

RAMANUJAN'S CONGRUENCES FOR THE PARTITION FUNCTION MODULO 5, 7, AND 11

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Abstract. Using Ramanujan's differential equations for Eisenstein series and an idea from Ramanujan's unpublished manuscript on the partition function $p(n)$ and the tau function $\tau(n)$, we provide simple proofs of Ramanujan's congruences for $p(n)$ modulo 5, 7, and 11.

1. INTRODUCTION

In his paper [3], [6, pp. 210–213], Ramanujan gave proofs, for the first time, of the the first two of his three famous congruences for the partition function $p(n)$, namely,

$$p(5n + 4) \equiv 0 \pmod{5}, \quad (1.1)$$

$$p(7n + 5) \equiv 0 \pmod{7}, \quad (1.2)$$

$$p(11n + 6) \equiv 0 \pmod{11}, \quad (1.3)$$

where n is any nonnegative integer. The congruence (1.3) was stated for the first time by Ramanujan in [4], [6, p. 230]. After Ramanujan died in 1920, G. H. Hardy [5], [6, pp. 232–238] extracted proofs of (1.1)–(1.3) from an unpublished manuscript of Ramanujan on $p(n)$ and $\tau(n)$. This manuscript was published for the first time in handwritten form in a volume [7] containing Ramanujan's lost notebook. An expanded and annotated version was prepared by the author and K. Ono [1]. The proofs in [5] employ Eisenstein series.

Our objective in this paper is to give new proofs of (1.1)–(1.3) that also utilize Eisenstein series. The principal idea behind the new proofs is actually due to Ramanujan in the aforementioned manuscript. To describe this idea, we need to define Ramanujan's Eisenstein series. Let

$$P := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}, \quad (1.4)$$

$$Q := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad (1.5)$$

and

$$R := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}, \quad (1.6)$$

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where here and in the sequel $|q| < 1$. These Eisenstein series satisfy Ramanujan's famous differential equations [2, eq. (30)]

$$q \frac{dP}{dq} = \frac{P^2 - Q}{12}, \quad q \frac{dQ}{dq} = \frac{PQ - R}{3}, \quad q \frac{dR}{dq} = \frac{PR - Q^2}{2}. \quad (1.7)$$

Moreover, Q and R satisfy the well-known discriminant relation [2, eq. (44)]

$$Q^3 - R^2 = 1728q(q; q)_\infty^{24} =: \sum_{n=1}^{\infty} \tau(n)q^n. \quad (1.8)$$

In his manuscript, Ramanujan proves or states further congruences for $p(n)$. In particular, he proves that $p(n) \equiv 11\tau(n) \pmod{13}$. To prove this congruence, Ramanujan establishes the “identity”

$$(q^{13}; q^{13})_\infty \sum_{n=1}^{\infty} p(13n - 7)q^n = 11 \sum_{n=1}^{\infty} \tau(n)q^n + 13J. \quad (1.9)$$

He then writes, “It is not necessary to know all the details above in order to prove (1.9). The proof can be very much simplified as follows; using (1.7) and ... we can show that

$$(Q^3 - R^2)^7 = q \frac{dJ}{dq} + 3(Q^3 - R^2) + 13J.” \quad (1.10)$$

Here $J = J(q)$ is a power series in q with integral coefficients. We use this notation of Ramanujan throughout the paper and emphasize that J is not necessarily the same at each occurrence. Apparently, Ramanujan realized this simplification at precisely this juncture while writing his paper, for he did not return to his proofs of (1.1)–(1.3) to utilize this observation and thereby simplify them. Ramanujan's idea allows one to give more natural proofs of (1.1)–(1.3) via Eisenstein series. Instead of using further (more complicated) Eisenstein series identities, we employ the differential equations (1.7). In his doctoral dissertation, J. M. Rushforth [8] used this idea to simplify Ramanujan's proof of (1.3), as extracted by Hardy for [5]. In this paper, we use Ramanujan's observation above along with (1.7) and (1.8) to give simplified proofs of (1.1)–(1.3), with the proof of (1.3) being precisely that of Rushforth.

For the proofs of (1.2) and (1.3), it is still necessary to use three further basic Eisenstein series from Table I of Ramanujan's paper [2], [6, p. 141] (but not nearly as many identities as used in [5]), namely,

$$Q^2 = 1 + 480 \sum_{n=1}^{\infty} \frac{n^7 q^n}{1 - q^n}, \quad (1.11)$$

$$QR = 1 - 264 \sum_{n=1}^{\infty} \frac{n^9 q^n}{1 - q^n}, \quad (1.12)$$

and

$$441Q^3 + 250R^2 = 691 + 65520 \sum_{n=1}^{\infty} \frac{n^{11} q^n}{1 - q^n}. \quad (1.13)$$

2. PROOFS OF (1.1)–(1.3)

Theorem 2.1. *For each nonnegative integer n ,*

$$p(5n + 4) \equiv 0 \pmod{5}. \quad (2.1)$$

Proof. From the definitions (1.5) and (1.6), respectively,

$$Q = 1 + 5J \quad \text{and} \quad R = P + 5J, \quad (2.2)$$

since $n^5 \equiv n$ by Fermat's little theorem. It follows from (2.2) and (1.7) that

$$\begin{aligned} Q^3 - R^2 &= Q(1 + 5J)^2 - (P + 5J)^2 = Q - P^2 + 5J \\ &= -12q \frac{dP}{dq} + 5J = 3q \frac{dP}{dq} + 5J. \end{aligned} \quad (2.3)$$

But, by (1.8) and the binomial theorem,

$$Q^3 - R^2 = 1728q(q; q)_\infty^{24} = 3q \frac{(q; q)_\infty^{25}}{(q; q)_\infty} + 5J = 3q \frac{(q^5; q^5)_\infty^5}{(q; q)_\infty} + 5J. \quad (2.4)$$

Combining (2.3) and (2.4) and using the generating function for $p(n)$, we find that

$$q(q^5; q^5)_\infty^5 \sum_{n=0}^{\infty} p(n)q^n = q \frac{dP}{dq} + 5J. \quad (2.5)$$

We now equate those terms on both sides of (2.5) whose powers are of the form q^{5n} to find that

$$(q^5; q^5)_\infty^5 \sum_{n=0}^{\infty} p(5n + 4)q^{5n+5} = 5J. \quad (2.6)$$

Ramanujan's congruence (2.1) follows immediately from (2.6). \square

Theorem 2.2. *For each nonnegative integer n ,*

$$p(7n + 5) \equiv 0 \pmod{7}. \quad (2.7)$$

Proof. The first two steps of our proof are the same as those of Ramanujan and Hardy [5], [6, p. 235]. From the definition of R , it is obvious that

$$R = 1 + 7J. \quad (2.8)$$

Using (1.11), Fermat's little theorem, and the definition (1.4), we also find that

$$Q^2 = P + 7J. \quad (2.9)$$

Hence, from (2.8), (2.9), and (1.7),

$$\begin{aligned} (Q^3 - R^2)^2 &= (PQ - 1 + 7J)^2 \\ &= P^2Q^2 - 2PQ + 1 + 7J \\ &= P(P^2 - Q) - PQ + R + 7J \\ &= 5Pq \frac{dP}{dq} - 3q \frac{dQ}{dq} + 7J \\ &= 6q \frac{d}{dq} P^2 - 3q \frac{dQ}{dq} + 7J = q \frac{dJ}{dq} + 7J. \end{aligned} \quad (2.10)$$

On the other hand, by (1.8) and the binomial theorem,

$$(Q^3 - R^2)^2 = q^2 \frac{(q; q)_\infty^{49}}{(q; q)_\infty} = q^2 \frac{(q^7; q^7)_\infty^7}{(q; q)_\infty} + 7J = (q^7; q^7)_\infty^7 \sum_{n=0}^{\infty} p(n)q^{n+2} + 7J. \quad (2.11)$$

We now equate the right sides of (2.10) and (2.11) and then extract those terms involving q^7 . Equating these terms, we find that

$$(q^7; q^7)_\infty^7 \sum_{n=0}^{\infty} p(7n+5)q^{7n+7} = 7J,$$

from whence (2.7) immediately follows. \square

Theorem 2.3. *For each nonnegative integer n ,*

$$p(11n+6) \equiv 0 \pmod{11}. \quad (2.12)$$

Proof. The beginning of the proof is identical to that of Ramanujan as related by Hardy [5], [6, pp. 235–236]. Let it suffice to say that using (1.13), one can easily show that

$$Q^3 - 3R^2 = -2P + 11J, \quad (2.13)$$

and that from (1.12), it is trivial that

$$QR = 1 + 11J. \quad (2.14)$$

Using (2.13) and (2.14), Ramanujan and Hardy [5] then prove that

$$(Q^3 - R^2)^5 = P^5 - 3P^3Q - 4P^2R + 6QR + 11J. \quad (2.15)$$

We refer readers to the details in [5], [6, p. 236] demonstrating (2.15).

Invoking Ramanujan's differential equations (1.7), we find that

$$\begin{aligned} P^5 - 3P^3Q - 4P^2R + 6QR &= (P^3 + 3PQ + 5R)(P^2 - Q) - 5P^2(PQ - R) \\ &\quad - 3P(PR - Q^2) + 11J \\ &= 12(P^3 + 3PQ + 5R)q \frac{dP}{dq} - 15P^2q \frac{dQ}{dq} - 6Pq \frac{dR}{dq} + 11J \\ &= (P^3 + 3PQ + 5R)q \frac{dP}{dq} - 4P^2q \frac{dQ}{dq} + 5Pq \frac{dR}{dq} + 11J \\ &= 3q \frac{d}{dq} P^4 - 4q \frac{d}{dq} (P^2Q) + 5q \frac{d}{dq} (PR) + 11J \\ &= q \frac{dJ}{dq} + 11J. \end{aligned} \quad (2.16)$$

Now, by (1.8), the binomial theorem, (2.15), and (2.16),

$$(Q^3 - R^2)^5 = q^5 \frac{(q; q)_\infty^{121}}{(q; q)_\infty} = q^5 (q^{11}; q^{11})_\infty^{11} \sum_{n=0}^{\infty} p(n)q^n = q \frac{dJ}{dq} + 11J. \quad (2.17)$$

Equating the terms involving the powers q^{11n} on both sides of (2.17), we easily deduce that

$$(q^{11}; q^{11})_{\infty}^{11} \sum_{n=0}^{\infty} p(11n+6)q^{11n+11} = 11J,$$

from which (2.12) is apparent. \square

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