(N). \quad (7.5)$$

**Example 7.3.** Let  $N = 11$ . Then  $A(11) = B(11) = 10$ , and

$$\begin{aligned} 20_o &= 18_o + 2_o = 16_o + 4_o = 14_o + 6_o = 14_o + 4_o + 2_o = 12_o + 8_o \\ &= 12_o + 6_o + 2_o = 10_o + 8_o + 2_o = 10_o + 6_o + 4_o = 8_o + 6_o + 4_o + 2_o, \\ 23_o &= 23_b = 19_o + 3_o + 1_o = 17_o + 5_o + 1_o = 15_o + 7_o + 1_o = 15_o + 5_o + 3_o \\ &= 13_o + 9_o + 1_o = 13_o + 7_o + 3_o = 11_o + 9_o + 3_o = 11_o + 7_o + 5_o. \end{aligned}$$

**Example 7.4.** Let  $N = 12$ . Then  $A(12) = B(12) = 12$ , and

$$\begin{aligned} 22_o &= 20_o + 2_o = 18_o + 4_o = 16_o + 6_o = 16_o + 4_o + 2_o = 14_o + 8_o = 14_o + 6_o + 2_o \\ &= 12_o + 10_o = 12_o + 8_o + 2_o = 12_o + 6_o + 4_o = 10_o + 8_o + 4_o = 10_o + 6_o + 4_o + 2_o, \\ 25_o &= 21_o + 3_o + 1_o = 19_o + 5_o + 1_o = 17_o + 7_o + 1_o = 17_o + 5_o + 3_o = 15_o + 7_o + 3_o \\ &= 15_o + 9_o + 1_o = 13_o + 11_o + 1_o = 13_o + 9_o + 3_o = 13_o + 7_o + 5_o = 11_o + 9_o + 5_o \\ &= 9_o + 7_o + 5_o + 3_o + 1_o. \end{aligned}$$

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