EISENSTEIN SERIES AND RAMANUJAN-TYPE SERIES FOR $1/\pi$

NAYANDEEP DEKA BARUAH[1] and BRUCE C. BERNDT[2]

Abstract. Using certain representations for Eisenstein series, we uniformly derive several Ramanujan-type series for $1/\pi$.

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1. Introduction

In his famous paper [37], [38, pp. 36–38], Ramanujan recorded 17 hypergeometric-like series representations for $1/\pi$. Proofs of the first three series representations were briefly sketched by Ramanujan [38, p. 36]. These three series belong to the classical theory of elliptic functions, while the latter fourteen series depend on Ramanujan's alternative theories of elliptic functions. The mathematical community seems to have forgotten that in 1928 S. Chowla [25], [26], [27, pp. 87–91, 116–119] gave the first published proof of a general series representation for $1/\pi$ and used it to derive the first of Ramanujan's series for $1/\pi$ [37, Eq. (28)]. It was not until 1987 that proofs of all 17 formulas were found by J. M. and P. B. Borwein [14]. These authors subsequently discovered many further series for $1/\pi$ [15], [16], [17], [18], [19], where [17] is coauthored with D. H. Bailey. D. V. Chudnovsky and G. V. Chudnovsky [28] independently proved several of Ramanujan's series representations for $1/\pi$ and established new ones as well. Further particular series representations for $1/\pi$ as well as some general formulas have subsequently been derived by Berndt and H. H. Chan [10], Berndt, Chan, and W.–C. Liaw [11], H. H. Chan, S. H. Chan, and Z. Liu [20], H. H. Chan and Liaw [21], H. H. Chan and K. P. Loo [23], H. H. Chan, Liaw, and V. Tan [22], and H. H. Chan and H. Verrill [24]. J. Guillera [29]–[33] discovered some beautiful series for $1/\pi$ as well as for $1/\pi^2$. Further work has been accomplished by W. Zudilin [41]–[43].

The purpose of this paper is to return to Ramanujan's ideas expressed in Section 13 of his fundamental paper [37], [38, p. 36] and use them in conjunction with twelve identities for Eisenstein series recorded without proofs in Section 10 of [37], [38, pp. 33–34] and with further identities of this type to reprove 13 of Ramanujan's 17 identities from [37] as well as to establish many new series representations for $1/\pi$. In particular, we rely on Ramanujan's initial ideas more so than previous authors. For example, the Borweins employ Legendre's relation between elliptic integrals; in our derivations, we do not need knowledge of elliptic integrals. However, in contrast to Ramanujan's

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proposed derivations, "from these [alternative] theories we can deduce further series for $1/\pi$," *we do not appeal to Ramanujan's alternative theories in this paper.* In another paper [2], we employ Ramanujan's ideas once again, but now with his alternative theories, to derive several new series representations for $1/\pi$.

The formulas from Section 10 to which we alluded above are also found in Ramanujan's second notebook [39] and were first proved by the second author in [6, Chapter 21]. Berndt's proofs of some of the formulas follow a hint given by Ramanujan at the beginning of Chapter 21, but unfortunately Berndt was not able to use Ramanujan's idea, or any idea with which Ramanujan might have been familiar, to prove most of the identities. Thus, Berndt resorted to the theory of modular forms to prove most of Ramanujan's formulas. It would be of enormous interest to discover how Ramanujan might have proved all twelve identities.

The authors are extremely indebted to Heng Huat Chan who provided this direction for us. At the request of the second author, Chan provided notes more fully developing Ramanujan's ideas in Section 13 of [37] and used them to give complete proofs of Ramanujan's formulas (28)–(30) in [37], [38, pp. 36–37]. These three series representations are, in fact, published with Ramanujan's lost notebook [40] but are clearly *not* from the last year of Ramanujan's life, but instead are from his earlier days in India or more likely from his first year at Cambridge.

2. Preliminary Definitions and Results

We use the standard shifted or rising factorial notation

$$(a)_0 := 1, \qquad (a)_n := a(a+1)(a+2)\cdots(a+n-1), \qquad n \geq 1.$$

The hypergeometric functions ${}_pF_{p-1}$, $p \geq 1$, are defined by

$${}_pF_{p-1}(a_1, \ldots, a_p; b_1, \ldots, b_{p-1}; x) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_{p-1})_n} \frac{x^n}{n!}, \qquad |x| < 1.$$

If

$$q = \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right),$$

then one of the fundamental results in the theory of elliptic functions [6, p. 101, Entry 6] is given by

$$\phi^2(q) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right), \tag{2.1}$$

where here, and for the sequel,

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \qquad \text{and} \qquad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \qquad |q| < 1. \tag{2.2}$$

We also need Ramanujan's function

$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{2.3}$$

where the latter identity is Euler's pentagonal number theorem, which is easily derived from Jacobi's triple product identity. Following Ramanujan, define

$$z := z(q) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) = \phi^2(q). \tag{2.4}$$

In the sequel, we often emphasize that x is also a function of q when writing $x = x(q)$.

The identity (2.1) enables one to derive formulas for Ramanujan's functions ϕ, ψ, and f at different arguments in terms of x, q, and z. In particular, Ramanujan recorded the following identities in his second notebook [39], [6, pp. 122–124].

Lemma 2.1. *We have*

$$\phi(q) = \sqrt{z}, \tag{2.5}$$
$$\phi(-q^2) = \sqrt{z}(1-x)^{1/8}, \tag{2.6}$$
$$\phi(q^2) = \sqrt{z}\left(\tfrac{1}{2}(1+\sqrt{1-x})\right)^{1/2}, \tag{2.7}$$
$$\psi(q) = \sqrt{\frac{z}{2}}\left(\frac{x}{q}\right)^{1/8}, \tag{2.8}$$
$$\psi(q^2) = \frac{\sqrt{z}}{2}\left(\frac{x}{q}\right)^{1/4}, \tag{2.9}$$
$$f(-q^2) = \frac{\sqrt{z}\{x(1-x)\}^{1/12}}{2^{1/3}q^{1/12}}. \tag{2.10}$$

We lastly define a modular equation as understood by Ramanujan [35, p. 214]. Suppose that the equality

$$n\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-k^2\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\ell^2\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \ell^2\right)} \tag{2.11}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and ℓ that is implied by (2.11). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α.

3. The Development of Ramanujan's Ideas

Ramanujan's series representations for $1/\pi$ depend upon Clausen's product formulas for hypergeometric series and Ramanujan's Eisenstein series

$$P(q) := 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \qquad |q| < 1. \tag{3.1}$$

More precisely, but briefly, by combining two different relations between $P(q)$ and $P(q^n)$, for certain positive integers n, along with a Clausen formula, we can obtain series representations for $1/\pi$.

As with the Borweins in their proofs, we begin with Clausen's formulas. We quote Theorems 5.7(i)–(vi) and Formula (5.5.9) in [14, pp. 180–181]. Let

$$A_k := \frac{\left(\frac{1}{2}\right)_k^3}{k!^3}, \quad B_k := \frac{\left(\frac{1}{4}\right)_k \left(\frac{1}{2}\right)_k \left(\frac{3}{4}\right)_k}{k!^3}, \quad C_k := \frac{\left(\frac{1}{6}\right)_k \left(\frac{1}{2}\right)_k \left(\frac{5}{6}\right)_k}{k!^3}, \tag{3.2}$$

and

$$X := 4x(1-x), \quad Y := \frac{4x}{(1-x)^2}, \quad U := \frac{x^2}{4(1-x)}, \quad V := \frac{4\sqrt{x}(1-x)}{(1+x)^2},$$

$$W := \frac{2\sqrt{X}}{1-X}, \quad L := \frac{27X^2}{(4-X)^3}, \quad \text{and} \quad M := \frac{27X}{(1-4X)^3}.$$

Then

$$z^2 = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; X\right) = \sum_{k=0}^{\infty} A_k X^k, \quad 0 \le x \le \tfrac{1}{2}, \tag{3.3}$$

$$= \frac{1}{1-x} \quad {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; -Y\right) = \frac{1}{1-x} \quad \sum_{k=0}^{\infty} (-1)^k A_k Y^k, \quad 0 \le x \le 3 - 2\sqrt{2}, \tag{3.4}$$

$$= \frac{1}{\sqrt{1-x}} \quad {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; -U\right) = \frac{1}{\sqrt{1-x}} \quad \sum_{k=0}^{\infty} (-1)^k A_k U^k, \ 0 \le x \le 2\sqrt{2} - 2, \tag{3.5}$$

$$= \frac{1}{1+x} \quad {}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; V^2\right) = \frac{1}{1+x} \quad \sum_{k=0}^{\infty} B_k V^{2k}, \quad 0 \le x \le 3 - 2\sqrt{2}, \tag{3.6}$$

$$= \frac{1}{1-2x} \quad {}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; -W^2\right) = \frac{1}{1-2x} \quad \sum_{k=0}^{\infty} (-1)^k B_k W^{2k}, \tag{3.7}$$

$$0 \le x \le \tfrac{1}{2}\left(1 - 2^{1/4}\sqrt{2-\sqrt{2}}\right),$$

$$= \frac{2}{\sqrt{4-X}} \quad {}_3F_2\left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; L\right) = \frac{2}{\sqrt{4-X}} \quad \sum_{k=0}^{\infty} C_k L^k, \quad 0 \le x \le \tfrac{1}{2}, \tag{3.8}$$

$$= \frac{1}{\sqrt{1-4X}} \quad {}_3F_2\left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; -M\right) = \frac{1}{\sqrt{1-4X}} \quad \sum_{k=0}^{\infty} (-1)^k C_k M^k, \ 0 \le x \le \tfrac{1}{2}. \tag{3.9}$$

Now, differentiating (3.3) with respect to x, we find that

$$2z\frac{dz}{dx} = \sum_{k=0}^{\infty} A_k k X^{k-1} \cdot 4(1-2x). \tag{3.10}$$

Next, recall the representation for $P(q^2)$ given in [6, p. 120, Entry 9(iv)], namely,

$$P(q^2) = (1-2x)z^2 + 6x(1-x)z\frac{dz}{dx}. \tag{3.11}$$

Employing (3.3) and (3.10) in (3.11), we find that

$$P(q^2) = (1-2x)\sum_{k=0}^{\infty}(3k+1)A_k X^k. \tag{3.12}$$

Now set

$$x_n := x(e^{-\pi\sqrt{n}}) \quad \text{and} \quad z_n := z(e^{-\pi\sqrt{n}}). \tag{3.13}$$

The numbers x_n are *singular moduli*. In his notebooks [39], Ramanujan calculated the values of many singular moduli, and in the sequel, we frequently appeal to Ramanujan's values, as recorded and proved in [8]. It also can be easily shown that [1, Chapter 15]

$$1 - x_n = x_{1/n} \quad \text{and} \quad z_{1/n} = \sqrt{n}z_n. \tag{3.14}$$

Setting $q = e^{-\pi\sqrt{n}}$ in (3.12), we deduce that

$$P(e^{-2\pi\sqrt{n}}) = (1-2x_n)\sum_{k=0}^{\infty}(3k+1)A_k X_n^k, \tag{3.15}$$

where $X_n = 4x_n(1-x_n)$.

Similarly, differentiating each of (3.4)–(3.9) with respect to x, and proceeding as above, we can find that

$$P(e^{-2\pi\sqrt{n}}) = \frac{1+x_n}{1-x_n}\sum_{k=0}^{\infty}(3k+1)(-1)^k A_k Y_n^k, \tag{3.16}$$

$$= \frac{2-x_n}{2\sqrt{1-x_n}}\sum_{k=0}^{\infty}(6k+1)(-1)^k A_k U_n^k, \tag{3.17}$$

$$= \sum_{k=0}^{\infty}\frac{3k(1-6x_n+x_n^2)+1-4x_n+x_n^2}{1+X_n}B_k V_n^{2k}, \tag{3.18}$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{3k(1+X_n)+1+X_n/2}{1-X_n}B_k W_n^{2k}, \tag{3.19}$$

$$= \frac{\sqrt{1-X_n}(X_n+8)}{(4-X_n)^{3/2}}\sum_{k=0}^{\infty}(6k+1)C_k L_n^k, \tag{3.20}$$

$$= \frac{\sqrt{1-X_n}}{(1-4X_n)^{3/2}}\sum_{k=0}^{\infty}(-1)^k\{3(1+8X_n)k+1+2X_n\}C_k M_n^k, \tag{3.21}$$

where

$$Y_n := \frac{4x_n}{(1-x_n)^2}, \quad U_n := \frac{x_n^2}{4(1-x_n)}, \quad V_n := \frac{4\sqrt{x_n}(1-x_n)}{(1+x_n)^2},$$

$$W_n := \frac{2\sqrt{X_n}}{1-X_n}, \quad L_n := \frac{27X_n^2}{(4-X_n)^3}, \quad \text{and} \quad M_n := \frac{27X_n}{(1-4X_n)^3}.$$

Next, we determine a transformation formula for $P(q)$. Recall the transformation formula for Ramanujan's function $f(-q)$ [6, p. 43, Entry 27(iii)]. If α, $\beta > 1$ with $\alpha\beta = \pi^2$, then

$$e^{-\alpha/12}\alpha^{1/4}f(-e^{-2\alpha}) = e^{-\beta/12}\beta^{1/4}f(-e^{-2\beta}). \tag{3.22}$$

Taking the logarithm of both sides of (3.22), we obtain

$$-\frac{\alpha}{12}+\frac{1}{4}\log\alpha+\sum_{k=1}^{\infty}\log\left(1-e^{-2k\alpha}\right)=-\frac{\beta}{12}+\frac{1}{4}\log\beta+\sum_{k=1}^{\infty}\log\left(1-e^{-2k\beta}\right). \qquad (3.23)$$

Differentiating both sides of (3.23) with respect to α, we find that

$$-\frac{1}{12}+\frac{1}{4\alpha}+\sum_{k=1}^{\infty}\frac{2ke^{-2k\alpha}}{1-e^{-2k\alpha}}=\frac{\beta}{12\alpha}-\frac{1}{4\alpha}-\sum_{k=1}^{\infty}\frac{(2k\beta/\alpha)e^{-2k\beta}}{1-e^{-2k\beta}}. \qquad (3.24)$$

Multiplying both sides of (3.24) by 12α, rearranging, and then employing the definition of $P(q)$ from (3.1), we deduce that

$$6-\alpha P(e^{-2\alpha})=\beta P(e^{-2\beta}). \qquad (3.25)$$

Setting $\alpha=\pi/\sqrt{n}$, so that $\beta=\pi\sqrt{n}$, in (3.25), we arrive at

$$n\,P(e^{-2\pi\sqrt{n}})+P(e^{-2\pi/\sqrt{n}})=\frac{6\sqrt{n}}{\pi}. \qquad (3.26)$$

If we set $n=1$ in (3.26), we deduce that

$$P(e^{-2\pi})=\frac{3}{\pi}, \qquad (3.27)$$

which has been established many, many times in the literature, including by Ramanujan in Section 8 of Chapter 14 of his second notebook [39]. See [5, p. 256] for references to several proofs.

In his paper [37], Ramanujan recorded twelve representations for

$$f_n(q):=nP(q^{2n})-P(q^2), \qquad (3.28)$$

corresponding to twelve values of n, namely, $n=2$, 3, 4, 5, 7, 11, 15, 17, 19, 23, 31, and 35. He also recorded the representations for $n=2$ and 4 in Chapter 17 and for the remaining ten values and for $n=9$ and $n=25$ in Chapter 21 of his second notebook [39]. These representations for $q=e^{-\pi/\sqrt{n}}$ combined with (3.26) and the identities (3.15)–(3.21) are the primary ingredients in our derivations of series representations for $1/\pi$ in the following sections.

In our derivations which follow, for each value of n, there are potentially seven series representations for $1/\pi$, each arising from one of the formulas (3.3)–(3.9). However, it may happen that one or more of these series diverge. Also, we have needed to make decisions about the elegance or inelegance of series representations, and consequently we have not listed certain representations which we think are inelegant.

4. Example: $n=2$

Theorem 4.1. *If A_k and C_k, $k\geq 0$, are defined by* (3.2)*, then*

$$\frac{1}{\pi}=\sum_{k=0}^{\infty}\{(8-5\sqrt{2})k+3-2\sqrt{2}\}A_k(2\sqrt{2}-2)^{3k}, \qquad (4.1)$$

$$\frac{2}{\pi}=\sum_{k=0}^{\infty}(-1)^k(4k+1)A_k, \qquad (4.2)$$

$$\frac{2\sqrt{\sqrt{2}-1}}{\pi}=\sum_{k=0}^{\infty}(-1)^k\{(4\sqrt{2}-2)k+\sqrt{2}-1\}A_k\left(\frac{\sqrt{2}-1}{2}\right)^{3k}, \tag{4.3}$$

$$\frac{5\sqrt{5}}{\pi}=\sum_{k=0}^{\infty}(28k+3)C_k\left(\frac{3}{5}\right)^{3k}. \tag{4.4}$$

The identity (4.2) was first proved by G. Bauer in 1859 [3]. Many years later, the formula (4.2) was communicated by Ramanujan in his first letter to G. H. Hardy [38, p. xxvi], [13, p. 25]. Ramanujan also recorded (4.2) as Example 14 in Section 7 of Chapter 10 in his second notebook [39], [5, pp. 23–24]. Ramanujan [39], W. N. Bailey [4, p. 96], and Hardy [34], [36, pp. 517–518] all observed that (4.2) is a limiting case of Dougall's theorem. See also [14, p. 184]. The identity (4.4) is due to the Borweins [15]. The remaining two identities are new.

Proof of (4.1). From, for example, [8, p. 281, Theorem 9.2], $x_2=(\sqrt{2}-1)^2$, and from [6, p. 127, Entry 13(ix)],

$$f_2(q)=2P(q^4)-P(q^2)=z^2(q)\left(1-\frac{x(q)}{2}\right). \tag{4.5}$$

Setting $q=e^{-\pi/\sqrt{2}}$ and using (3.14), we find that $z(e^{-\pi/\sqrt{2}})=\sqrt{2}z(e^{-\pi\sqrt{2}})=\sqrt{2}z_2$ and $x(e^{-\pi/\sqrt{2}})=1-x(e^{-\pi\sqrt{2}})=1-x_2$. Hence, we deduce that

$$f_2(e^{-\pi/\sqrt{2}})=2P(e^{-2\pi\sqrt{2}})-P(e^{-2\pi/\sqrt{2}})=z_2^2(1+x_2)=(4-2\sqrt{2})z_2^2. \tag{4.6}$$

Setting $n=2$ in (3.26), we find that

$$2\,P(e^{-2\pi\sqrt{2}})+P(e^{-2\pi/\sqrt{2}})=\frac{6\sqrt{2}}{\pi}. \tag{4.7}$$

From (4.6) and (4.7), we deduce that

$$P(e^{-2\pi\sqrt{2}})=\frac{3}{\pi\sqrt{2}}+\frac{\sqrt{2}-1}{\sqrt{2}}z_2^2. \tag{4.8}$$

With the help of (3.3), we can rewrite (4.8) in the form

$$P(e^{-2\pi\sqrt{2}})=\frac{3}{\pi\sqrt{2}}+\frac{\sqrt{2}-1}{\sqrt{2}}\sum_{k=0}^{\infty}A_kX_2^k, \tag{4.9}$$

where $X_2=4x_2(1-x_2)=8(\sqrt{2}-1)^3$.

Now, setting $n=2$ in (3.15), we find that

$$P(e^{-2\pi\sqrt{2}})=(1-2x_2)\sum_{k=0}^{\infty}(3k+1)A_kX_2^k=(4\sqrt{2}-5)\sum_{k=0}^{\infty}(3k+1)A_kX_2^k. \tag{4.10}$$

From (4.9) and (4.10), we readily deduce (4.1). □

Proof of (4.2). With the help of (3.4), we can rewrite (4.8) as

$$\begin{aligned} P(e^{-2\pi\sqrt{2}}) &= \frac{3}{\pi\sqrt{2}} + \frac{\sqrt{2}-1}{\sqrt{2}(1-x_2)}\sum_{k=0}^{\infty}(-1)^k A_k Y_2^k \\ &= \frac{3}{\pi\sqrt{2}} + \frac{1}{2\sqrt{2}}\sum_{k=0}^{\infty}(-1)^k A_k, \end{aligned} \tag{4.11}$$

where we also used the fact that $Y_2 = 1$.

Setting $n = 2$ in (3.16), we find that

$$P(e^{-2\pi\sqrt{2}}) = \frac{1+x_2}{1-x_2}\sum_{k=0}^{\infty}(3k+1)A_k = \sqrt{2}\sum_{k=0}^{\infty}(3k+1)A_k. \tag{4.12}$$

From (4.11) and (4.12), we easily arrive at (4.2). □

Proof of (4.3). Employing (3.5) in (4.8), we obtain the equality

$$\begin{aligned} P(e^{-2\pi\sqrt{2}}) &= \frac{3}{\pi\sqrt{2}} + \frac{\sqrt{2}-1}{\sqrt{2}\sqrt{1-x_2}}\sum_{k=0}^{\infty}(-1)^k A_k U_2^k \\ &= \frac{3}{\pi\sqrt{2}} + \frac{\sqrt{\sqrt{2}-1}}{2}\sum_{k=0}^{\infty}(-1)^k A_k U_2^k, \end{aligned} \tag{4.13}$$

where $U_2 = \{(\sqrt{2}-1)/2\}^3$.

Also, setting $n = 2$ in (3.17), we deduce that

$$\begin{aligned} P(e^{-2\pi\sqrt{2}}) &= \frac{2-x_2}{2\sqrt{1-x_2}}\sum_{k=0}^{\infty}(6k+1)(-1)^k A_k U_2^k \\ &= \frac{2\sqrt{2}-1}{2\sqrt{2}\sqrt{\sqrt{2}-1}}\sum_{k=0}^{\infty}(6k+1)(-1)^k A_k U_2^k. \end{aligned} \tag{4.14}$$

From (4.13) and (4.14), we deduce (4.3). □

Proof of (4.4). Using (3.8) in (4.8), we find that

$$\begin{aligned} P(e^{-2\pi\sqrt{2}}) &= \frac{3}{\pi\sqrt{2}} + \frac{2(\sqrt{2}-1)}{\sqrt{2}\sqrt{4-X_2}}\sum_{k=0}^{\infty}C_k L_2^k \\ &= \frac{3}{\pi\sqrt{2}} + \frac{1}{\sqrt{10}}\sum_{k=0}^{\infty}C_k L_2^k, \end{aligned} \tag{4.15}$$

where $L_2 = (3/5)^3$.

Also, setting $n = 2$ in (3.20), we find that

$$\begin{aligned} P(e^{-2\pi\sqrt{2}}) &= \frac{\sqrt{1-X_2}(X_2+8)}{(4-X_2)^{3/2}}\sum_{k=0}^{\infty}(6k+1)C_k L_2^k \\ &= \frac{7\sqrt{2}}{5\sqrt{5}}\sum_{k=0}^{\infty}(6k+1)C_k L_2^k. \end{aligned} \tag{4.16}$$

From (4.15) and (4.16), we easily deduce (4.4). ☐

5. Example: $n = 4$

Theorem 5.1. *If A_k, B_k, and C_k, $k \geq 0$, are defined by* (3.2), *then*

$$\frac{1}{\pi} = \sum_{k=0}^{\infty}\{(48\sqrt{2}-66)k+20\sqrt{2}-28\}A_k(1584\sqrt{2}-2240)^k, \tag{5.1}$$

$$\frac{2\sqrt{2}}{\pi} = \sum_{k=0}^{\infty}(-1)^k(6k+1)A_k\left(\frac{1}{8}\right)^k, \tag{5.2}$$

$$\frac{2\cdot 2^{1/4}}{\pi} = \sum_{k=0}^{\infty}(-1)^k\{(18-6\sqrt{2})k+5-3\sqrt{2}\}A_k\left(\frac{(\sqrt{2}-1)^6}{16\sqrt{2}}\right)^k, \tag{5.3}$$

$$\frac{9}{2\pi} = \sum_{k=0}^{\infty}(7k+1)B_k\left(\frac{32}{81}\right)^k, \tag{5.4}$$

$$\frac{\sqrt{33}}{\pi} = \sum_{k=0}^{\infty}(126k+10)C_k\left(\frac{2}{11}\right)^{3k+1}. \tag{5.5}$$

The identities (5.4) and (5.5) are due to Berndt, Chan, and Liaw [11] and the Borweins [15], respectively. Guillera [31] proved the identity (5.2) by the WZ-method. The remaining two identities seem to be new.

Proof of (5.1). We note from [8, p. 284] that $x_4 = (\sqrt{2}-1)^4 = 17-12\sqrt{2}$. We set $q = e^{-\pi}$ in (4.5) and then use (3.27) to obtain

$$2P(e^{-4\pi}) - \frac{3}{\pi} = z^2(e^{-\pi})\left(1-\frac{x(e^{-\pi})}{2}\right). \tag{5.6}$$

Upon using the trivial evaluation $x(e^{-\pi}) = x_1 = \frac{1}{2}$ in (5.6), we deduce that

$$P(e^{-4\pi}) = \frac{3}{2\pi} + \frac{3}{8}z_1^2. \tag{5.7}$$

Now, from (2.5) and (2.7) in Lemma 2.1, we find that

$$\phi^4(q) = \frac{4}{(1+\sqrt{1-x(q)})^2}\phi^4(q^2). \tag{5.8}$$

Setting $q = e^{-\pi}$ in (5.8), we find that

$$z_1^2 = \frac{4}{(1+\sqrt{1-x_1})^2}z_4^2 = 8(3-2\sqrt{2})z_4^2. \tag{5.9}$$

Employing (5.9) in (5.7), we arrive at

$$P(e^{-4\pi}) = \frac{3}{2\pi} + 3(3-2\sqrt{2})z_4^2. \tag{5.10}$$

Now, using (3.3) in (5.10), we find that

$$P(e^{-4\pi}) = \frac{3}{2\pi} + 3(3-2\sqrt{2})\sum_{k=0}^{\infty} A_k X_4^k, \tag{5.11}$$

where

$$X_4 = 4x_4(1-x_4) = 16\sqrt{2}(\sqrt{2}-1)^6. \tag{5.12}$$

Setting $n = 4$ in (3.15), we obtain

$$\begin{aligned} P(e^{-4\pi}) &= (1-2x_4)\sum_{k=0}^{\infty}(3k+1)A_k X_4^k \\ &= (24\sqrt{2}-33)\sum_{k=0}^{\infty}(3k+1)A_k X_4^k. \end{aligned} \tag{5.13}$$

From (5.11) and (5.13), we easily deduce (5.1). □

Proof of (5.2). Employing (3.4) in (5.10), we find that

$$\begin{aligned} P(e^{-4\pi}) &= \frac{3}{2\pi} + \frac{3(3-2\sqrt{2})}{1-x_4}\sum_{k=0}^{\infty}(-1)^k A_k Y_4^k \\ &= \frac{3}{2\pi} + \frac{3}{4\sqrt{2}}\sum_{k=0}^{\infty}(-1)^k A_k Y_4^k, \end{aligned} \tag{5.14}$$

where

$$Y_4 = \frac{4x_4}{(1-x_4)^2} = \frac{1}{8}.$$

Setting $n = 4$ in (3.16), we find that

$$\begin{aligned} P(e^{-4\pi}) &= \frac{1+x_4}{1-x_4}\sum_{k=0}^{\infty}(-1)^k(3k+1)A_k Y_4^k \\ &= \frac{3}{2\sqrt{2}}\sum_{k=0}^{\infty}(-1)^k(3k+1)A_k Y_4^k. \end{aligned} \tag{5.15}$$

From (5.14) and (5.15), we easily arrive at (5.2). □

Proof of (5.3). We employ (3.4) to rewrite (5.10) as

$$\begin{aligned} P(e^{-4\pi}) &= \frac{3}{2\pi} + \frac{3(3-2\sqrt{2})}{\sqrt{1-x}}\sum_{k=0}^{\infty}(-1)^k A_k U_4^k \\ &= \frac{3}{2\pi} + \frac{3(\sqrt{2}-1)}{2\cdot 2^{1/4}}\sum_{k=0}^{\infty}(-1)^k A_k U_4^k, \end{aligned} \tag{5.16}$$

where

$$U_4 = \frac{x_4^2}{4(1-x_4)} = \frac{(\sqrt{2}-1)^6}{16\sqrt{2}}.$$

Setting $n=4$ in (3.17), we find that

$$P(e^{-4\pi}) = \frac{2-x_4}{2\sqrt{1-x_4}}\sum_{k=0}^{\infty}(-1)^k(6k+1)A_kU_4^k$$
$$= \frac{3(4\sqrt{2}-5)}{4\cdot 2^{1/4}(\sqrt{2}-1)}\sum_{k=0}^{\infty}(-1)^k(6k+1)A_kU_4^k. \qquad (5.17)$$

From (5.16) and (5.17), we readily deduce (5.3). □

Proof of (5.4). Using (3.6) in (5.10), we find that

$$P(e^{-4\pi}) = \frac{3}{2\pi} + \frac{3(3-2\sqrt{2})}{1+x}\sum_{k=0}^{\infty}B_kV_4^{2k}$$
$$= \frac{3}{2\pi} + \frac{1}{2}\sum_{k=0}^{\infty}B_kV_4^{2k}, \qquad (5.18)$$

where

$$V_4 = \frac{4\sqrt{x_4}(1-x_4)}{(1+x_4)^2} = \frac{4\sqrt{2}}{9}.$$

Next, setting $n=4$ in (3.18), we find that

$$P(e^{-4\pi}) = \sum_{k=0}^{\infty}\sum_{k=0}^{\infty}\frac{3k(1-6x_4+x_4^2)+1-X_4}{1+X_4}B_kV_4^{2k}$$
$$= \sum_{k=0}^{\infty}(-1)^k\left(\frac{7k}{3}+56\right)B_kV_4^{2k}, \qquad (5.19)$$

where we have also used (5.12). From (5.18) and (5.19), we arrive at (5.4). □

Proof of (5.5). With the aid of (3.6), we can rewrite (5.10) as

$$P(e^{-4\pi}) = \frac{3}{2\pi} + \frac{6(3-2\sqrt{2})}{\sqrt{4-X_4}}\sum_{k=0}^{\infty}C_kL_4^k$$
$$= \frac{3}{2\pi} + \frac{3}{\sqrt{33}}\sum_{k=0}^{\infty}C_kL_4^k, \qquad (5.20)$$

where

$$L_4 = \frac{27X_4^2}{(4-X_4)^3} = \left(\frac{2}{11}\right)^3.$$

Next, setting $n=4$ in (3.20), we find that

$$P(e^{-4\pi}) = \frac{\sqrt{1-X_4}(8+X_4)}{(4-X_4)^{3/2}}\sum_{k=0}^{\infty}\sum_{k=0}^{\infty}(6k+1)C_kL_4^k$$
$$= \frac{63}{11\sqrt{33}}\sum_{k=0}^{\infty}(-1)^k(6k+1)C_kL_4^k, \qquad (5.21)$$

where we have also used (5.12). From (5.20) and (5.21), we readily deduce (5.5). □

6. Example: $n = 6$

Theorem 6.1. *If A_k and B_k, $k \geq 0$, are defined by* (3.2), *then*

$$\frac{\sqrt{6}+\sqrt{2}+1}{\pi} = \sum_{k=0}^{\infty}\{(6\sqrt{3}+3\sqrt{6}-6)k+2\sqrt{3}+\sqrt{6}-3-\sqrt{2}\} \times A_k\{8(\sqrt{2}+1)^2(\sqrt{3}-\sqrt{2})^3(2-\sqrt{3})^3\}^k, \tag{6.1}$$

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k\{(12\sqrt{2}-12)k+4\sqrt{2}-5\}A_k(\sqrt{2}-1)^{4k}, \tag{6.2}$$

$$\frac{2\sqrt{2\sqrt{3}+2\sqrt{2}}}{\pi} = \sum_{k=0}^{\infty}\{(4\sqrt{6}+6\sqrt{3}-6)k+3\sqrt{2}-3\sqrt{3}+5-\sqrt{6}\} \times (-1)^k A_k\{8(\sqrt{2}+1)^2(\sqrt{3}+\sqrt{2})^3(2+\sqrt{3})^3\}^{-k}, \tag{6.3}$$

$$\frac{2\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}(8k+1)B_k\left(\frac{1}{9}\right)^k. \tag{6.4}$$

The last identity was recorded by Ramanujan in his paper [37, Eq. (40)], [38, p. 38]. To the best of our knowledge, the remaining identities are new.

Proof. First of all, we derive an expression for $f_6(e^{-\pi/\sqrt{6}}) = 6P(e^{-2\pi\sqrt{6}}) - P(e^{-2\pi/\sqrt{6}})$. To this end, from Entry 3(ii) in Chapter 21 of Ramanujan's second notebook [39], [6, p. 460], we record that

$$1+12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 36\sum_{k=1}^{\infty}\frac{kq^{6k}}{1-q^{6k}} = \phi^2(q)\phi^2(q^3) - 4q\psi^2(-q)\psi^2(-q^3), \tag{6.5}$$

where $\phi(q)$ and $\psi(q)$ are defined by (2.2). Now, replacing q by $-q^2$ in (6.5) and then employing the definition of P in (3.1), we find that

$$3P(q^{12}) - P(q^4) = 2\phi^2(-q^2)\phi^2(-q^6) + 4q^2\psi^2(q^2)\psi^2(q^6). \tag{6.6}$$

Transcribing (6.6) with the aid of (2.5), (2.6), (2.8), and (2.9) of Lemma 2.1, we deduce that

$$3P(q^{12}) - P(q^4) = z(q)z(q^6)\{2(1-x(q))^{1/4}(1-x(q^6))^{1/2} + x^{1/2}(q)x^{1/4}(q^6)\}. \tag{6.7}$$

Multiplying (6.7) by 2 and adding the resulting equality to (4.5), we find that

$$6P(q^{12}) - P(q^2) = z^2(q)\left(1-\frac{x(q)}{2}\right) + 2z(q)z(q^6)\{2(1-x(q))^{1/4}(1-x(q^6))^{1/2} + x^{1/2}(q)x^{1/4}(q^6)\}. \tag{6.8}$$

Now set $q = e^{-\pi/\sqrt{6}}$, so that, by (3.14), $x(q) = x(e^{-\pi/\sqrt{6}}) = 1 - x(e^{-\pi\sqrt{6}}) = 1 - x_6$, and $z(q) = z(e^{-\pi/\sqrt{6}}) = \sqrt{6}z(e^{-\pi\sqrt{6}}) = \sqrt{6}z_6$. We therefore deduce that

$$6P(e^{-2\pi\sqrt{6}}) - P(e^{-2\pi/\sqrt{6}}) = \{3(1+x_6) + 6\sqrt{6}x_6^{1/4}(1-x_6)^{1/2}\}z_6^2. \tag{6.9}$$

Now, the singular modulus x_6 is given by [8, p. 282]

$$x_6 = (2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2 = \frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{6}+\sqrt{2}-1},$$

so that

$$1 - x_6 = 2(\sqrt{2}+1)^2(2-\sqrt{3})(\sqrt{3}-\sqrt{2}) \tag{6.10}$$

and

$$1 + x_6 = 1 - x_6 + 2x_6 = 2\sqrt{6}(\sqrt{2}+1)(2-\sqrt{3})(\sqrt{3}-\sqrt{2}). \tag{6.11}$$

Thus, from (6.9), we deduce that

$$6P(e^{-2\pi\sqrt{6}}) - P(e^{-2\pi/\sqrt{6}}) = 6\sqrt{6}(\sqrt{2}+1)^2(2-\sqrt{3})(\sqrt{3}-\sqrt{2})z_6^2. \tag{6.12}$$

Next, setting $n = 6$ in (3.26), we find that

$$6P(e^{-2\pi\sqrt{6}}) + P(e^{-2\pi/\sqrt{6}}) = \frac{6\sqrt{6}}{\pi}. \tag{6.13}$$

Adding (6.12) and (6.13), we obtain the identity

$$P(e^{-2\pi\sqrt{6}}) = \frac{\sqrt{3}}{\pi\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}}(\sqrt{2}+1)^2(2-\sqrt{3})(\sqrt{3}-\sqrt{2})z_6^2. \tag{6.14}$$

Now we are ready to prove (6.1). Using (3.3) in (6.14), we find that

$$P(e^{-2\pi\sqrt{6}}) = \frac{\sqrt{3}}{\pi\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}}(\sqrt{2}+1)^2(2-\sqrt{3})(\sqrt{3}-\sqrt{2})\sum_{k=0}^{\infty} A_k X_6^k, \tag{6.15}$$

where $X_6 = 4x_6(1-x_6) = 8(\sqrt{2}+1)^2(2-\sqrt{3})^3(\sqrt{3}-\sqrt{2})^3$.

Next, setting $n = 6$ in (3.15), we find that

$$\begin{aligned} P(e^{-2\pi\sqrt{6}}) &= (1-2x_6)\sum_{k=0}^{\infty}(3k+1)A_k X_6^k \\ &= (\sqrt{2}+1)(2-\sqrt{3})(\sqrt{3}-\sqrt{2})(3\sqrt{2}+3-\sqrt{6})\sum_{k=0}^{\infty}(3k+1)A_k X_6^k. \end{aligned} \tag{6.16}$$

From (6.15) and (6.16), we deduce (6.1).

Similarly, employing (3.4), (3.5), and (3.6) in (6.14) and setting $n = 6$ in each of (3.16), (3.17), and (3.18), we can derive the series identities in (6.2)–(6.4). □

7. Example: $n = 5$

Theorem 7.1. *If A_k and B_k, $k \geq 0$, are defined by* (3.2), *then*

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} \{4\sqrt{5}k + \sqrt{5} - 1\} A_k (\sqrt{5} - 2)^{2k+1/2}, \tag{7.1}$$

$$\begin{aligned}\frac{8}{\pi} = \sum_{k=0}^{\infty} (-1)^k \Bigg[& 2\{(15 + 5\sqrt{5})\sqrt{\sqrt{5}+1} - 7\sqrt{10} - 5\sqrt{2}\}k + (9 + 3\sqrt{5})\sqrt{\sqrt{5}+1} \\ & - 7\sqrt{2} - 5\sqrt{10}\Bigg] A_k \left(\frac{\sqrt{5}-1}{4}\right)^{3k} \left(\frac{\sqrt{5}+1}{2} - \sqrt{\frac{\sqrt{5}+1}{2}}\right)^{6k},\end{aligned} \tag{7.2}$$

$$\frac{8}{\pi} = \sum_{k=0}^{\infty} (-1)^k (20k + 3) B_k \frac{1}{4^k}. \tag{7.3}$$

The last identity was recorded by Ramanujan [37, Eq. (35)], [38, p. 38]. The other identities appear to be new.

Proof of (7.1). From Entry 4(iii) in Chapter 21 of Ramanujan's second notebook [39], [6, p. 464], we see that

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} \frac{kq^{2k}}{1-q^{2k}} - 30 \sum_{k=1}^{\infty} \frac{kq^{10k}}{1-q^{10k}} & \\ = \frac{1}{4\sqrt{2}} \phi^2(q)\phi^2(q^5) & \{3 + \sqrt{x(q)x(q^5)} + \sqrt{(1-x(q))(1-x(q^5))}\} \\ \times & \left\{1 + \sqrt{x(q)x(q^5)} + \sqrt{(1-x(q))(1-x(q^5))}\right\}^{1/2}. \end{aligned} \tag{7.4}$$

With the help of (3.1) we can rewrite (7.4) in the form

$$\begin{aligned} 5P(q^{10}) - P(q^2) & \\ = \frac{1}{\sqrt{2}} \phi^2(q)\phi^2(q^5) & \{3 + \sqrt{x(q)x(q^5)} + \sqrt{(1-x(q))(1-x(q^5))}\} \\ \times & \left\{1 + \sqrt{x(q)x(q^5)} + \sqrt{(1-x(q))(1-x(q^5))}\right\}^{1/2}. \end{aligned} \tag{7.5}$$

Now we set $q = e^{-\pi/\sqrt{5}}$ and use (3.14) and (2.4) to deduce that $x(q) = x_{1/5} = 1 - x_5$, $x(q^5) = x_5$, and $\phi^2(e^{-\pi/\sqrt{5}}) = \sqrt{5}\phi^2(e^{-\pi\sqrt{5}}) = \sqrt{5}z_5$. Thus, from (7.5), we find that

$$\begin{aligned} 5P(e^{-2\pi\sqrt{5}}) - P(e^{-2\pi/\sqrt{5}}) &= \sqrt{5}z_5^2\{3 + 2(x_5(1-x_5))^{1/2}\}\{\tfrac{1}{2}(1 + 2(x_5(1-x_5))^{1/2})\}^{1/2} \\ &= \sqrt{5}z_5^2\{3 + \sqrt{X_5}\}\left\{\tfrac{1}{2}(1 + \sqrt{X_5})\right\}^{1/2}, \end{aligned} \tag{7.6}$$

where $X_5 = 4x_5(1 - x_5)$.

But, by [9], the singular modulus x_5 is given by

$$x_5 = \frac{1}{2} - \left(\frac{\sqrt{5}-1}{2}\right)^{3/2},$$

so that

$$X_5 = 9 - 4\sqrt{5} \quad \text{and} \quad \sqrt{X_5} = \sqrt{5} - 2. \tag{7.7}$$

Thus, from (7.6), we find that

$$5P(e^{-2\pi\sqrt{5}}) - P(e^{-2\pi/\sqrt{5}}) = \sqrt{5}(\sqrt{5}+1)\left(\frac{\sqrt{5}-1}{2}\right)^{1/2} z_5^2. \tag{7.8}$$

Next, setting $n = 5$ in (3.26), we find that

$$5P(e^{-2\pi\sqrt{5}}) + P(e^{-2\pi/\sqrt{5}}) = \frac{6\sqrt{5}}{\pi}. \tag{7.9}$$

Adding (7.8) and (7.9), we deduce that

$$P(e^{-2\pi\sqrt{5}}) = \frac{3}{\pi\sqrt{5}} + \frac{\sqrt{5}+1}{\sqrt{5}} \cdot \left(\frac{\sqrt{5}-1}{2}\right)^{1/2} z_5^2. \tag{7.10}$$

Now, employing (3.3) in (7.10), we deduce the identity

$$P(e^{-2\pi\sqrt{5}}) = \frac{3}{\pi\sqrt{5}} + \frac{\sqrt{5}+1}{\sqrt{5}} \cdot \left(\frac{\sqrt{5}-1}{2}\right)^{1/2} \sum_{k=0}^{\infty} A_k X_5^k. \tag{7.11}$$

Next, setting $n = 5$ in (3.15), we find that

$$\begin{aligned} P(e^{-2\pi\sqrt{5}}) &= (1-2x_5)\sum_{k=0}^{\infty}(3k+1)A_k X_5^k \\ &= 2(\sqrt{5}-2)^{1/2}\sum_{k=0}^{\infty}(3k+1)A_k X_5^k. \end{aligned} \tag{7.12}$$

Using (7.11) and (7.12), we arrive at (7.1). □

Proof of (7.3). Employing (3.7) in (7.10), we find that

$$\begin{aligned} P(e^{-2\pi\sqrt{5}}) &= \frac{3}{\pi\sqrt{5}} + \frac{\sqrt{5}+1}{\sqrt{5}(1-2x_5)} \cdot \left(\frac{\sqrt{5}-1}{2}\right)^{1/2} \sum_{k=0}^{\infty}(-1)^k B_k W_5^{2k} \\ &= \frac{3}{\pi\sqrt{5}} + \frac{3+\sqrt{5}}{4\sqrt{5}}\sum_{k=0}^{\infty}(-1)^k B_k W_5^{2k}, \end{aligned} \tag{7.13}$$

where

$$W_5 = \frac{2\sqrt{X_5}}{1-X_5} = \frac{1}{2}.$$

Next, setting $n = 5$ in (3.19), we find that

$$P(e^{-2\pi\sqrt{5}}) = \sum_{k=0}^{\infty} \frac{3k(1+X_n) + 1 + X_n/2}{1-X_n} B_k (-1)^k W_n^{2k}$$
$$= \left(\frac{3\sqrt{5}}{2}k + \frac{3\sqrt{5}+2}{8}\right)(-1)^k B_k W_5^{2k}. \tag{7.14}$$

From (7.13) and (7.14), we readily arrive at (7.3). Thus, we complete the proof.

The proof of (7.2) is similar. □

8. Example: $n = 3$

Theorem 8.1. *If A_k, B_k, and C_k, $k \geq 0$, are defined by* (3.2), *then*

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} (6k+1) A_k \frac{1}{4^k}, \tag{8.1}$$

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} (-1)^k \{(15\sqrt{3} - 24)k + 6\sqrt{3} - 10\} A_k 2^k (\sqrt{3}-1)^{6k}, \tag{8.2}$$

$$\frac{4\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} (-1)^k \{(30 - 6\sqrt{3})k + 7 - 3\sqrt{3}\} A_k \frac{(2-\sqrt{3})^{3k}}{2^{4k}}, \tag{8.3}$$

$$\frac{8\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \{(85\sqrt{3} - 135)k + 8\sqrt{3} - 12\} B_k \left(\frac{8\sqrt{2}}{51\sqrt{3} - 75}\right)^{2k+1}, \tag{8.4}$$

$$\frac{5\sqrt{5}}{2\pi\sqrt{3}} = \sum_{k=0}^{\infty} (11k+1) C_k \left(\frac{4}{125}\right)^k. \tag{8.5}$$

The identities (8.1) and (8.5) are due to Ramanujan [37, Eqs. (28), (33), resp.], [38, pp. 36–37]. The remaining identities are new. Because the proofs are similar to those in previous sections, we do not give them. We note that $f_3(e^{-\pi/\sqrt{3}}) = 3P(e^{-2\pi\sqrt{3}}) - P(e^{-2\pi/\sqrt{3}}) = \frac{3\sqrt{3}}{2}\, z_3^2$.

9. Example: $n = 7$

Theorem 9.1. *If A_k, B_k, and C_k, $k \geq 0$, are defined by* (3.2), *then*

$$\frac{16}{\pi} = \sum_{k=0}^{\infty} (42k+5) A_k \frac{1}{2^{6k}}, \tag{9.1}$$

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} (-1)^k \{(255\sqrt{7} - 672)k + 112\sqrt{7} - 296\} A_k (32 - 12\sqrt{7})^{3k}, \tag{9.2}$$

$$\frac{8\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \{(102\sqrt{7} - 210)k + 35\sqrt{7} - 89\} (-1)^k A_k \left(\frac{8-3\sqrt{7}}{4}\right)^{3k}, \tag{9.3}$$

$$\frac{29241}{\pi} = \sum_{k=0}^{\infty}\{(76160-455\sqrt{7})k+784\sqrt{7}+6728\}B_k\left(\frac{8\sqrt{2}(325+119\sqrt{7})}{29241}\right)^{2k}, \tag{9.4}$$

$$\frac{9\sqrt{7}}{\pi} = \sum_{k=0}^{\infty}(65k+8)(-1)^k B_k\left(\frac{16}{63}\right)^{2k}, \tag{9.5}$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}} = \sum_{k=0}^{\infty}(133k+8)C_k\left(\frac{4}{85}\right)^{3k}, \tag{9.6}$$

$$\frac{5\sqrt{15}}{\pi} = \sum_{k=0}^{\infty}(63k+8)(-1)^k C_k\left(\frac{4}{5}\right)^{3k}. \tag{9.7}$$

The identities (9.1) and (9.6) are due to Ramanujan [37, Eqs. (29), (34), resp.], [38, pp. 36–37], and (9.5) is due to Berndt, Chan, and Liaw [11]. The other four identities seem to be new. Because the proofs are similar to those in previous sections, we do not record them. We note that $f_7(e^{-\pi/\sqrt{7}}) = 7P(e^{-2\pi\sqrt{7}}) - P(e^{-2\pi/\sqrt{7}}) = \frac{27\sqrt{7}}{8}\ z_7^2$.

10. Example: $n = 9$

Theorem 10.1. *If A_k and B_k, $k \geq 0$, are defined by* (3.2), *then*

$$\frac{12^{1/4}}{\pi} = \sum_{k=0}^{\infty}\{(24\sqrt{3}-36)k+9\sqrt{3}-15\}A_k(2-\sqrt{3})^{4k}, \tag{10.1}$$

$$\frac{4}{\pi} = \sum_{k=0}^{\infty}[6\{9\sqrt{2}+7\sqrt{6}-3^{1/4}(5\sqrt{3}+11)\}k+15\sqrt{6}+21\sqrt{2}-3^{1/4}(13\sqrt{3}+27)] \times(-1)^k A_k\left(\frac{(\sqrt{3}+1)^{1/3}(\sqrt{2}-3^{1/4})}{2^{2/3}}\right)^{6k}, \tag{10.2}$$

$$\frac{16}{\pi\sqrt{3}} = \sum_{k=0}^{\infty}(-1)^k(28k+3)B_k\frac{1}{48^k}. \tag{10.3}$$

The identity (10.1) is due to J. M. and P. B. Borwein [14], [15]; (10.3) is due to Ramanujan [37, Eq. (36)], [38, p. 38]; and (10.2) is new. The proofs are similar to those in previous sections. We note that

$$f_9(e^{-\pi/3}) = 9P(e^{-6\pi}) - P(e^{-2\pi/3}) = 9\sqrt{2}\cdot 3^{1/4}(\sqrt{3}-1)\ z_9^2.$$

11. Example: $n = 10$

Theorem 11.1. *If A_k and B_k, $k \geq 0$, are defined by* (3.2)*, then*

$$\frac{3\sqrt{2}+\sqrt{5}+2}{\pi} = \sum_{k=0}^{\infty}\{(15\sqrt{2}+6\sqrt{10}-6\sqrt{5})k+2\sqrt{10}-3\sqrt{5}+5\sqrt{2}-4\} \times A_k\{(3+\sqrt{5})(2+\sqrt{5})(3\sqrt{2}-\sqrt{5}-2)\}^{3k}, \tag{11.1}$$

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k\{(60-24\sqrt{5})k+23-10\sqrt{5}\}A_k(\sqrt{5}-2)^{4k}, \tag{11.2}$$

$$\frac{9}{2\pi\sqrt{2}} = \sum_{k=0}^{\infty}(10k+1)B_k\frac{1}{9^{2k}}, \tag{11.3}$$

$$\frac{6}{\pi} = \sum_{k=0}^{\infty}(-1)^k[6\{a(90\sqrt{2}+12\sqrt{10}-30-36\sqrt{5})+b(90+12\sqrt{5}-30\sqrt{2}-36\sqrt{10})\}k + a(27\sqrt{2}-29+5\sqrt{10}-12\sqrt{5})+b(39-9\sqrt{10}+8\sqrt{5}-17\sqrt{2})] \times A_k\left(\frac{(\sqrt{10}-3)(3-\sqrt{5})(3-2\sqrt{2})}{4}\right)^{3k}, \tag{11.4}$$

where

$$a = \sqrt{\frac{\sqrt{10}+1}{2}} \quad \textit{and} \quad b = \sqrt{\frac{\sqrt{10}-1}{2}}.$$

The identity (11.3) is due to Ramanujan [37, Eq. (41)], [38, p. 38]. The remaining three are new. As in the foregoing sections, we do not give the proofs. We note that $f_{10}(e^{-\pi/\sqrt{10}}) = 10P(e^{-2\pi\sqrt{10}}) - P(e^{-2\pi/\sqrt{10}}) = (4\sqrt{5}+5)\left(1+(\sqrt{10}-3)^2(3-2\sqrt{2})^2\right) z_{10}^2$.

12. Example: $n = 13$

Theorem 12.1. *If A_k and B_k, $k \geq 0$, are defined by* (3.2)*, then*

$$\frac{\sqrt{5\sqrt{13}-17}}{\pi} = \sum_{k=0}^{\infty}\{(273\sqrt{2}-75\sqrt{26})k+112\sqrt{2}-31\sqrt{26}\}A_k(5\sqrt{13}-18)^{2k}, \tag{12.1}$$

$$\frac{12}{\pi} = \sum_{k=0}^{\infty}(-1)^k\{12\sqrt{13}(1+2(c-d)^4)k+3\sqrt{13}-7+(9\sqrt{13}+7)(c-d)^4\} \times A_k(\sqrt{13}+3)^{3k}\left(\sqrt{\frac{19+5\sqrt{13}}{2}}-\sqrt{\frac{17+5\sqrt{13}}{2}}\right)^{6k}, \tag{12.2}$$

$$\frac{72}{\pi} = \sum_{k=0}^{\infty}(-1)^k(260k+23)B_k\frac{1}{18^{2k}}, \tag{12.3}$$

where

$$c = \sqrt{\frac{19+5\sqrt{13}}{2}} \quad \textit{and} \quad d = \sqrt{\frac{17+5\sqrt{13}}{2}}.$$

The identity (12.3) is due to Ramanujan [37, Eq. (37)], [38, p. 38]. The remaining identities are new.

Proof. Ramanujan did not record any expression for $f_{13}(q) = 13P(q^{26}) - P(q^2)$. So our first task is to find an expression for $f_{13}(e^{-\pi/\sqrt{13}})$.

From (2.3) and (2.10) of Lemma 2.1, we find that

$$\frac{q^{p/12}f(-q^{2p})}{q^{1/12}f(-q^2)} = \frac{1}{\sqrt{m}}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}, \tag{12.4}$$

where β has degree p over α, and $m = \phi^2(q)/\phi^2(q^p)$. Taking logarithms on both sides of (12.4), differentiating with respect to q, and then using the definition of $P(q)$ from (3.1), we arrive at

$$\begin{aligned}\frac{1}{12q}\{pP(q^{2p}) - P(q^2)\} &= \frac{1}{12\beta(1-\beta)}\cdot\frac{d}{dq}(\beta(1-\beta)) - \frac{1}{12\alpha(1-\alpha)}\cdot\frac{d}{dq}(\alpha(1-\alpha)) \\ &\quad - \frac{1}{2m}\frac{dm}{dq} \qquad (12.5)\\ &= \frac{1-2\beta}{12\beta(1-\beta)}\cdot\frac{d\beta}{d\alpha}\cdot\frac{d\alpha}{dq} - \frac{1-2\alpha}{12\alpha(1-\alpha)}\cdot\frac{d\alpha}{dq} \\ &\quad - \frac{1}{2m}\frac{dm}{d\alpha}\cdot\frac{d\alpha}{dq}. \qquad (12.6)\end{aligned}$$

Now, from Entry 9(i) of Chapter 17 and Entry 24(vi) in Chapter 18 in Ramanujan's second notebook [39], [6, pp. 120, 217], we find that

$$\frac{d\alpha}{dq} = \frac{\alpha(1-\alpha)\phi^4(q)}{q} \quad \text{and} \quad \frac{d\beta}{d\alpha} = \frac{\beta(1-\beta)}{\alpha(1-\alpha)}\cdot\frac{p}{m^2}. \tag{12.7}$$

Employing (12.7) in (12.5) and then simplifying, we find that

$$pP(q^{2p}) - P(q^2) = \phi^2(q)\phi^2(q^p)\left\{\frac{p(1-2\beta)}{m} - (1-2\alpha)m - 6\alpha(1-\alpha)\frac{dm}{d\alpha}\right\}. \tag{12.8}$$

Next, we record the following modular equation of degree 13 from Chapter 20 of Ramanujan's second notebook [39], [6, p. 376, Entry 8(iii)]. If β has degree 13 over α, then

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}. \tag{12.9}$$

Differentiating (12.9) with respect to α, we find that

$$\begin{aligned}\frac{dm}{d\alpha} &= \frac{1}{\sqrt{\alpha}}\left\{\frac{\alpha^{1/4}}{4\beta^{3/4}}\cdot\frac{d\beta}{d\alpha} - \frac{\beta^{1/4}}{4\alpha^{3/4}}\right\} + \frac{1}{\sqrt{1-\alpha}}\left\{\frac{-(1-\alpha)^{1/4}}{4(1-\beta)^{3/4}}\cdot\frac{d\beta}{d\alpha} + \frac{(1-\beta)^{1/4}}{4(1-\alpha)^{3/4}}\right\} \\ &\quad - \frac{1}{\sqrt{\alpha(1-\alpha)}}\left\{\frac{(\alpha(1-\alpha))^{1/4}(1-2\beta)}{4(\beta(1-\beta))^{3/4}}\cdot\frac{d\beta}{d\alpha} - \frac{(\beta(1-\beta))^{1/4}(1-2\alpha)}{4(\alpha(1-\alpha))^{3/4}}\right\}\end{aligned}$$

$$-\frac{4}{(\alpha(1-\alpha))^{1/3}}\left\{\frac{(\alpha(1-\alpha))^{1/6}(1-2\beta)}{6(\beta(1-\beta))^{5/6}}\cdot\frac{d\beta}{d\alpha}-\frac{(\beta(1-\beta))^{1/6}(1-2\alpha)}{6(\alpha(1-\alpha))^{5/6}}\right\}. \tag{12.10}$$

Next, if $q=e^{-\pi/\sqrt{13}}$, then, by (3.14), $\alpha=1-\beta=1-x_{13}$, $m=\sqrt{13}$, and, by (12.7),

$$\left[\frac{d\beta}{d\alpha}\right]_{q=e^{-\pi/\sqrt{13}}}=1.$$

Thus, setting $q=e^{-\pi/\sqrt{13}}$ in (12.10), we deduce that

$$\left[\frac{dm}{d\alpha}\right]_{q=e^{-\pi/\sqrt{13}}}=\frac{1-2x_{13}}{4x_{13}(1-x_{13})}\left\{\left(\frac{x_{13}}{1-x_{13}}\right)^{1/4}+\left(\frac{1-x_{13}}{x_{13}}\right)^{1/4}\right\}-\frac{11(1-2x_{13})}{6x_{13}(1-x_{13})}. \tag{12.11}$$

Now, setting $p=13$ and $q=e^{-\pi/\sqrt{13}}$ in (12.8), and then using (12.11), we find that

$$\begin{aligned}f_{13}(e^{-\pi/\sqrt{13}})&=13P(e^{-2\pi\sqrt{13}})-P(e^{-2\pi/\sqrt{13}})\\&=\sqrt{13}(1-2x_{13})\left[(11+2\sqrt{13})-\frac{3}{2}\left\{\left(\frac{x_{13}}{1-x_{13}}\right)^{1/4}+\left(\frac{1-x_{13}}{x_{13}}\right)^{1/4}\right\}\right]z_{13}^2.\end{aligned} \tag{12.12}$$

From [12], we note that

$$x_{13}=\frac{1}{2}\left(\frac{\sqrt{13}-3}{2}\right)^3\left(\sqrt{\frac{7+\sqrt{13}}{4}}-\sqrt{\frac{3+\sqrt{13}}{4}}\right)^4,$$

so that

$$1-x_{13}=\frac{1}{2}\left(\frac{\sqrt{13}-3}{2}\right)^3\left(\sqrt{\frac{7+\sqrt{13}}{4}}+\sqrt{\frac{3+\sqrt{13}}{4}}\right)^4,$$

$$\begin{aligned}1-2x_{13}&=1-x_{13}-x_{13}\\&=\frac{1}{2}\left(\frac{\sqrt{13}-3}{2}\right)^3\\&\quad\times\left\{\left(\sqrt{\frac{7+\sqrt{13}}{4}}+\sqrt{\frac{3+\sqrt{13}}{4}}\right)^4-\left(\sqrt{\frac{7+\sqrt{13}}{4}}-\sqrt{\frac{3+\sqrt{13}}{4}}\right)^4\right\}\\&=(7\sqrt{13}-25)\sqrt{\frac{17+5\sqrt{13}}{2}},\end{aligned} \tag{12.13}$$

and

$$\left(\frac{x_{13}}{1-x_{13}}\right)^{1/4}+\left(\frac{1-x_{13}}{x_{13}}\right)^{1/4}=5+\sqrt{13}. \tag{12.14}$$

With the help of (12.13) and (12.14), we can rewrite (12.12) in the form

$$\begin{aligned} f_{13}(e^{-\pi/\sqrt{13}}) &= 13P(e^{-2\pi\sqrt{13}}) - P(e^{-2\pi/\sqrt{13}}) \\ &= (7\sqrt{13}-25)\frac{13+7\sqrt{13}}{2}\cdot\sqrt{\frac{17+5\sqrt{13}}{2}}z_{13}^2. \end{aligned} \tag{12.15}$$

Now, we prove (12.1). Setting $n=13$ in (3.26), we find that

$$13P(e^{-2\pi\sqrt{13}}) + P(e^{-2\pi/\sqrt{13}}) = \frac{6\sqrt{13}}{\pi}. \tag{12.16}$$

Adding (12.15) and (12.16), we obtain

$$P(e^{-2\pi\sqrt{13}}) = \frac{3}{\pi\sqrt{13}} + (7\sqrt{13}-25)\frac{7+\sqrt{13}}{4\sqrt{13}}\cdot\sqrt{\frac{17+5\sqrt{13}}{2}}z_{13}^2. \tag{12.17}$$

Employing (3.3) in (12.17), we find that

$$P(e^{-2\pi\sqrt{13}}) = \frac{3}{\pi\sqrt{13}} + (7\sqrt{13}-25)\frac{7+\sqrt{13}}{4\sqrt{13}}\cdot\sqrt{\frac{17+5\sqrt{13}}{2}}\sum_{k=0}^{\infty}A_kX_{13}^k, \tag{12.18}$$

where $X_{13} = 4x_{13}(1-x_{13}) = \{(\sqrt{13}-3)/2\}^6$.

Next, setting $n=13$ in (3.15), we obtain

$$\begin{aligned} P(e^{-2\pi\sqrt{13}}) &= (1-2x_{13})\sum_{k=0}^{\infty}(3k+1)A_kX_{13}^k \\ &= (7\sqrt{13}-25)\sqrt{\frac{17+5\sqrt{13}}{2}}\sum_{k=0}^{\infty}(3k+1)A_kX_{13}^k. \end{aligned} \tag{12.19}$$

From (12.18) and (12.19), we readily arrive at (12.1).

Similarly, we can prove (12.2) and (12.3). □

Because the proofs in the closing Sections 13–18 follow along the same lines as those in previous sections, we omit the proofs.

13. Example: $n = 14$

Theorem 13.1. *If B_k, $k \geq 0$, is defined by* (3.2), *then*

$$\frac{1}{\pi\sqrt{14}\sqrt{44\sqrt{2}-50}} = \sum_{k=0}^{\infty}\left\{\frac{2}{7}k + \frac{3}{196}(3-\sqrt{2})\right\}B_k\left(\frac{1}{11+8\sqrt{2}}\right)^{2k}. \tag{13.1}$$

This identity is due to Berndt, Chan, and Liaw [11]. We note that

$$\begin{aligned} f_{14}(e^{-\pi/\sqrt{14}}) &= 14P(e^{-2\pi\sqrt{14}}) - P(e^{-2\pi/\sqrt{14}}) \\ &= \{7(1+x_{14}) + 6\sqrt{14}\ \left(3+2\sqrt{2}\right)\ x_{14}^{1/4}(1-x_{14})^{1/2}\}\ z_{14}^2, \end{aligned}$$

where

$$x_{14} = 995 + 704\sqrt{2} + 8\sqrt{30926 + 21868\sqrt{2}} \\ - 2\left\{990130 + 700128\sqrt{2} + 4(6218652 + 4397251\sqrt{2})\sqrt{1562\sqrt{2} - 2209}\right\}^{1/2}.$$

14. Example: $n = 15$

Theorem 14.1. *If A_k and B_k, $k \geq 0$, are defined by* (3.2), *then*

$$\frac{32}{\pi} = \sum_{k=0}^{\infty}\{(42\sqrt{5}+30)k + 5\sqrt{5} - 1\}A_k \frac{1}{2^{6k}}\left(\frac{\sqrt{5}-1}{2}\right)^{8k}, \tag{14.1}$$

$$\frac{121\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}(-1)^k\{(1365\sqrt{5}-2240)k + 300\sqrt{5} - 604\}B_k\left(\frac{4}{45}\right)^{k}. \tag{14.2}$$

The first identity is due to Ramanujan [37, Eq. (30)], [38, p. 37], and the other is new. We note that

$$f_{15}(e^{-\pi/\sqrt{15}}) = 15P(e^{-2\pi\sqrt{15}}) - P(e^{-2\pi/\sqrt{15}}) = \frac{3\sqrt{15}(11+9\sqrt{5})}{16}\; z_{15}^2.$$

15. Example: $n = 17$

Theorem 15.1. *If B_k, $k \geq 0$, is defined by* (3.2), *then*

$$\frac{16\sqrt{38+10\sqrt{17}}}{\pi} = \sum_{k=0}^{\infty}(-1)^k\{(60\sqrt{17}+340)k + 33 + 3\sqrt{17}\} \\ \times B_k\left(\frac{25\sqrt{17}-103}{128}\right)^{k}. \tag{15.1}$$

The representation (15.1) is new. We note that

$$f_{17}(e^{-\pi/\sqrt{17}}) = 17P(e^{-2\pi\sqrt{17}}) - P(e^{-2\pi/\sqrt{17}}) = \frac{\sqrt{17}}{2}\left(\sqrt{17} + 3\sqrt{4+\sqrt{17}}\right)(1-2x_{17})\; z_{17}^2,$$

where

$$x_{17} = \frac{1}{2}\left(\sqrt{\frac{7+\sqrt{17}}{4}} - \sqrt{\frac{3+\sqrt{17}}{4}}\right)^4\left(\sqrt{\frac{3+\sqrt{4+\sqrt{17}}}{4}} - \sqrt{\frac{\sqrt{4+\sqrt{17}}-1}{4}}\right)^8.$$

16. Example: $n = 18$

Theorem 16.1. *If A_k and B_k, $k \geq 0$, are defined by* (3.2), *then*

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k(84k + 21 - 6\sqrt{6})A_k(\sqrt{3}-\sqrt{2})^{8k+2}, \tag{16.1}$$

$$\frac{2\sqrt{\sqrt{2}-1}}{3\pi} = \sum_{k=0}^{\infty}(-1)^k\{2(106+45\sqrt{6}-65\sqrt{3}-72\sqrt{2})k+90+38\sqrt{6}$$
$$-55\sqrt{3}-62\sqrt{2}\}A_k\left(\frac{(5\sqrt{2}-7)^3(7-4\sqrt{3})^3(5-2\sqrt{6})^2}{8}\right)^k, \quad (16.2)$$

$$\frac{1}{3\pi\sqrt{3}} = \sum_{k=0}^{\infty}(40k+3)B_k\frac{1}{49^{2k}}. \quad (16.3)$$

The identity (16.1) is due to the Borwein brothers [14], [15], and (16.3) is due to Ramanujan [37, Eq. (42)], [38, p. 38]. Identity (16.2) is new. We note that

$$f_{18}(e^{-\pi/3\sqrt{2}}) = 18P(e^{-6\pi\sqrt{2}}) - P(e^{-2\pi/3\sqrt{2}})$$
$$= 18\sqrt{2}(5+2\sqrt{6})(7+6\sqrt{6})(5\sqrt{2}-7)(7-4\sqrt{3})\ z_{18}^2.$$

17. Example: $n = 22$

Theorem 17.1. *If A_k and B_k, $k \geq 0$, are defined by* (3.2), *then*

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k\{(660\sqrt{2}-924)k+284\sqrt{2}-401\}A_k(\sqrt{2}-1)^{12k}, \quad (17.1)$$

$$\frac{2}{\pi\sqrt{11}} = \sum_{k=0}^{\infty}(280k+19)B_k\ \frac{1}{99^{2k+1}}. \quad (17.2)$$

Identity (17.2) is due to Ramanujan [37, Eq. (43)], [38, p. 38], while (17.1) is new. We note that

$$f_{22}\big(e^{-\pi/\sqrt{22}}\big) = 22P(e^{-2\pi\sqrt{22}}) - P(e^{-2\pi/\sqrt{22}})$$
$$= 6\sqrt{22}(11+17\sqrt{2})(10-3\sqrt{11})(3\sqrt{11}-7\sqrt{2})(5\sqrt{2}+7)\ z_{22}^2.$$

18. Example: $n = 25$

Theorem 18.1. *If A_k and B_k, $k \geq 0$, are defined by* (3.2), *then*

$$\frac{5^{1/4}}{\pi} = \sum_{k=0}^{\infty}\{(540\sqrt{5}-1200)k+235\sqrt{5}-525\}A_k(\sqrt{5}-2)^{8k}, \quad (18.1)$$

$$\frac{288}{\pi\sqrt{5}} = \sum_{k=0}^{\infty}(-1)^k(644k+41)B_k\frac{1}{(72\sqrt{5})^{2k}}. \quad (18.2)$$

Identity (18.2) is due to Ramanujan [37, Eq. (38)]; the identity (18.1) is new. We note that $f_{25}(e^{-\pi/5}) = 25P(e^{-10\pi}) - P(e^{-2\pi/5}) = 300 \cdot 5^{1/4}(\sqrt{5}-2)^{5/3}\ z_{25}^2$.

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Department of Mathematical Sciences, Tezpur University, Napaam-784028, Sonitpur, Assam, India

Department of Mathematics, University of Illinois, 1409 West Green St., Urbana, IL 61801, USA

E-mail address: nayan@tezu.ernet.in

E-mail address: berndt@illinois.edu