

RAMANUJAN'S SERIES FOR $1/\pi$ ARISING FROM HIS CUBIC AND QUARTIC THEORIES OF ELLIPTIC FUNCTIONS

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Abstract. Using certain representations for Eisenstein series, we derive several of Ramanujan's series for $1/\pi$ arising from his cubic and quartic theories of elliptic functions.

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1. INTRODUCTION

In his famous paper [16], [17, pp. 36–38], Ramanujan recorded 17 hypergeometric-like series representations for $1/\pi$. Proofs of the first three series representations were briefly sketched by Ramanujan [17, p. 36]. These three series belong to the classical theory of elliptic functions, while the latter fourteen series depend on Ramanujan's alternative theories of elliptic functions in which q is replaced by one or other of the functions

$$q_r := q_r(x) := \exp \left(-\pi \csc(\pi/r) \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)} \right), \quad (1.1)$$

where $r = 3, 4$, or 6 , and where ${}_2F_1$ denotes the classical hypergeometric function defined in Section 2. In the classical theory of elliptic functions, the variable $q = q_2$. Ramanujan recorded 3 series for $1/\pi$ associated to q_2 , 2 series for q_3 , 10 series for q_4 , and 2 series for q_6 . It was not until 1987 that proofs of all 17 series representations for $1/\pi$ were found by J. M. and P. B. Borwein [7]. These authors and several other authors in the past two decades have found several new series for $1/\pi$; references can be found in our paper [1].

In his second notebook [18, pp. 257–262], Ramanujan recorded without proofs his theorems in alternative bases, and these were first proved in 1995 by Berndt, S. Bhargava, and F. G. Garvan [5], who gave these theories the appellation, the theories of signature r ($r = 3, 4, 6$). An account of this work may also be found in Berndt's book [4, Chap. 33]. The cubic and quartic theories were further developed to obtain particular series representations for $1/\pi$ by H. H. Chan, W.-C. Liaw, and V. Tan [11], Berndt, Chan, and Liaw [6], and by Chan and Liaw [10]. Applications of the alternative theories to deriving series for $1/\pi$ have also been made by the Borweins in their

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book [7, pp. 177–191] and by D. V. and G. V. Chudnovsky [12], [13]. Some beautiful conjectured series representations for $1/\pi$ have been made by J. Guillera [14].

In [1], we employed Ramanujan’s ideas expressed in Section 13 of his fundamental paper [16], [17, p. 36] and used them in conjunction with twelve identities for Eisenstein series recorded without proofs in Section 10 of [16], [17, pp. 33–34], but later proved in [3, Chapter 21], and with further identities of this type to prove 13 of Ramanujan’s 17 identities from [16] as well as to establish many new series representations for $1/\pi$. In particular, we rely on Ramanujan’s initial ideas more so than previous authors. However, in that paper, in contrast to Ramanujan’s proposed derivations, “from these [alternative] theories we can deduce further series for $1/\pi$,” we did not appeal to Ramanujan’s alternative theories. In this paper, we appeal to Ramanujan’s cubic and quartic theory of elliptic functions to derive several series for $1/\pi$, including some of Ramanujan and the Borweins.

2. PRELIMINARY DEFINITIONS AND RESULTS

We use the standard shifted or rising factorial notation

$$(a)_0 := 1, \quad (a)_n := a(a+1)(a+2)\cdots(a+n-1), \quad n \geq 1.$$

The hypergeometric functions ${}_pF_{p-1}$, $p \geq 1$, are defined by

$${}_pF_{p-1}(a_1, \dots, a_p; b_1, \dots, b_{p-1}; x) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_{p-1})_n} \frac{x^n}{n!}, \quad |x| < 1.$$

If

$$q = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}\right),$$

then one of the fundamental results in the theory of elliptic functions [3, p. 101, Entry 6] is given by

$$\phi^2(q) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right), \quad (2.1)$$

where here, and for the sequel,

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \quad |q| < 1. \quad (2.2)$$

We also need Ramanujan’s function

$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}, \quad (2.3)$$

where the latter identity is Euler’s famous pentagonal number theorem. Following Ramanujan, define

$$z := z(q) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) = \phi^2(q), \quad (2.4)$$

by (2.1). In the sequel, we often emphasize that x is also a function of q when writing $x = x(q)$.

We now define the three cubic theta functions introduced by J. M. and P. B. Borwein in [8] and further studied by them with F. Garvan in [9]. For $\omega = \exp(2\pi i/3)$, let

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \quad (2.5)$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \quad (2.6)$$

and

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}. \quad (2.7)$$

Ramanujan [18, p. 258] recorded the fundamental inversion formula

$$z(q) := z(3; q) := {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right) = a(q), \quad (2.8)$$

where $q = q_3$ is given by (1.1). This theorem was first proved in print by Berndt, Bhargava, and Garvan [5], [4, p. 99].

In the quartic theory, or the theory of signature 4, $a(q)$, $b(q)$, and $c(q)$ in the cubic theory are replaced by the functions

$$A(q) := \phi^4(q) + 16q\psi^4(q^2), \quad B(q) := \phi^4(q) - 16q\psi^4(q^2), \quad (2.9)$$

and

$$C(q) := 8\sqrt{q}\phi^2(q)\psi^2(q^2), \quad (2.10)$$

where $\phi(q)$ and $\psi(q)$ are defined by (2.2).

In [6], Berndt, Chan, and Liaw proved the inversion formula in the quartic theory

$$z(q) := z(4; q) := {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; x\right) = \sqrt{A(q)}, \quad (2.11)$$

where $q = q_4$ is given by (1.1).

Berndt, Bhargava, and Garvan [5], [4, p. 146, Eq. (9.7)] established the following transfer principle by which formulas in the quartic theory can be derived from formulas in the classical theory.

Lemma 2.1. *Suppose that we have a formula*

$$\Omega(x, q^2, z) = 0 \quad (2.12)$$

in the classical base. Then in the quartic base, we have the identity

$$\Omega\left(\frac{2\sqrt{x(q)}}{1+\sqrt{x(q)}}, q_4, \sqrt{1+\sqrt{x(q)}} z(q)\right) = 0, \quad (2.13)$$

where $q = q_4$ and $x(q) = x(4; q)$ are the variables in the quartic theory.

In establishing series representations for $1/\pi$ in the quartic theory, we employ modular equations in the variables α and β in the classical theory and transfer them into modular equations in the quartic theory. Using (2.13), we summarize the substitutions that we need [4, p. 153]:

$$\alpha \mapsto \frac{2\sqrt{\alpha}}{1+\sqrt{\alpha}}, \quad 1-\alpha \mapsto \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}, \quad \beta \mapsto \frac{2\sqrt{\beta}}{1+\sqrt{\beta}}, \quad 1-\beta \mapsto \frac{1-\sqrt{\beta}}{1+\sqrt{\beta}}, \quad (2.14)$$

$$m \mapsto \sqrt{\frac{1+\sqrt{\alpha}}{1+\sqrt{\beta}}} m(4; q), \quad (2.15)$$

where m and $m(4; q) = m(q)$ are the multipliers in the classical and quartic bases, respectively.

In the sequel, when it is clear that we are working with either the cubic or the quartic theory, then we use the abridged notations $z(q)$ for $z(3; q)$ or $z(4; q)$, respectively, and likewise we use $x(q)$ for $x(3; q)$ or $x(4; q)$, respectively.

3. THE DEVELOPMENT OF RAMANUJAN'S IDEAS

Ramanujan's series representations for $1/\pi$ depend upon Clausen's product formulas for hypergeometric series and Ramanujan's Eisenstein series

$$P(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}, \quad |q| < 1. \quad (3.1)$$

More precisely, but briefly, by combining two different relations between $P(q)$ and $P(q^n)$, for certain positive integers n , along with a Clausen formula, we [1] obtain series representations for $1/\pi$. In this paper, we derive several series arising from Ramanujan's cubic and quartic theories developed in [5], [4], [6], [10], [11], and [15].

4. SERIES FOR $1/\pi$ ARISING FROM RAMANUJAN'S QUARTIC THEORY

We begin with Clausen's formula [7, p. 178, Proposition 5.6(b)]. Let

$$B_k := \frac{\left(\frac{1}{4}\right)_k \left(\frac{1}{2}\right)_k \left(\frac{3}{4}\right)_k}{k!^3} \quad \text{and} \quad H := H(q) := 4x(q)(1-x(q)). \quad (4.1)$$

Then

$$z^2 := z(q)^2 = {}_3F_2 \left(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}; 1, 1; H \right) = \sum_{k=0}^{\infty} B_k H^k(q), \quad 0 \leq x \leq \frac{1}{2}. \quad (4.2)$$

The functions $z := z(q)$ and $P(q)$ are related by [5, Eq. (9.10)], [4, p. 149, Eq. (9.10)]

$$P(q) = 12x(1-x) \frac{dz}{dx} + (1-3x)z^2, \quad (4.3)$$

where $x := x(q)$.

Differentiating (4.2) with respect to $x = x(q)$, employing (4.1), and substituting in (4.3), we find that

$$P(q) = \sum_{k=0}^{\infty} \{6(1-2x)k + 1 - 3x\} B_k H^k(q). \quad (4.4)$$

Now set

$$x_n := x(e^{-2\pi\sqrt{n/2}}) \quad \text{and} \quad z_n := z(e^{-2\pi\sqrt{n/2}}). \quad (4.5)$$

The numbers x_n are *quartic singular moduli*. It can be shown that [6, Eqs. (4.22), (4.24)]

$$1 - x_n = x_{1/n}, \quad z_{1/n} = \sqrt{n}z_n, \quad \text{and} \quad m(x_{1/n}) = \sqrt{n}. \quad (4.6)$$

Setting $q = e^{-2\pi\sqrt{n/2}}$ in (4.4), we deduce that

$$P(e^{-2\pi\sqrt{n/2}}) = \sum_{k=0}^{\infty} \{6(1-2x_n)k + 1 - 3x_n\} B_k H_n^k, \quad (4.7)$$

where $H_n = 4x_n(1-x_n)$.

Next, we recall from [6, Eq. (4.30)] a transformation formula for $P(q)$. If z_n is defined by (4.5), then

$$nP(e^{-2\pi\sqrt{n/2}}) + P(e^{-2\pi/\sqrt{2n}}) = \frac{6\sqrt{2n}}{\pi} - nz_n^2. \quad (4.8)$$

In his paper [16], Ramanujan recorded twelve representations for

$$f_n(q) := nP(q^{2n}) - P(q^2) \quad (4.9)$$

in the classical base for the values $n = 2, 3, 4, 5, 7, 11, 15, 17, 19, 23, 31$, and 35 . He also recorded representations for $n = 2$ and 4 in Chapter 17 and for the remaining ten values and for $n = 9$ and $n = 25$ in Chapter 21 of his second notebook [18]. By Lemma 2.1 we can transfer these into the quartic base. These representations for $q = e^{-2\pi/\sqrt{2n}}$ in conjunction with (4.8) and (4.7) are the primary ingredients in our derivations of series representations for $1/\pi$ in the following sections.

4.1. Example: $n = 2$.

Theorem 4.1. *If B_k , $k \geq 0$, is defined by (4.1), then*

$$\frac{9}{2\pi} = \sum_{k=0}^{\infty} (7k+1) B_k \left(\frac{32}{81}\right)^k. \quad (4.10)$$

The identity (4.10) is due to the Berndt, Chan, and Liaw [6]. Surprisingly, it was not recorded by Ramanujan in his paper [16].

Proof of (4.10). Setting $n = 2$ in (4.7), we obtain

$$P(e^{-2\pi}) = \sum_{k=0}^{\infty} \{6(1-2x_2)k + 1 - 3x_2\} B_k H_2^k, \quad (4.11)$$

where $H_2 = 4x_2(1-x_2)$.

But, it is well known that [2, p. 256]

$$P(e^{-2\pi}) = \frac{3}{\pi}. \quad (4.12)$$

From (4.11) and (4.12), we find that

$$\frac{3}{\pi} = \sum_{k=0}^{\infty} \{6(1 - 2x_2)k + 1 - 3x_2\} B_k H_2^k. \quad (4.13)$$

Now, we need to calculate x_2 . From [3, p. 214, Eq. (24.17)], we recall the following modular equation of degree 2 in classical base. If β has degree 2 over α and m is the corresponding multiplier, then

$$\frac{m^2}{2} = \frac{1 + \beta}{2 - \alpha}. \quad (4.14)$$

With the help of the transfer rules given in (2.14) and (2.15), the modular equation (4.14) in the classical base can be transformed into a modular equation in the quartic base. If β has degree 2 over α and m is the corresponding multiplier in the quartic base, then

$$m^2 = 1 + 3\sqrt{\beta}. \quad (4.15)$$

Setting $q = e^{-2\pi/\sqrt{4}} = e^{-\pi}$ in (4.15), so that $\beta = x_2$ and $m = \sqrt{2}$ by (4.6), we obtain

$$1 + 3\sqrt{x_2} = 2, \quad (4.16)$$

which immediately implies that

$$x_2 = \frac{1}{9}. \quad (4.17)$$

Thus,

$$H_2 = 4x_2(1 - x_2) = \frac{32}{81}. \quad (4.18)$$

Employing (4.17) and (4.18) in (4.13), we arrive at (4.10) to complete the proof. \square

4.2. Example: $n = 3$.

Theorem 4.2. *If B_k , $k \geq 0$, is defined by (4.1), then*

$$\frac{2\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} (8k + 1) B_k \left(\frac{1}{9}\right)^k. \quad (4.19)$$

The identity (4.19) is due to Ramanujan [16].

Proof of (4.19). Setting $n = 3$ in (4.7), we obtain

$$P(e^{-2\pi\sqrt{3/2}}) = \sum_{k=0}^{\infty} \{6(1 - 2x_3)k + 1 - 3x_3\} B_k H_3^k, \quad (4.20)$$

where $H_3 = 4x_3(1 - x_3)$.

To calculate x_3 and hence H_3 , first of all, we record the following modular equation of degree 3 in the quartic base [4, p. 153]. If β has degree 3 over α in the quartic base, then

$$(\alpha\beta)^{1/2} + \{(1 - \alpha)(1 - \beta)\}^{1/2} + 4\{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/2} = 1. \quad (4.21)$$

Setting $q = e^{-2\pi/\sqrt{3\cdot 2}} = e^{-2\pi/\sqrt{6}}$ in (4.21), so that $\alpha = x_{1/3} = 1 - x_3$ and $\beta = x_3$, we deduce that

$$\{x_3(1 - x_3)\}^{1/2} = \frac{1}{6}, \quad (4.22)$$

from which we readily obtain

$$x_3 = \frac{3 - 2\sqrt{2}}{6}, \quad (4.23)$$

and thus

$$H_3 = \frac{1}{9}. \quad (4.24)$$

With the help of (4.23) and (4.24), we can rewrite (4.20) in the form

$$P(e^{-2\pi\sqrt{3/2}}) = \sum_{k=0}^{\infty} \left\{ 4\sqrt{2}k + \frac{2\sqrt{2}-1}{2} \right\} B_k \left(\frac{1}{9} \right)^k. \quad (4.25)$$

Next, in the classical base, from [3, p. 460],

$$3P(q^6) - P(q^2) = \phi^2(q)\phi^2(q^3) \left\{ 1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} \right\}, \quad (4.26)$$

where ϕ is defined in (2.2). With the aid of the transfer rules (2.14) and (2.15) and Lemma 2.1, the identity (4.26) in the classical base can be transformed into the identity in quartic base

$$3P(q^3) - P(q) = \frac{z^2}{m} \left\{ \sqrt{(1+\sqrt{\alpha})(1+\sqrt{\beta})} + 2\sqrt{\sqrt{\alpha\beta}} + \sqrt{(1-\sqrt{\alpha})(1-\sqrt{\beta})} \right\}. \quad (4.27)$$

Setting $q = e^{-2\pi/\sqrt{6}}$ in (4.27), so that $\alpha = x_{1/3} = 1 - x_3$, $\beta = x_3$, and $m = \sqrt{3}$ by (4.6), and then using (4.23), we find that

$$3P(e^{-2\pi\sqrt{3/2}}) - P(e^{-2\pi/\sqrt{6}}) = 3\sqrt{2}z_3^2. \quad (4.28)$$

Setting $n = 3$ in (4.8), we find that

$$3P(e^{-2\pi\sqrt{3/2}}) + P(e^{-2\pi/\sqrt{6}}) = \frac{6\sqrt{6}}{\pi} - 3z_3^2. \quad (4.29)$$

From (4.28) and (4.29), we deduce that

$$P(e^{-2\pi\sqrt{3/2}}) = \frac{\sqrt{6}}{\pi} + \frac{\sqrt{2}-1}{2}z_3^2. \quad (4.30)$$

With the help of (4.2) and (4.24), we can rewrite (4.30) in the form

$$\begin{aligned} P(e^{-2\pi\sqrt{3/2}}) &= \frac{\sqrt{6}}{\pi} + \frac{\sqrt{2}-1}{2} \sum_{k=0}^{\infty} B_k H_3^k \\ &= \frac{\sqrt{6}}{\pi} + \frac{\sqrt{2}-1}{2} \sum_{k=0}^{\infty} B_k \left(\frac{1}{9} \right)^k. \end{aligned} \quad (4.31)$$

From (4.25) and (4.31), we readily arrive at (4.19) to finish the proof. \square

4.3. Example: $n = 4$.

Theorem 4.3. *If B_k , $k \geq 0$, is defined by (4.1), then*

$$\frac{49}{\pi} = \sum_{k=0}^{\infty} \{8(160 - 65\sqrt{2})k + 260 - 144\sqrt{2}\} B_k \left(\frac{32(325\sqrt{2} - 457)}{2401} \right)^k. \quad (4.32)$$

Proof of (4.32). Setting $n = 4$ in (4.7), we obtain

$$P(e^{-2\pi\sqrt{2}}) = \sum_{k=0}^{\infty} \{6(1 - 2x_4)k + 1 - 3x_4\} B_k H_4^k, \quad (4.33)$$

where $H_4 = 4x_4(1 - x_4)$.

To calculate x_4 and H_4 , we recall the following modular equation of degree 4 in the classical base [3, p. 215, Entry 24(iii)]. If β has degree 4 over α and m is the corresponding multiplier in the classical base, then

$$\frac{m}{2} = \frac{1 + \sqrt{\beta}}{1 + \sqrt{1 - \alpha}}. \quad (4.34)$$

With the help of the transfer rules (2.14) and (2.15), the modular equation (4.34) in the classical base can be transformed, after some simplification, into the following modular equation in the quartic base. If β has degree 4 over α and m is the corresponding multiplier in the quartic base, then

$$\frac{m}{2} = \frac{\sqrt{1 + \sqrt{\beta}} + \sqrt{2}\beta^{1/4}}{\sqrt{1 + \sqrt{\alpha}} + \sqrt{1 - \sqrt{\alpha}}}. \quad (4.35)$$

Setting $q = e^{-2\pi/\sqrt{8}}$ in (4.35), so that $\alpha = x_{1/4} = 1 - x_4$, $\beta = x_4$, and $m = 2$ by (4.6), we deduce that

$$\sqrt{1 + \sqrt{1 - x_4}} + \sqrt{1 - \sqrt{1 - x_4}} = \sqrt{1 + \sqrt{x_4}} + \sqrt{2}x_4^{1/4}. \quad (4.36)$$

Squaring both sides of (4.36) and then simplifying, we deduce that

$$1 - \sqrt{x_4} = 2\sqrt{2}x_4^{1/4}\sqrt{1 + \sqrt{x_4}}. \quad (4.37)$$

Squaring both sides of (4.37), we find that

$$1 - 7x_4 = 10\sqrt{x_4}, \quad (4.38)$$

which implies that

$$49x_4^2 - 114x_4 + 1 = 0. \quad (4.39)$$

Since $0 < x_4 < 1$, we conclude that

$$x_4 = \frac{57 - 40\sqrt{2}}{49}. \quad (4.40)$$

We also can deduce that

$$H_4 = 4x_4(1 - x_4) = \frac{32(325\sqrt{2} - 457)}{2401}. \quad (4.41)$$

Next, in the classical base, from [3, p. 127, Entries 13(viii) and (ix)] and the process of duplication [3, p. 125], we find that

$$4P(q^8) - P(q^2) = \frac{3z^2}{4}(2 - \alpha + 2\sqrt{1 - \alpha}). \quad (4.42)$$

With the aid of the transfer rules (2.14) and (2.15) and Lemma 2.1, the identity (4.42) in the classical base can be transformed into the identity in the quartic base

$$4P(q^4) - P(q) = \frac{3}{2}z^2(1 + \sqrt{1 - \alpha}). \quad (4.43)$$

Setting $q = e^{-2\pi/\sqrt{8}}$ in (4.43), so that $\alpha = x_{1/4} = 1 - x_4$, we find that

$$4P(e^{-2\pi\sqrt{2}}) - P(e^{-\pi/\sqrt{2}}) = 6(1 + \sqrt{x_4})z_4^2. \quad (4.44)$$

Setting $n = 4$ in (4.8), we find that

$$4P(e^{-2\pi\sqrt{2}}) + P(e^{-\pi/\sqrt{2}}) = \frac{12\sqrt{2}}{\pi} - 4z_4^2. \quad (4.45)$$

Adding (4.44) and (4.45), and then employing (4.2), we deduce that

$$\begin{aligned} P(e^{-2\pi\sqrt{2}}) &= \frac{\sqrt{3}}{\pi\sqrt{2}} + \frac{1 + 3\sqrt{x_4}}{4}z_4^2 \\ &= \frac{\sqrt{3}}{\pi\sqrt{2}} + \frac{1 + 3\sqrt{x_4}}{4} \sum_{k=0}^{\infty} B_k H_4^k. \end{aligned} \quad (4.46)$$

From (4.33) and (4.46), we arrive at

$$\frac{3}{\pi\sqrt{2}} = \sum_{k=0}^{\infty} \left\{ 6(1 - x_4)k + 1 - 3x_4 - \frac{1 + 3\sqrt{x_4}}{4} \right\} B_k H_4^k. \quad (4.47)$$

Employing (4.40) and (4.41) in (4.47), we readily deduce (4.32). \square

4.4. Example: $n = 5$.

Theorem 4.4. *If B_k , $k \geq 0$, is defined by (4.1), then*

$$\frac{9}{2\pi\sqrt{2}} = \sum_{k=0}^{\infty} (10k + 1)B_k \left(\frac{1}{81}\right)^k. \quad (4.48)$$

The identity (4.48) is due to Ramanujan [16].

Proof of (4.48). Setting $n = 5$ in (4.7), we obtain

$$P(e^{-2\pi\sqrt{5/2}}) = \sum_{k=0}^{\infty} \{6(1 - 2x_5)k + 1 - 3x_5\} B_k H_5^k, \quad (4.49)$$

where $H_5 = 4x_5(1 - x_5)$.

To calculate x_5 and hence H_5 , we recall the following modular equation of degree 5 in the quartic base [4, p. 154]. If β has degree 5 over α in the quartic base, then

$$\begin{aligned} & (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} \\ & + 8\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} \left((\alpha\beta)^{1/6} + \{(1-\alpha)(1-\beta)\}^{1/6} \right) = 1. \end{aligned} \quad (4.50)$$

Setting $q = e^{-2\pi/\sqrt{5^2}} = e^{-2\pi/\sqrt{10}}$ in (4.50), so that $\alpha = x_{1/5} = 1 - x_5$ and $\beta = x_5$, we see that

$$\{x_5(1-x_5)\}^{1/2} = \frac{1}{18}, \quad (4.51)$$

from which we arrive at

$$x_5 = \frac{9-4\sqrt{5}}{18}. \quad (4.52)$$

Therefore,

$$H_5 = \frac{1}{81}. \quad (4.53)$$

With the help of (4.52) and (4.53), we rewrite (4.49) as

$$P(e^{-2\pi\sqrt{5/2}}) = \sum_{k=0}^{\infty} \left\{ \frac{8\sqrt{5}}{3}k + \frac{2\sqrt{5}}{3} - \frac{1}{2} \right\} B_k \left(\frac{1}{81} \right)^k. \quad (4.54)$$

Next, in the classical base, from [3, p. 464, Entry 4(iii)],

$$\begin{aligned} 5P(q^{10}) - P(q^2) &= \phi^2(q)\phi^2(q^5) \left\{ 3 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} \right\} \\ &\times \left\{ \frac{1}{2} \left(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} \right) \right\}^{1/2}, \end{aligned} \quad (4.55)$$

where ϕ is defined in (2.2). With the help of (2.14), (2.15), and Lemma 2.1, the identity (4.55) in the classical base can be transformed into the identity in the quartic base

$$\begin{aligned} 5P(q^5) - P(q) &= \frac{z^2}{m\sqrt{2}} \sqrt{(1+\sqrt{\alpha})(1+\sqrt{\beta})} \\ &\times \left\{ 3 + \frac{2\sqrt{\sqrt{\alpha}\beta}}{\sqrt{(1+\sqrt{\alpha})(1+\sqrt{\beta})}} + \frac{\sqrt{(1-\sqrt{\alpha})(1-\sqrt{\beta})}}{\sqrt{(1+\sqrt{\alpha})(1+\sqrt{\beta})}} \right\} \\ &\times \left\{ 1 + \frac{2\sqrt{\sqrt{\alpha}\beta}}{\sqrt{(1+\sqrt{\alpha})(1+\sqrt{\beta})}} + \frac{\sqrt{(1-\sqrt{\alpha})(1-\sqrt{\beta})}}{\sqrt{(1+\sqrt{\alpha})(1+\sqrt{\beta})}} \right\}^{1/2}. \end{aligned} \quad (4.56)$$

Setting $q = e^{-2\pi/\sqrt{10}}$ in (4.56), so that $\alpha = x_{1/5} = 1 - x_5$, $\beta = x_5$, and $m = \sqrt{5}$ by (4.6), and then using (4.52), we find that

$$5P(e^{-2\pi\sqrt{5/2}}) - P(e^{-2\pi/\sqrt{10}}) = 4\sqrt{5}z_5^2. \quad (4.57)$$

Setting $n = 5$ in (4.8), we find that

$$5P(e^{-2\pi\sqrt{5/2}}) + P(e^{-2\pi/\sqrt{10}}) = \frac{6\sqrt{10}}{\pi} - 5z_5^2. \quad (4.58)$$

From (4.57) and (4.58), we deduce that

$$P(e^{-2\pi\sqrt{5/2}}) = \frac{6}{\pi\sqrt{10}} + \left(\frac{2}{\sqrt{5}} - \frac{1}{2}\right) z_5^2. \quad (4.59)$$

With the aid of (4.2) and (4.53), we can rewrite (4.59) in the form

$$\begin{aligned} P(e^{-2\pi\sqrt{5/2}}) &= \frac{6}{\pi\sqrt{10}} + \left(\frac{2}{\sqrt{5}} - \frac{1}{2}\right) \sum_{k=0}^{\infty} B_k H_5^k \\ &= \frac{6}{\pi\sqrt{10}} + \left(\frac{2}{\sqrt{5}} - \frac{1}{2}\right) \sum_{k=0}^{\infty} B_k \left(\frac{1}{81}\right)^k. \end{aligned} \quad (4.60)$$

From (4.54) and (4.60), we readily deduce (4.48) to complete the proof. \square

4.5. Example: $n = 7$.

Theorem 4.5. *If B_k , $k \geq 0$, is defined by (4.1), then*

$$\frac{1}{\pi\sqrt{14}\sqrt{-50 + 44\sqrt{2}}} = \sum_{k=0}^{\infty} \left\{ \frac{2}{7}k + \frac{3(3 - \sqrt{2})}{196} \right\} B_k \left(\frac{1}{11 + 8\sqrt{2}} \right)^{2k}. \quad (4.61)$$

The identity (4.61) is due to Berndt, Chan, and Liaw [6].

Proof of (4.61). We use the quartic modular equation of degree 7 from [4, p. 155, Theorem 10.3] to evaluate x_7 and use Entry 5(iii) of [3, p. 468] to evaluate the representation for $7P(e^{-2\pi\sqrt{7/2}}) - P(e^{-2\pi/\sqrt{14}})$. The remainder of the proof follows along the same lines as that in the previous section, and so we omit the detailed proof. \square

4.6. Example: $n = 9$.

Theorem 4.6. *If B_k , $k \geq 0$, is defined by (4.1), then*

$$\frac{1}{3\pi\sqrt{3}} = \sum_{k=0}^{\infty} (40k + 3) B_k \left(\frac{1}{49} \right)^{2k+1}. \quad (4.62)$$

The identity (4.62) is due to Ramanujan [16].

Proof of (4.62). The proof of (4.62) follows along the same lines as those in Sections 4.2–4.5, and so we mention only two notable points. We use the quartic modular equation of degree 9 from [4, p. 158, Theorem 10.7] to evaluate x_9 , and we employ Entry 10(ii) in [3, p. 482] to evaluate the representation for $9P(e^{-2\pi\sqrt{9/2}}) - P(e^{-2\pi/\sqrt{18}})$. We forego additional details. \square

4.7. Example: $n = 11$.

Theorem 4.7. *If B_k , $k \geq 0$, is defined by (4.1), then*

$$\frac{2}{\pi\sqrt{11}} = \sum_{k=0}^{\infty} (280k + 19) B_k \left(\frac{1}{99} \right)^{2k+1}. \quad (4.63)$$

The identity (4.63) is also due to Ramanujan [16].

Proof of (4.63). Our proof of the identity above also follows along the same lines as those in Sections 4.2–4.5. In this case, we use the quartic modular equation of degree 11 from [4, p. 155, Theorem 10.4] to evaluate x_{11} and use Entry 8(ii) in [3, p. 480] to evaluate the representation for $11P(e^{-2\pi\sqrt{11/2}}) - P(e^{-2\pi/\sqrt{22}})$. We omit further details of the proof. \square

5. SERIES FOR $1/\pi$ ARISING FROM RAMANUJAN'S CUBIC THEORY

As in Section 4, we begin with Clausen's formula [7, pp. 178, Proposition 5.6(b)]. Let

$$C_k := \frac{\left(\frac{1}{3}\right)_k \left(\frac{1}{2}\right)_k \left(\frac{2}{3}\right)_k}{k!^3} \quad \text{and} \quad L := 4x(3)(1 - x(3)). \quad (5.1)$$

If $z := z(3; q)$ is defined by (2.8), then

$$z^2 = {}_3F_2\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}; 1, 1; L\right) = \sum_{k=0}^{\infty} C_k L^k, \quad 0 \leq x \leq \frac{1}{2}. \quad (5.2)$$

Let $q = q_3$ be defined by (1.1), then the functions z and $P(q)$ are related by [4, p. 105, Lemma 4.1]

$$P(q) = 12x(1 - x) \frac{dz}{dx} + (1 - 4x)z^2 \quad (5.3)$$

where $x := x(4; q)$.

Differentiating (5.2) with respect to $x = x(q)$, employing (5.1), and substituting in (5.3), Chan, Liaw, and Tan [11, Theorem 2.7] showed that

$$P(q) = \sum_{k=0}^{\infty} \{6(1 - 2x)k + 1 - 4x\} C_k L^k(q). \quad (5.4)$$

Now set

$$x_n := x(e^{-2\pi\sqrt{n/3}}) \quad \text{and} \quad z_n := z(e^{-2\pi\sqrt{n/3}}). \quad (5.5)$$

The numbers x_n are *cubic singular moduli*. Chan, Liaw, and Tan also showed that [11, Eqs. (3.11) and (3.7)]

$$1 - x_n = x_{1/n}, \quad z_{1/n} = \sqrt{n}z_n, \quad \text{and} \quad m(x_{1/n}) = \sqrt{n}, \quad (5.6)$$

where $m(x(q)) = m(x(q), x(q^n))$ is the multiplier connecting $x(q)$ and $x(q^n)$, defined by

$$m = \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x(q)\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x(q^n)\right)}. \quad (5.7)$$

Furthermore, setting $q = e^{-2\pi\sqrt{n/3}}$ in (5.4), they arrive at

$$P(e^{-2\pi\sqrt{n/3}}) = \sum_{k=0}^{\infty} \{6(1 - 2x_n)k + 1 - 4x_n\} C_k L_n^k, \quad (5.8)$$

where $L_n = 4x_n(1 - x_n)$.

Next, we recall two further identities from [11, Eqs. (3.12) and (3.17)], namely,

$${}_n P(e^{-2\pi\sqrt{n/3}}) - P(e^{-2\pi/\sqrt{3n}}) = 4z_n^2 \{n(1-2x_n) - 3\sqrt{n}x_n(1-x_n)\} \frac{dm}{dx}(1-x_n, x_n) \quad (5.9)$$

and

$${}_n P(e^{-2\pi\sqrt{n/3}}) + P(e^{-2\pi/\sqrt{2n}}) = \frac{6\sqrt{3n}}{\pi} - 2nz_n^2. \quad (5.10)$$

For certain values of n in (5.8), (5.9), and (5.10), we can obtain several series for $1/\pi$. We demonstrate this by giving some examples in the remaining sub-sections.

5.1. Example: $n = 2$.

Theorem 5.1. *If C_k , $k \geq 0$, is defined by (5.1), then*

$$\frac{3\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} (6k+1)C_k \frac{1}{2^k}. \quad (5.11)$$

The identity (5.11) is due to the Borwein brothers [7].

Proof of (5.11). Setting $n = 2$ in (5.8), we obtain

$$P(e^{-2\pi\sqrt{2/3}}) = \sum_{k=0}^{\infty} \{6(1-2x_2)k + 1 - 4x_2\} C_k L_2^k, \quad (5.12)$$

where $L_2 = 4x_2(1-x_2)$.

To calculate x_2 and hence L_2 , we recall the following cubic modular equation of degree 2 [4, p. 120, Theorem 7.1(i)]. If β has degree 2 over α in the theory of signature 3, then

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} = 1. \quad (5.13)$$

Setting $q = e^{-2\pi/\sqrt{3 \cdot 2}} = e^{-2\pi/\sqrt{6}}$ in (5.13), so that $\alpha = x_{1/2} = 1 - x_2$ and $\beta = x_2$, we obtain

$$\{x_2(1-x_2)\}^{1/3} = \frac{1}{2}, \quad (5.14)$$

from which we readily arrive at

$$x_2 = \frac{\sqrt{2}-1}{2\sqrt{2}} \quad \text{and} \quad L_2 = \frac{1}{2}. \quad (5.15)$$

Employing (5.15) in (5.12), we find that

$$P(e^{-2\pi\sqrt{2/3}}) = \sum_{k=0}^{\infty} (3\sqrt{2}k + \sqrt{2} - 1) C_k \frac{1}{2^k}. \quad (5.16)$$

Next, setting $n = 2$ in (5.9), we obtain

$$2P(e^{-2\pi\sqrt{2/3}}) - P(e^{-2\pi/\sqrt{6}}) = 4z_2^2 \{2(1-2x_2) - 3\sqrt{2}x_2(1-x_2)\} \frac{dm}{dx}(1-x_2, x_2). \quad (5.17)$$

To evaluate $\frac{dm}{dx}(1-x_2, x_2)$ we recall from Theorem 7.1(iii) in [4, p. 120] that

$$m = \frac{(1-x(q^2))^{2/3}}{(1-x(q))^{1/3}} - \frac{x^{2/3}(q^2)}{x^{1/3}(q)}, \quad (5.18)$$

where m is the multiplier connecting $x(q)$ and $x(q^2)$. Differentiating (5.18) with respect to $x := x(q)$, we find that

$$\begin{aligned} \frac{dm}{dx} &= \frac{-\frac{2}{3} \left(\frac{1-x}{1-x(q^2)} \right)^{1/3} \frac{dx(q^2)}{dx} + \frac{1}{3} \left(\frac{1-x(q^2)}{1-x} \right)^{2/3}}{(1-x)^{2/3}} \\ &\quad - \frac{\frac{2}{3} \left(\frac{x}{x(q^2)} \right)^{1/3} \frac{dx(q^2)}{dx} - \frac{1}{3} \left(\frac{x(q^2)}{x} \right)^{2/3}}{x^{2/3}}. \end{aligned} \quad (5.19)$$

But, by Theorem 2.3 of [10],

$$\frac{dx(q^n)}{dx(q)} = \frac{n}{m^2} \cdot \frac{x(q^n)(1-x(q^n))}{x(q)(1-x(q))}. \quad (5.20)$$

In particular, if $n = 2$, then

$$\frac{dx(q^2)}{dx} = \frac{2}{m^2} \cdot \frac{x(q^2)(1-x(q^2))}{x(1-x)}. \quad (5.21)$$

Setting $q = e^{-2\pi/\sqrt{6}}$ in (5.21), so that $x = x_{1/2} = 1 - x_2$, $x(q^2) = x_2$, and $m = \sqrt{2}$ by (5.6), we deduce that

$$\left[\frac{dx(q^2)}{dx} \right]_{q=e^{-2\pi/\sqrt{6}}} = 1. \quad (5.22)$$

Therefore, setting $q = e^{-2\pi/\sqrt{6}}$ in (5.19) and then using (5.22), we find that

$$\left[\frac{dm}{dx} \right]_{q=e^{-2\pi/\sqrt{6}}} = \frac{4}{3}. \quad (5.23)$$

Employing (5.23) and (5.15) in (5.17), we deduce that

$$2P(e^{-2\pi\sqrt{2/3}}) - P(e^{-2\pi/\sqrt{6}}) = 2\sqrt{2}z_2^2. \quad (5.24)$$

With the help of (5.2) and (5.15), we can rewrite (5.24) in the form

$$2P(e^{-2\pi\sqrt{2/3}}) - P(e^{-2\pi/\sqrt{6}}) = 2\sqrt{2} \sum_{k=0}^{\infty} C_k \frac{1}{2^k}. \quad (5.25)$$

Setting $n = 2$ in (5.10) and then invoking (5.15) and (5.2), we find that

$$2P(e^{-2\pi\sqrt{2/3}}) + P(e^{-2\pi/\sqrt{6}}) = \frac{6\sqrt{6}}{\pi} - 4z_2^2 = \frac{6\sqrt{6}}{\pi} - 4 \sum_{k=0}^{\infty} C_k \frac{1}{2^k}. \quad (5.26)$$

Adding (5.25) and (5.26), we deduce that

$$P(e^{-2\pi\sqrt{2/3}}) = \frac{3\sqrt{3}}{2\pi\sqrt{2}} + \frac{1-\sqrt{2}}{\sqrt{2}} \sum_{k=0}^{\infty} C_k \frac{1}{2^k}. \quad (5.27)$$

From (5.16) and (5.27), we readily arrive at (5.11) to complete the proof. \square

5.2. Example: $n = 3$.

Theorem 5.2. *If C_k , $k \geq 0$, is defined by (5.1), then*

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \{(7 - 3\sqrt{3})k + 2 - \sqrt{3}\} C_k \left(\frac{3\sqrt{3}(2 - \sqrt{3})^2}{2} \right)^k. \quad (5.28)$$

The identity (5.28) is also due to the Borwein brothers [7].

Proof of (5.28). Setting $n = 3$ in (5.8), we find that

$$P(e^{-2\pi}) = \sum_{k=0}^{\infty} \{6(1 - 2x_3)k + 1 - 4x_3\} C_k L_3^k, \quad (5.29)$$

where $L_3 = 4x_3(1 - x_3)$. Employing (4.12) in (5.29), we find that

$$\frac{3}{\pi} = \sum_{k=0}^{\infty} \{6(1 - 2x_3)k + 1 - 4x_3\} C_k L_3^k. \quad (5.30)$$

To evaluate x_3 we recall from [4, p. 123, Lemma 7.4] that

$$m = 1 + 2x^{1/3}(q^3), \quad (5.31)$$

where m is the multiplier connecting $x(q)$ and $x(q^3)$. Setting $q = e^{-2\pi/3}$ in (5.31), so that $x(q^3) = x_3$ and $m = \sqrt{3}$ by (5.6), we deduce that

$$x_3 = \frac{3\sqrt{3} - 5}{4}. \quad (5.32)$$

Therefore,

$$L_3 = 4x_3(1 - x_3) = \frac{3\sqrt{3}(2 - \sqrt{3})^2}{2}. \quad (5.33)$$

Employing (5.32) and (5.33) in (5.30), we readily arrive at (5.28). \square

5.3. Example: $n = 5$.

Theorem 5.3. *If C_k , $k \geq 0$, is defined by (5.1), then*

$$\frac{15\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} (33k + 4) C_k \left(\frac{4}{125} \right)^k. \quad (5.34)$$

The identity (5.34) is due to Ramanujan [16].

Proof of (5.34). Setting $n = 5$ in (4.7), we see that

$$P(e^{-2\pi\sqrt{5/3}}) = \sum_{k=0}^{\infty} \{6(1 - 2x_5)k + 1 - 4x_5\} C_k L_5^k, \quad (5.35)$$

where $L_5 = 4x_5(1 - x_5)$.

To calculate x_5 and L_5 , we first recall from [4, p. 153] the following modular equation of degree 5 in the theory of signature 3. If β has degree 5 over α in the cubic base, then

$$(\alpha\beta)^{1/3} + \{(1 - \alpha)(1 - \beta)\}^{1/3} + 3\{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/6} = 1. \quad (5.36)$$

Setting $q = e^{-2\pi/\sqrt{5\cdot 3}} = e^{-2\pi/\sqrt{15}}$ in (5.36), so that $\alpha = x_{1/5} = 1 - x_5$ and $\beta = x_5$, we find that

$$\{x_5(1 - x_5)\}^{1/3} = \frac{1}{5}, \quad (5.37)$$

from which we readily obtain

$$x_5 = \frac{5\sqrt{5} - 11}{10\sqrt{5}} \quad \text{and} \quad L_5 = \frac{4}{125}. \quad (5.38)$$

Employing (5.38) in (5.35), we deduce that

$$P(e^{-2\pi\sqrt{5/3}}) = \sum_{k=0}^{\infty} \left\{ \frac{66k + 22 - 5\sqrt{5}}{5\sqrt{5}} \right\} C_k \left(\frac{4}{125} \right)^k. \quad (5.39)$$

Next, Z.-G. Liu [15, Eq. (1.14)] proved that

$$5P(q^5) - P(q) = 4\{a(q)a(q^5) + 2c(q)c(q^5)\}, \quad (5.40)$$

where $q = q_3$ is defined by (1.1) and $a(q)$ and $c(q)$ are defined in (2.5) and (2.7).

Now, from Corollary 3.2 in [4, p. 102], we note that

$$b(q) = x^{1/3}(q)z(q). \quad (5.41)$$

Employing (2.8) and (5.41) in (5.40), we find that

$$5P(q^5) - P(q) = 4z(q)z(q^5)\{1 + 2(x(q)x(q^5))^{1/3}\}. \quad (5.42)$$

Setting $q = e^{-2\pi/\sqrt{15}}$ in (5.42), so that $x(q) = x_{1/5} = 1 - x_5$, $x(q^5) = x_5$, and $m = \sqrt{5}$ by (5.6), and then using (5.38), we deduce that

$$5P(e^{-2\pi\sqrt{5/3}}) - P(e^{-2\pi/\sqrt{15}}) = \frac{28}{\sqrt{5}}z_5^2. \quad (5.43)$$

Setting $n = 5$ in (5.10), we deduce that

$$5P(e^{-2\pi\sqrt{5/3}}) + P(e^{-2\pi/\sqrt{15}}) = \frac{6\sqrt{15}}{\pi} - 10z_5^2. \quad (5.44)$$

Adding (5.43) and (5.44), we arrive at

$$P(e^{-2\pi\sqrt{5/3}}) = \frac{63\sqrt{3}}{\pi} + \left(\frac{14}{5\sqrt{5}} - 1 \right) z_5^2. \quad (5.45)$$

With the help of (5.2) and (5.38), we can rewrite (5.45) in the form

$$\begin{aligned} P(e^{-2\pi\sqrt{5/3}}) &= \frac{63\sqrt{3}}{\pi} + \left(\frac{14}{5\sqrt{5}} - 1 \right) \sum_{k=0}^{\infty} C_k L_5^k \\ &= \frac{63\sqrt{3}}{\pi} + \left(\frac{14}{5\sqrt{5}} - 1 \right) \sum_{k=0}^{\infty} C_k \left(\frac{4}{125} \right)^k. \end{aligned} \quad (5.46)$$

From (5.39) and (5.46), we can easily deduce (5.34) to finish the proof. \square

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