

# AN INTEGRAL ANALOGUE OF THETA FUNCTIONS AND GAUSS SUMS IN RAMANUJAN'S LOST NOTEBOOK

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**Abstract.** One page in Ramanujan's lost notebook is devoted to claims about a certain integral with two parameters. One claim gives an inversion formula for the integral that is similar to the transformation formula for theta functions. Other claims are remindful of Gauss sums. In this paper, we prove all the claims made by Ramanujan about this integral.

## 1. INTRODUCTION

In two papers [3], [4], [5, pp. 59–67, 202–207], Ramanujan examined the properties of two integrals

$$\phi_w(t) := \int_0^\infty \frac{\cos(\pi tx)}{\cosh(\pi x)} e^{-\pi wx^2} dx, \quad (1.1)$$

$$\psi_w(t) := \int_0^\infty \frac{\sin(\pi tx)}{\sinh(\pi x)} e^{-\pi wx^2} dx, \quad (1.2)$$

which can be regarded as continuous integral analogues of Gauss sums from one point of view or of theta functions from another point of view. These integrals are also briefly discussed in a two-page fragment copied by G. N. Watson from Ramanujan's "loose papers" and published with Ramanujan's lost notebook [6, pp. 221–222]. A thorough discussion of this two-page fragment can be found in the fourth book by G. E. Andrews and the first author on Ramanujan's lost notebook [1].

Page 198 of [6] is an isolated page that is actually part of the original lost notebook, and its contents are related to the two aforementioned papers by Ramanujan and the fragment on pages 221–222 of [6]. On this page, Ramanujan records theorems, much in the spirit of those for  $\phi_w(t)$  and  $\psi_w(t)$ , for the function

$$F_w(t) := \int_0^\infty \frac{\sin(\pi tx)}{\tanh(\pi x)} e^{-\pi wx^2} dx. \quad (1.3)$$

The formulas claimed by Ramanujan on page 198 without proofs are difficult to read, partly because the original page was perhaps a thin, colored piece of paper, for example, a piece of parchment paper, that was difficult for the photographers of [6] to photocopy. Since Ramanujan never discussed the results on page 198 in any of his papers and since no one else has apparently ever discussed them as well, it is the objective of this paper to prove them and to also convince readers that  $F_w(t)$  is a beautiful continuous integral analogue of either theta functions or Gauss sums.

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## 2. SOME THEOREMS ABOUT $\phi_w(t)$ AND $\psi_w(t)$

Recall that  $\phi_w(t)$  and  $\psi_w(t)$  are defined by (1.1) and (1.2), respectively. We first note that

$$\phi_w(t) = \phi_w(-t) \quad \text{and} \quad \psi_w(t) = -\psi_w(-t). \quad (2.1)$$

In this brief section we recall some of Ramanujan's theorems for these two functions from [3] and [4]. More detailed proofs can be found in [1].

**Theorem 2.1.** [5, p. 202, eq. (1)] *For  $w > 0$ ,*

$$\phi_w(t) = \frac{1}{\sqrt{w}} e^{-\pi t^2/(4w)} \phi_{1/w}(it/w). \quad (2.2)$$

**Theorem 2.2.** [5, p. 202] *We have*

$$e^{\pi(t+w)^2/(4w)} \phi_w(t+w) = e^{\pi t^2/(4w)} \left( \frac{1}{2} + \psi_w(t) \right). \quad (2.3)$$

**Theorem 2.3.** [5, p. 203, eq. (4)] *We have*

$$\frac{1}{2} + \psi_w(t+i) = \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{ \frac{1}{2} - \psi_{1/w} \left( \frac{it}{w} + i \right) \right\}. \quad (2.4)$$

**Theorem 2.4.** [5, p. 203, eq. (5)] *We have*

$$\phi_w(t+i) + \phi_w(t-i) = \frac{1}{\sqrt{w}} e^{-\pi t^2/(4w)}. \quad (2.5)$$

**Theorem 2.5.** [5, p. 203, eq. (8)] *We have*

$$e^{\pi(t+w)^2/(4w)} \left\{ \frac{1}{2} - \psi_w(t+w) \right\} = e^{\pi(t-w)^2/(4w)} \left\{ \frac{1}{2} + \psi_w(t-w) \right\}. \quad (2.6)$$

**Theorem 2.6.** [5, p. 203, eq. (10)] *If  $n$  is any positive integer,*

$$\psi_w(t) - \psi_w(t+2ni) = -\frac{i}{\sqrt{w}} \sum_{k=0}^{n-1} e^{-\pi(t+(2k+1)i)^2/(4w)}. \quad (2.7)$$

Observe that Theorem 2.6 indicates that the function  $\psi_w(t)$  possesses a “quasi-period”  $2i$ , and that the right-hand side of (2.7) is an analogue of a Gauss sum.

## 3. THE CLAIMS ON PAGE 198 IN THE LOST NOTEBOOK

We note immediately from the definition (1.3) that

$$F_w(t) = -F_w(-t). \quad (3.1)$$

**Entry 3.1.** *We have*

$$F_w(t) = -\frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} F_{1/w}(it/w). \quad (3.2)$$

The beautiful transformation formula (3.2) shows that  $F_w(t)$  is an integral analogue of theta functions.

*Proof.* Write

$$\begin{aligned}
F_w(t) &= \int_0^\infty \frac{\sin(\pi tx) \cosh(\pi x)}{\sinh(\pi x)} e^{-\pi wx^2} dx \\
&= \int_0^\infty \frac{\sin(\pi tx) \cos(i\pi x)}{\sinh(\pi x)} e^{-\pi wx^2} dx \\
&= \frac{1}{2} \int_0^\infty \frac{\sin(t+i)\pi x + \sin(t-i)\pi x}{\sinh(\pi x)} e^{-\pi wx^2} dx \\
&= \frac{1}{2} \{ \psi_w(t+i) + \psi_w(t-i) \}, \tag{3.3}
\end{aligned}$$

by (1.2). Recall from (2.4) that

$$\frac{1}{2} + \psi_w(t+i) = \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{ \frac{1}{2} - \psi_{1/w} \left( \frac{it}{w} + i \right) \right\}. \tag{3.4}$$

Since  $\psi(t)$  is odd, we find from (3.4) that

$$-\frac{1}{2} + \psi_w(t-i) = -\frac{1}{2} - \psi_w(-t+i) = -\frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{ \frac{1}{2} - \psi_{1/w} \left( -\frac{it}{w} + i \right) \right\}. \tag{3.5}$$

Hence, from (3.3)–(3.5),

$$\begin{aligned}
F_w(t) &= \frac{1}{2} \left\{ \frac{1}{2} + \psi_w(t+i) - \frac{1}{2} + \psi_w(t-i) \right\} \\
&= \frac{1}{2} \left( \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{ \frac{1}{2} - \psi_{1/w} \left( \frac{it}{w} + i \right) \right\} \right. \\
&\quad \left. - \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{ \frac{1}{2} - \psi_{1/w} \left( -\frac{it}{w} + i \right) \right\} \right) \\
&= \frac{i}{2\sqrt{w}} e^{-\pi t^2/(4w)} \left( -\psi_{1/w} \left( \frac{it}{w} + i \right) + \psi_{1/w} \left( -\frac{it}{w} + i \right) \right) \\
&= -\frac{i}{2\sqrt{w}} e^{-\pi t^2/(4w)} \left( \psi_{1/w} \left( \frac{it}{w} + i \right) + \psi_{1/w} \left( \frac{it}{w} - i \right) \right) \\
&= -\frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} F_{1/w}(it/w),
\end{aligned}$$

by (3.3), and this completes the proof.  $\square$

**Entry 3.2.** *If  $n$  is any positive integer, then*

$$F_w(t) - F_w(t+2ni) = -\frac{i}{\sqrt{w}} \sum_{j=0}^{n'} e^{-\pi(t+2ji)^2/(4w)}, \tag{3.6}$$

where the prime  $'$  on the summation sign indicates that the terms with  $j=0, n$  are to be multiplied by  $\frac{1}{2}$ .

Entry 3.2 is an analogue of Theorem 2.6 and demonstrates that  $F_w(t)$  has a quasi-period  $2i$ .

*Proof.* Recall from (3.3) that

$$F_w(t) = \frac{1}{2}\{\psi_w(t+i) + \psi_w(t-i)\}, \quad (3.7)$$

and so

$$\begin{aligned} & F_w(t) - F_w(t+2ni) \\ &= \frac{1}{2}\{\psi_w(t+i) - \psi_w(t+(2n+1)i)\} + \frac{1}{2}\{\psi_w(t-i) - \psi_w(t+(2n-1)i)\}. \end{aligned}$$

Applying Entry 2.6 on the right side above, we see that

$$\begin{aligned} & F_w(t) - F_w(t+2ni) \\ &= \frac{1}{2} \left\{ -\frac{i}{\sqrt{w}} \sum_{k=0}^{n-1} e^{-\pi(t+(2k+2)i)^2/(4w)} - \frac{i}{\sqrt{w}} \sum_{k=0}^{n-1} e^{-\pi(t+2ki)^2/(4w)} \right\} \\ &= -\frac{i}{\sqrt{w}} \sum_{j=0}^{n'} e^{-\pi(t+2ji)^2/(4w)}. \end{aligned}$$

This concludes the proof.  $\square$

**Entry 3.3.** *If  $n$  is a positive integer, then*

$$F_w(t) - e^{\pi n(t+nw)} F_w(t+2nw) = -e^{-\pi t^2/(4w)} \sum_{j=0}^{n'} e^{\pi(t+2jw)^2/(4w)}, \quad (3.8)$$

where the prime on the summation sign has the same meaning as in Entry 3.2.

Observe that Entry 3.3 indicates that  $2w$  is a quasi-period for  $F_w(t)$ . The functions  $\phi_w(t)$  and  $\psi_w(t)$  also possess the same quasi-period.

*Proof.* Replacing  $t$  by  $t+i$  and  $t-i$  in Theorem 2.5, we deduce, respectively, that

$$\begin{aligned} & e^{\pi(t+i+w)^2/4w} \psi_w(t+i+w) + e^{\pi(t+i-w)^2/4w} \psi_w(t+i-w) \\ &= \frac{1}{2}(e^{\pi(t+i+w)^2/4w} - e^{\pi(t+i-w)^2/4w}), \quad (3.9) \end{aligned}$$

and

$$\begin{aligned} & e^{\pi(t-i+w)^2/4w} \psi(t-i+w) + e^{\pi(t-i-w)^2/4w} \psi(t-i-w) \\ &= \frac{1}{2}(e^{\pi(t-i+w)^2/4w} - e^{\pi(t-i-w)^2/4w}). \quad (3.10) \end{aligned}$$

Now observe that  $e^{4\pi i(t+w)/4w} = e^{4\pi i(t-w)/4w}$ . We multiply  $e^{\pi(t-i+w)^2/4w}$  in its two appearances in (3.10) by  $e^{4\pi i(t+w)/4w}$ , and we multiply  $e^{\pi(t-i-w)^2/4w}$  in its two appearances in (3.10) by  $e^{4\pi i(t-w)/4w}$ . Thus, (3.10) can be recast in the form

$$\begin{aligned} & e^{\pi(t+i+w)^2/4w} \psi(t-i+w) + e^{\pi(t+i-w)^2/4w} \psi(t-i-w) \\ &= \frac{1}{2}(e^{\pi(t+i+w)^2/4w} - e^{\pi(t+i-w)^2/4w}). \quad (3.11) \end{aligned}$$

Using (3.7), we find that

$$\begin{aligned}
& e^{\pi(t+i+w)^2/(4w)} F_w(t+w) + e^{\pi(t+i-w)^2/(4w)} F_w(t-w) \\
&= \frac{1}{2} \{ e^{\pi(t+i+w)^2/4w} \psi_w(t+i+w) + e^{\pi(t+i+w)^2/4w} \psi_w(t-i+w) \\
&\quad + e^{\pi(t+i-w)^2/4w} \psi_w(t+i-w) + e^{\pi(t+i-w)^2/4w} \psi_w(t-i-w) \} \\
&= \frac{1}{2} (e^{\pi(t+i+w)^2/4w} - e^{\pi(t+i-w)^2/4w}). \tag{3.12}
\end{aligned}$$

We now apply (3.12) with  $t$  successively replaced by  $t+w, t+3w, \dots, t+(2n-1)w$  to deduce the  $n$  equations

$$\begin{aligned}
& e^{\pi(t+i+2w)^2/(4w)} F_w(t+2w) + e^{\pi(t+i)^2/(4w)} F_w(t) \\
&\quad = \frac{1}{2} (e^{\pi(t+i+2w)^2/(4w)} - e^{\pi(t+i)^2/(4w)}), \\
& e^{\pi(t+i+4w)^2/(4w)} F_w(t+4w) + e^{\pi(t+i+2w)^2/(4w)} F_w(t+2w) \\
&\quad = \frac{1}{2} (e^{\pi(t+i+4w)^2/(4w)} - e^{\pi(t+i+2w)^2/(4w)}), \\
&\quad \quad \quad \vdots \\
& e^{\pi(t+i+2nw)^2/(4w)} F_w(t+2nw) + e^{\pi(t+i+(2n-2)w)^2/(4w)} F_w(t+(2n-2)w) \\
&\quad = \frac{1}{2} (e^{\pi(t+i+2nw)^2/(4w)} - e^{\pi(t+i+(2n-2)w)^2/(4w)}).
\end{aligned}$$

Alternately adding and subtracting the identities above, we conclude that

$$\begin{aligned}
& e^{\pi(t+i)^2/(4w)} F_w(t) + (-1)^{n+1} e^{\pi(t+i+2nw)^2/(4w)} F_w(t+2nw) \\
&\quad = \sum_{j=0}^{n-1} (-1)^{j+1} e^{\pi(t+i+2jw)^2/(4w)},
\end{aligned}$$

that is to say,

$$F_w(t) - e^{\pi n(t+2nw)^2/(4w)} F_w(t+2nw) = -e^{-\pi t^2/(4w)} \sum_{j=0}^{n-1} e^{\pi(t+2jw)^2/(4w)},$$

which completes our proof.  $\square$

**Entry 3.4.** Let  $s = t + 2\eta_1 m w + 2\eta_2 n i$ , where  $\eta_1^2 = \eta_2^2 = 1$ , and where  $m$  and  $n$  are positive integers. Then

$$\begin{aligned}
F_w(s) + (-1)^{mn-1} e^{-\frac{1}{2}\pi\eta_1 m(s+t)} F_w(t) &= \eta_1 e^{-\pi s^2/(4w)} \sum_{j=0}^{m-1} e^{\pi(s-2j\eta_1 w)^2/(4w)} \\
&\quad + \eta_2 (-1)^{mn} \frac{i}{\sqrt{w}} e^{-\frac{1}{2}\pi\eta_1 m(s+t)} \sum_{j=0}^{n-1} e^{-\pi(t+2\eta_2 j i)^2/(4w)}, \tag{3.13}
\end{aligned}$$

where the prime on the summation signs has the same meaning as in the two previous entries.

*Proof.* If we examine the proof of Entry 3.3 with  $n$  replaced by  $-n$ , we see that we obtain the same result, but with the right-hand side multiplied by  $-1$ . Thus, we shall apply Entry 3.3 with  $n$  replaced by  $m$  and  $t$  replaced by  $t + 2\eta_2ni$ , but since in the notation of the present entry  $m > 0$ , we must multiply the right-hand side by  $\eta_1$ , and so we obtain

$$\begin{aligned} F_w(t + 2\eta_2ni) - e^{\pi\eta_1m(t+2\eta_2ni+\eta_1mw)} F_w(t + 2\eta_2ni + 2\eta_1mw) \\ = -\eta_1 e^{-\pi(t+2\eta_2ni)^2/(4w)} \sum_{j=0}^{m'} e^{\pi(t+2\eta_2ni+2\eta_1jw)^2/(4w)}. \end{aligned}$$

Using the definition of  $s$ , we can reformulate the foregoing equality as

$$\begin{aligned} F_w(t + 2\eta_2ni) - e^{\pi\eta_1m(s-\eta_1mw)} F_w(s) \\ = -\eta_1 e^{-\pi(s-2\eta_1mw)^2/(4w)} \sum_{j=0}^{m'} e^{\pi(s-2\eta_1mw+2\eta_1jw)^2/(4w)} \\ = -\eta_1 e^{-\pi(s-2\eta_1mw)^2/(4w)} \sum_{j=0}^{m'} e^{\pi(s-2\eta_1jw)^2/(4w)}. \end{aligned}$$

If we examine the proof of Entry 3.2 with  $n$  replaced by  $-n$ , we see that the identity still holds except that the right-hand side must now be multiplied by  $-1$ . Since in the present notation  $n > 0$ , then if we apply Entry 3.2 to  $F_w(t + 2\eta_2ni)$  above, we must multiply the right-hand side by  $\eta_2$ . Hence,

$$\begin{aligned} F_w(t) - e^{\pi\eta_1m(s-\eta_1mw)} F_w(s) \\ = -\eta_1 e^{-\pi(s-2\eta_1mw)^2/(4w)} \sum_{j=0}^{m'} e^{\pi(s-2\eta_1jw)^2/(4w)} - \eta_2 \frac{i}{\sqrt{w}} \sum_{j=0}^{n'} e^{-\pi(t+2\eta_2ji)^2/(4w)}. \end{aligned}$$

Upon multiplying both sides above by  $e^{-\pi\eta_1m(s-\eta_1mw)}$  and simplifying, we find that

$$\begin{aligned} F_w(s) + (-1)^{mn-1} e^{-\frac{1}{2}\pi\eta_1m(s+t)} F_w(t) = -\eta_1 e^{-\pi s^2/(4w)} \sum_{j=0}^{m'} e^{\pi(s-2j\eta_1w)^2/(4w)} \\ + \eta_2 (-1)^{mn} \frac{i}{\sqrt{w}} e^{-\frac{1}{2}\pi\eta_1m(s+t)} \sum_{j=0}^{n'} e^{\pi(t+2\eta_2ji)^2/(4w)}, \end{aligned}$$

where we used the fact that

$$(-1)^{mn} e^{-\frac{1}{2}\pi\eta_1m(s+t)} = e^{-\pi\eta_1m(s-\eta_1mw)}.$$

This completes our proof.  $\square$

## 4. EXAMPLES

If we set  $s = t$  in Entry 3.4, it follows that  $w = -(\eta_2 ni)/(\eta_1 m)$ . If we further suppose that both  $m$  and  $n$  are odd, then (3.13) reduces to the identity

$$(1 + e^{-\pi\eta_1 mt})F_w(t) = \eta_1 e^{-\pi t^2/4w} \sum_{j=0}^{m'} e^{\pi(t-2j\eta_1 w)^2/(4w)} - \eta_2 \frac{i}{\sqrt{w}} e^{-\pi\eta_1 mt} \sum_{j=0}^{n'} e^{-\pi(t+2\eta_2 j)^2/(4w)}.$$

In the identity above, first let  $\eta_1 = 1, \eta_2 = -1$  and multiply both sides by  $e^{mt}$ . Secondly, let  $\eta_1 = -1, \eta_2 = 1$  and multiply both sides by  $e^{-mt}$ . Replace  $t$  by  $2t/\pi$  in each identity. We then respectively obtain the two identities

$$\begin{aligned} & 2 \cosh(mt) \int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)} e^{-\frac{\pi n x^2}{m} i} dx & (4.1) \\ &= \frac{1}{2} e^{mt} + e^{(m-2)t + \frac{\pi n}{m} i} + e^{(m-4)t + \frac{4\pi n}{m} i} + \dots + \frac{1}{2} e^{-mt + \pi m n i} \\ &+ \sqrt{\frac{m}{n}} \left\{ \frac{1}{2} e^{-mt + (\frac{mt^2}{\pi n} + \frac{\pi}{4})i} + e^{(\frac{2}{n}-1)mt + [(\frac{t^2}{\pi^2}-1)\frac{\pi m}{n} + \frac{\pi}{4}]i} + \dots + \frac{1}{2} e^{mt + [(\frac{t^2}{\pi^2}-n^2)\frac{\pi m}{n} + \frac{\pi}{4}]i} \right\} \end{aligned}$$

and

$$\begin{aligned} & 2 \cosh(mt) \int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)} e^{-\frac{\pi n x^2}{m} i} dx & (4.2) \\ &= -\frac{1}{2} e^{-mt} - e^{(2-m)t + \frac{\pi n}{m} i} - e^{(4-m)t + \frac{4\pi n}{m} i} + \dots - \frac{1}{2} e^{mt + \pi m n i} \\ &- \sqrt{\frac{m}{n}} \left\{ \frac{1}{2} e^{mt + (\frac{mt^2}{\pi n} + \frac{\pi}{4})i} + e^{(1-\frac{2}{n})mt + [(\frac{t^2}{\pi^2}-1)\frac{\pi m}{n} + \frac{\pi}{4}]i} + \dots + \frac{1}{2} e^{-mt + [(\frac{t^2}{\pi^2}-n^2)\frac{\pi m}{n} + \frac{\pi}{4}]i} \right\}. \end{aligned}$$

Next add (4.1) and (4.2), divide both sides by 2, and equate the real and imaginary parts on both sides to obtain the two identities

$$\begin{aligned} & 2 \cosh(mt) \int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)} \cos \frac{\pi n x^2}{m} dx \\ &= \frac{1}{2} \sinh\{mt\} + \sinh\{(m-2)t\} \cos \frac{\pi n}{m} + \sinh\{(m-4)t\} \cos \frac{4\pi n}{m} \\ &+ \dots + \frac{1}{2} \sinh\{-mt\} \cos(\pi m n) \\ &+ \sqrt{\frac{m}{n}} \left\{ \frac{1}{2} \sinh\{-mt\} \cos \left( \frac{mt^2}{\pi n} + \frac{\pi}{4} \right) + \sinh \left\{ \left( \frac{2}{n} - 1 \right) mt \right\} \cos \left( \left( \frac{t^2}{\pi^2} - 1 \right) \frac{\pi m}{n} + \frac{\pi}{4} \right) \right. \\ &+ \dots + \left. \frac{1}{2} \sinh\{mt\} \cos \left( \left( \frac{t^2}{\pi^2} - n^2 \right) \frac{\pi m}{n} + \frac{\pi}{4} \right) \right\} & (4.3) \end{aligned}$$

and

$$\begin{aligned}
& -2 \cosh(mt) \int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)} \sin \frac{\pi n x^2}{m} dx \\
= & \sinh\{(m-2)t\} \sin \frac{\pi n}{m} + \sinh\{(m-4)t\} \sin \frac{4\pi n}{m} + \cdots + \frac{1}{2} \sinh\{-mt\} \sin(\pi mn) \\
& + \sqrt{\frac{m}{n}} \left\{ \frac{1}{2} \sinh\{-mt\} \sin \left( \frac{mt^2}{\pi n} + \frac{\pi}{4} \right) + \sinh \left\{ \left( \frac{2}{n} - 1 \right) mt \right\} \sin \left( \left( \frac{t^2}{\pi^2} - 1 \right) \frac{\pi m}{n} + \frac{\pi}{4} \right) \right. \\
& \left. + \cdots + \frac{1}{2} \sinh\{mt\} \sin \left( \left( \frac{t^2}{\pi^2} - n^2 \right) \frac{\pi m}{n} + \frac{\pi}{4} \right) \right\}. \tag{4.4}
\end{aligned}$$

Using (4.3) and (4.4), we can evaluate several definite integrals. For example, if we set  $m = n = 1$  in (4.3) and (4.4), we find that, respectively,

$$\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)} \cos(\pi x^2) dx = \frac{\sinh t}{2 \cosh t} \left( 1 - \cos \left( \frac{t^2}{\pi} + \frac{\pi}{4} \right) \right),$$

and

$$\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)} \sin(\pi x^2) dx = \frac{\sinh t}{2 \cosh t} \sin \left( \frac{t^2}{\pi} + \frac{\pi}{4} \right).$$

These evaluations can be found in [2, p. 542, formulas 3.991, nos. 1, 2], respectively. No further cases of (4.3) and (4.4) can be found in [2].

## 5. ONE FURTHER INTEGRAL

There is one further integral, namely,

$$G_w(t) := \int_0^\infty \frac{\sin(\pi t x)}{\coth(\pi x)} e^{-\pi w x^2} dx,$$

which can be placed in the theory of  $\phi_w(t)$ ,  $\psi_w(t)$ , and  $F_w(t)$ . First,

$$\begin{aligned}
G_w(t) &= \int_0^\infty \frac{\sin(\pi t x) \sinh(\pi x)}{\cosh(\pi x)} e^{-\pi w x^2} dx \\
&= -i \int_0^\infty \frac{\sin(\pi t x) \sin(i\pi x)}{\cosh(\pi x)} e^{-\pi w x^2} dx \\
&= \frac{i}{2} \int_0^\infty \frac{\cos\{\pi x(t+i)\} - \cos\{\pi x(t-i)\}}{\cosh(\pi x)} e^{-\pi w x^2} dx \\
&= \frac{i}{2} \{\phi_w(t+i) - \phi_w(t-i)\}, \tag{5.1}
\end{aligned}$$

by (1.1). The formula (5.1) should be compared with (2.5).

The next result shows that the theory of  $G_w(t)$  can be recast in the theory of  $\psi_w(t)$ .

**Theorem 5.1.** *If  $w' := 1/w$ , then*

$$G_w(t) = \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \psi_{w'}(-itw').$$

*Proof.* Using, in order, (5.1), Theorem 2.1, (2.1), Theorem 2.2, and (2.1), we find that

$$\begin{aligned}
G_w(t) &= \frac{i}{2\sqrt{w}} e^{-\pi(t+i)^2/(4w)} \phi_{1/w} \left( \frac{it}{w} - \frac{1}{w} \right) - \frac{i}{2\sqrt{w}} e^{-\pi(t-i)^2/(4w)} \phi_{1/w} \left( \frac{it}{w} + \frac{1}{w} \right) \\
&= \frac{i}{2\sqrt{w}} e^{-\pi(t^2-1)/(4w)} \left\{ e^{-i\pi t/(2w)} \phi_{1/w} \left( -\frac{it}{w} + \frac{1}{w} \right) - e^{i\pi t/(2w)} \phi_{1/w} \left( \frac{it}{w} + \frac{1}{w} \right) \right\} \\
&= \frac{i}{2\sqrt{w}} e^{-\pi(t^2-1)/(4w)} \left\{ e^{-\pi/(4w)} \left( \frac{1}{2} + \psi_{1/w} \left( -\frac{it}{w} \right) \right) - e^{-\pi/(4w)} \left( \frac{1}{2} + \psi_{1/w} \left( \frac{it}{w} \right) \right) \right\} \\
&= \frac{i}{2\sqrt{w}} e^{-\pi t^2/(4w)} \{ \psi_{w'}(-itw') - \psi_{w'}(itw') \} \\
&= \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \psi_{w'}(-itw'),
\end{aligned}$$

which completes the proof. □

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