IDENTITIES FOR PARTITIONS WITH DISTINCT COLORS

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ABSTRACT. We establish several identities for partitions with distinct colors that arise from Ramanujan's formulas for multipliers in the theory of modular equations. In the course of our investigations, we prove three conjectures made by C. Sandon and F. Zanella. However, combinatorial proofs of these three conjectures and most of the theorems in this paper remain to be given.

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1. INTRODUCTION

Our study has its origin in the work of H. M. Farkas and I. Kra [12], who observed that certain theta function identities can be transformed into remarkable identities for partition functions. Perhaps their most interesting partition identity is given in the following theorem.

Theorem 1.1. *Let S denote the set consisting of one copy of the positive integers and one additional copy of those positive integers that are multiples of 7. Then for each positive integer k , the number of partitions of $2k$ into even elements of S is equal to the number of partitions of $2k + 1$ into odd elements of S .*

M. D. Hirschhorn [14] gave a simple proof of Theorem 1.1, and the referee of [14] observed that the partition identity was equivalent to a modular equation of degree 7 due to Guetzlaff [13], [7, Chapter 19, Entry 19(i)].

The first author [9] of the present paper noted that Theorem 1.1 is one of five theorems of this sort that can be derived from modular equations of Russell-type, first studied extensively by H. Schröter [23], and later by S. Ramanujan [17] and R. Russell [19], [20]. Each of the five theorems is slightly different from the others, and it is not difficult to show that each is unique, i.e., there are no further theorems in which, for example, “7” in Theorem 1.1, can be replaced by another prime.

N. D. Baruah and the first author continued this study [4], [5] by examining further modular equations of Ramanujan from his notebooks [17] and demonstrating how they can be recast into interesting partition identities. Most of these modular equations were either of Russell-type or of a kind that can be derived from general formulas of Schröter [23]; see also [7, pp. 65–76].

Farkas and Kra [12] were also the first to raise the natural question of finding a bijective proof of the partition identity of Theorem 1.1. In that direction, greatly extending an idea of

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E. M. Wright [25], S. O. Warnaar [24] combinatorially established a very general theta function identity that includes that employed by Farkas and Kra [12]. Baruah and the first author [4] later showed that Warnaar's general theta function identity is equivalent to an identity of Ramanujan from his second notebook [17], [7, p. 47]. Warnaar further emphasized that bijective proofs of the resulting partition identities were greatly desired. Building on the work of Warnaar, the first bijective proof of Theorem 1.1 was constructed by S. Kim [15], who also found a bijective proof of the partition identity corresponding to Legendre's modular equation of degree 3 [16], which was rediscovered by Ramanujan [17], [7, pp. 230, 232]. C. Sandon and F. Zanello [21] extended S. Kim's ideas and consequently found bijective proofs of partition identities arising from modular equations of degrees 5 and 11. The final partition identity from [9], arising from a modular equation of degree 23, remains without a bijective proof.

Continuing their study, Sandon and Zanello [22] discovered, but did not prove, a panoply of fascinating partition identities. Three are proved in the present paper; others are proved in [10]. Their conjectures are formulated in terms of certain sets of integers satisfying the conditions of their Theorem 2.1; the partition-theoretic interpretations are stated as corollaries. We emphasize that these conjectures and corollaries are different formulations of the same phenomena; their corollaries are *not* less general than the corresponding conjectures. However, these conjectures are still open for bijective proofs.

After we define below a modular equation, as Ramanujan would have understood it, we shall inform readers of the purpose of this paper. The complete elliptic integral of the first kind is defined for $|k| < 1$ by

$$K := K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

The number k is called the modulus. The complementary modulus k' is defined by $k' = \sqrt{1 - k^2}$. Set $K' = K(k')$. Expanding the integrand in a binomial series and integrating termwise, one finds that

$$K = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

where ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$ denotes the ordinary hypergeometric function. Let K , K' , L , and L' , denote the complete elliptic integrals of the first kind associated with the moduli k , k' , ℓ , and $\ell' := \sqrt{1 - \ell^2}$, respectively. Suppose that the equality

$$n \frac{K'}{K} = \frac{L'}{L} \tag{1.1}$$

holds for some positive integer n . A relation between k and ℓ induced by (1.1) is called a modular equation of degree n . Ramanujan recorded his modular equations in terms of α and β , where $\alpha = k^2$ and $\beta = \ell^2$. We often say that β has degree n over α .

If

$$q := \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}\right) = \exp\left(-\pi \frac{K'}{K}\right), \tag{1.2}$$

then one of the primary theorems in the theory of elliptic functions [7, p. 101, Entry 6] asserts that

$$\varphi^2(q) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right) =: z, \quad (1.3)$$

where

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad |q| < 1.$$

If we further set $z_n := \varphi^2(q^n)$, then the multiplier m of degree n is defined by

$$m := \frac{z_1}{z_n}. \quad (1.4)$$

We are now ready to apprise readers of the purpose of our paper. In his notebooks [17], Ramanujan offered several beautiful formulas for multipliers of various degrees. Ramanujan's derivations of these formulas have remained elusive. The first author was able to prove some of them using analysis with which Ramanujan would have been familiar. However, to prove the others, it was necessary for this author to invoke the theory of modular forms [7]. A summary of most of Ramanujan's formulas for multipliers is also given in his lost notebook [18], [2, pp. 391, 392]. In this paper, we examine all of these formulas for multipliers. We convert each of them into a modular equation, write the modular equation in terms of q -products, and then interpret the identity of q -products in terms of colored partitions. (We emphasize that in the aforementioned papers, the authors primarily examined modular equations in the spirit of Russell and Schröter for their combinatorial interpretations.)

We complete our introduction with notation, definitions, and a lemma that will be used repeatedly in the remainder of the paper. For any complex number a and $|q| < 1$, define

$$(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n).$$

Recall that Ramanujan's theta function $f(-q)$ and his function $\chi(q)$ are defined by

$$f(-q) := (q; q)_{\infty}, \quad (1.5)$$

$$\chi(q) := (-q; q^2)_{\infty}. \quad (1.6)$$

The following evaluations of Ramanujan may be found in [7, p. 124] or [8, p. 123].

Lemma 1.2. *If α , q , and z are related by (1.2) and (1.3), then*

$$f(-q^2) = 2^{-1/3} \sqrt{z} \{\alpha(1 - \alpha)/q\}^{1/12}, \quad (1.7)$$

$$\chi(q) = 2^{1/6} \{\alpha(1 - \alpha)/q\}^{-1/24}, \quad (1.8)$$

$$\chi(-q) = 2^{1/6} (1 - \alpha)^{1/12} (\alpha/q)^{-1/24}, \quad (1.9)$$

$$\chi(-q^2) = 2^{1/3} (1 - \alpha)^{1/24} (\alpha/q)^{-1/12}. \quad (1.10)$$

Suppose that β has degree n over α . If we replace q by q^n above, then the same evaluations hold with α replaced by β and z replaced by z_n .

In the following proofs, we frequently employ Euler's famous identity (see [1, p. 5], [3, p. 3], [11, p. 15])

$$\frac{1}{(q; q^2)_\infty} = (-q; q)_\infty. \quad (1.11)$$

i.e., the number of partitions of a positive integer n into odd parts is identical to the number of partitions of n into distinct parts.

2. PARTITIONS WITHOUT MULTIPLES OF A CERTAIN PRIME

Theorems 2.1, 2.3, 3.1, and 5.1 are special cases of identity (5) in Warnaar's paper [24]. We shall give a detailed proof of Theorem 2.1 below, but for the remaining three theorems, we shall only provide Ramanujan's formula for the relevant multiplier and the q -product identity from which the desired partition identity arises. We emphasize that the proofs that we omit follow along similar lines to other proofs in this paper.

Theorem 2.1. *Let S denote the set of partitions whose parts occur in four different colors such that none of the parts are congruent to $0 \pmod{3}$. Let $A(N)$ be the number of partitions of $2N$ into an odd number of even parts in S . Let $B(N)$ be the number of partitions of $2N - 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.*

Proof. Consider the following modular equation for degree 3 (see [7, p. 230, Entry 5 (vii)] and [8, p. 146, Theorem 6.3.4 (vi)]):

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/2}, \quad (2.1)$$

where m is defined by (1.4). Multiply both sides of the identity (2.1) by $q^{2/3} \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/3}$. Then, upon rearrangement,

$$\begin{aligned} & \frac{2^{-4/3} z_1^2 \{\alpha(1-\alpha)/q\}^{1/3}}{2^{-4/3} z_3^2 \{\beta(1-\beta)/q^3\}^{1/3}} \\ &= q \frac{2^{2/3} (1-\alpha)^{1/3} (\alpha/q)^{-1/6}}{2^{2/3} (1-\beta)^{1/3} (\beta/q^3)^{-1/6}} + \frac{2^{4/3} (1-\beta)^{1/6} (\beta/q^3)^{-1/3}}{2^{4/3} (1-\alpha)^{1/6} (\alpha/q)^{-1/3}} - q \frac{2^{2/3} \{\alpha(1-\alpha)/q\}^{-1/6}}{2^{2/3} \{\beta(1-\beta)/q^3\}^{-1/6}}. \end{aligned}$$

Employing all the evaluations in Lemma 1.2, we have

$$\frac{f^4(-q^2)}{f^4(-q^6)} = q \frac{\chi^4(-q)}{\chi^4(-q^3)} + \frac{\chi^4(-q^6)}{\chi^4(-q^2)} - q \frac{\chi^4(q)}{\chi^4(q^3)},$$

which can be transformed into

$$\frac{(-q^2; q^2)_\infty^4}{(-q^6; q^6)_\infty^4} - \frac{(q^2; q^2)_\infty^4}{(q^6; q^6)_\infty^4} = q \left\{ \frac{(-q; q^2)_\infty^4}{(-q^3; q^6)_\infty^4} - \frac{(q; q^2)_\infty^4}{(q^3; q^6)_\infty^4} \right\},$$

by the definitions of f and χ in (1.5) and (1.6), respectively, and Euler's identity (1.11). Extracting the coefficients of $q^{2N}/2$ on both sides of the equation, we can finish the proof. $\square$

Example 2.2. *Let $N = 3$ in Theorem 2.1. Then $A(3) = B(3) = 4$, and the relevant partitions in the colors orange, blue, red, and green are*

$$\begin{aligned} 2_o + 2_b + 2_r &= 2_o + 2_b + 2_g = 2_o + 2_r + 2_g = 2_b + 2_r + 2_g; \\ 5_o &= 5_b = 5_r = 5_g. \end{aligned}$$

Theorem 2.3. *Let S denote the set of partitions whose parts can occur in two distinct colors such that none of the parts are congruent to $0 \pmod{5}$. Let $A(N)$ be the number of partitions of $2N$ into an odd number of even parts in S . Let $B(N)$ be the number of partitions of $2N - 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.*

Proof. As indicated at the beginning of this section, we forego almost all details in this proof. Recall the modular equation for degree 5 from [7, p. 281, Entry 13 (xii)],

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/4}, \quad (2.2)$$

where m is defined by (1.4). Multiply both sides of the identity (2.2) by $q^{2/3}\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/6}$ to obtain

$$\begin{aligned} & \frac{2^{-2/3}z_1\{\alpha(1-\alpha)/q\}^{1/6}}{2^{-2/3}z_5\{\beta(1-\beta)/q^5\}^{1/6}} \\ &= q \frac{2^{1/3}(1-\alpha)^{1/6}(\alpha/q)^{-1/12}}{2^{1/3}(1-\beta)^{1/6}(\beta/q^5)^{-1/12}} + \frac{2^{2/3}(1-\beta)^{1/12}(\beta/q^5)^{-1/6}}{2^{2/3}(1-\alpha)^{1/12}(\alpha/q)^{-1/6}} - q \frac{2^{1/3}\{\alpha(1-\alpha)/q\}^{-1/12}}{2^{1/3}\{\beta(1-\beta)/q^5\}^{-1/12}}. \end{aligned}$$

Utilizing all the representations in Lemma 1.2, we obtain the equivalent form

$$\frac{f^2(-q^2)}{f^2(-q^{10})} = q \frac{\chi^2(-q)}{\chi^2(-q^5)} + \frac{\chi^2(-q^{10})}{\chi^2(-q^2)} - q \frac{\chi^2(q)}{\chi^2(q^5)}.$$

Then employing the definitions of f and χ in (1.5) and (1.6), respectively, and invoking Euler's identity (1.11), we find that The identity (2.2) is equivalent to the q -product identity

$$\frac{(-q^2; q^2)_\infty^2}{(-q^{10}; q^{10})_\infty^2} - \frac{(q^2; q^2)_\infty^2}{(q^{10}; q^{10})_\infty^2} = q \left\{ \frac{(-q; q^2)_\infty^2}{(-q^5; q^{10})_\infty^2} - \frac{(q; q^2)_\infty^2}{(q^5; q^{10})_\infty^2} \right\}.$$

Equating the coefficients of $q^{2N}/2$ on both sides of the equation, we complete the proof. $\square$

Example 2.4. *Set $N = 3$ in Theorem 2.3. Then $A(3) = B(3) = 2$, and the relevant partitions into orange and blue parts are*

$$6_o = 6_b;$$

$$3_o + 1_o + 1_b = 3_b + 1_o + 1_b.$$

Theorem 2.5. *Let S denote the set of partitions such that parts appear no more than once in each of four colors and such that none is a multiple of 7. Let $A(N)$ be the number of partitions of $2N + 2$ into an odd number of even parts in S . Let $B(N)$ be the number of partitions of $2N - 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.*

Proof. Consider the multiplier of degree 7 given by [7, p. 314, Entry 19 (v)]

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left\{\frac{(1-\beta)}{(1-\alpha)}\right\}^{1/2} - \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/2} - 8\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/3}, \quad (2.3)$$

where m is defined by (1.4). Multiply both sides of the identity (2.3) by $q^2\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/3}$. Thus,

$$\frac{2^{-4/3}z_1^2\{\alpha(1-\alpha)/q\}^{1/3}}{2^{-4/3}z_7^2\{\beta(1-\beta)/q^7\}^{1/3}}$$

$$= q^3 \frac{2^{2/3}(1-\alpha)^{1/3}(\alpha/q)^{-1/6}}{2^{2/3}(1-\beta)^{1/3}(\beta/q^7)^{-1/6}} + \frac{2^{4/3}(1-\beta)^{1/6}(\beta/q^7)^{-1/3}}{2^{4/3}(1-\alpha)^{1/6}(\alpha/q)^{-1/3}} - q^3 \frac{2^{2/3}\{\alpha(1-\alpha)/q\}^{-1/6}}{2^{2/3}\{\beta(1-\beta)/q^7\}^{-1/6}} - 8q^2,$$

which can be transformed into

$$\frac{f^4(-q^2)}{f^4(-q^{14})} = q^3 \frac{\chi^4(-q)}{\chi^4(-q^7)} + \frac{\chi^4(-q^{14})}{\chi^4(-q^2)} - q^3 \frac{\chi^4(q)}{\chi^4(q^7)} - 8q^2,$$

by (1.7)–(1.10) in Lemma 1.2. Making use of the definitions of f and χ in (1.5) and (1.6), respectively, and employing Euler's identity (1.11), we deduce that

$$\frac{(-q^2; q^2)_\infty^4}{(-q^{14}; q^{14})_\infty^4} - \frac{(q^2; q^2)_\infty^4}{(q^{14}; q^{14})_\infty^4} - 8q^2 = q^3 \left\{ \frac{(-q; q^2)_\infty^4}{(-q^7; q^{14})_\infty^4} - \frac{(q; q^2)_\infty^4}{(q^7; q^{14})_\infty^4} \right\}.$$

Extracting the coefficients of $q^{2N+2}/2$ on both sides of the identity, we complete the proof. $\square$

Theorem 2.5 is equivalent to Conjecture 3.27 in Sandon and Zanello's paper [22].

Example 2.6. Let $N = 4$ in Theorem 2.5. Then $A(4) = B(4) = 4 + 4\binom{4}{2} + 4\binom{4}{2} = 52$, and the relevant partitions in colors orange, blue, red, and green are

$$10_o = 10_b = 10_r = 10_g = 6_o + 2_o + 2_b = \cdots = 4_o + 4_b + 2_o = \cdots;$$

$$5_o + 1_o + 1_b = \cdots = 3_o + 3_b + 1_o = \cdots = 3_o + 1_o + 1_b + 1_r + 1_g = \cdots.$$

Theorem 2.7. Let S denote the set of partitions whose parts appear at most once in each of two distinct colors, with none of the parts being multiples of 13. Let $A(N)$ be the number of partitions of $2N + 2$ into an odd number of even parts in S . Let $B(N)$ be the number of partitions of $2N - 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.

Proof. Consider the modular equation for degree 13 given by [2, p. 391, Entry 17.3.25],

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left\{\frac{(1-\beta)}{(1-\alpha)}\right\}^{1/4} - \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/4} - 4\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/6}, \quad (2.4)$$

where m is defined by (1.4). Multiply both sides of the identity (2.4) by $q^2 \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/6}$. Then, after some rearrangement,

$$\frac{2^{-2/3}z_1\{\alpha(1-\alpha)/q\}^{1/6}}{2^{-2/3}z_{13}\{\beta(1-\beta)/q^{13}\}^{1/6}} = q^3 \frac{2^{1/3}(1-\alpha)^{1/6}(\alpha/q)^{-1/12}}{2^{1/3}(1-\beta)^{1/6}(\beta/q^{13})^{-1/12}} + \frac{2^{2/3}(1-\beta)^{1/12}(\beta/q^{13})^{-1/6}}{2^{2/3}(1-\alpha)^{1/12}(\alpha/q)^{-1/6}} - q^3 \frac{2^{1/3}\{\alpha(1-\alpha)/q\}^{-1/12}}{2^{1/3}\{\beta(1-\beta)/q^{13}\}^{-1/12}} - 4q^2,$$

which can be transformed into

$$\frac{f^2(-q^2)}{f^2(-q^{26})} = q^3 \frac{\chi^2(-q)}{\chi^2(-q^{13})} + \frac{\chi^2(-q^{26})}{\chi^2(-q^2)} - q^3 \frac{\chi^2(q)}{\chi^2(q^{13})} - 4q^2,$$

by the identities in Lemma 1.2. Applying (1.5), (1.6), and Euler's identity (1.11), we see that

$$\frac{(-q^2; q^2)_\infty^2}{(-q^{26}; q^{26})_\infty^2} - \frac{(q^2; q^2)_\infty^2}{(q^{26}; q^{26})_\infty^2} - 4q^2 = q^3 \left\{ \frac{(-q; q^2)_\infty^2}{(-q^{13}; q^{26})_\infty^2} - \frac{(q; q^2)_\infty^2}{(q^{13}; q^{26})_\infty^2} \right\}.$$

Equate the coefficients of $q^{2N+2}/2$ on both sides of the foregoing identity to finish the proof. $\square$

Theorem 2.7 is equivalent to Conjecture 3.25 in Sandon and Zanello's paper [22].

Example 2.8. Let $N = 7$ in Theorem 2.7. Then $A(7) = B(7) = 2 + 2 + 2^3 + 2^3 + 2 + 2 = 24$. The relevant partitions into orange and blue parts are given by

$$\begin{aligned} 16_o &= 16_b = 12_o + 2_o + 2_b = 12_b + 2_o + 2_b = 10_o + 4_o + 2_o = \cdots = 8_o + 6_o + 2_o = \cdots \\ &= 8_o + 4_o + 4_b = 8_b + 4_o + 4_b = 6_o + 6_b + 4_o = 6_o + 6_b + 4_b; \\ 11_o + 1_o + 1_b &= 11_b + 1_o + 1_b = 9_o + 3_o + 1_o = \cdots = 7_o + 5_o + 1_o = \cdots = 7_o + 3_o + 3_b \\ &= 7_b + 3_o + 3_b = 5_o + 5_b + 3_o = 5_o + 5_b + 3_b = 5_o + 3_o + 3_b + 1_o + 1_b = 5_b + 3_o + 3_b + 1_o + 1_b. \end{aligned}$$

3. PARTITIONS WITHOUT MULTIPLES OF 9 OR 25

Theorem 3.1. Let S denote the set of partitions into distinct parts, none of which are congruent to 0 (mod 3). Let $A(N)$ denote the number of partitions of $2N$ into an odd number of even parts in S , and let $B(N)$ denote the number of partitions of $2N - 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.

Proof. As mentioned at the beginning of Section 2, we forego almost all details in this proof. Recall the formula for the multiplier of degree 9 [2, p. 391, Entry 17.3.24],

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/8}, \quad (3.1)$$

where m is defined by (1.4). The identity (3.1) can be converted into the q -product identity

$$\frac{(-q^2; q^2)_\infty}{(-q^{18}; q^{18})_\infty} - \frac{(q^2; q^2)_\infty}{(q^{18}; q^{18})_\infty} = q \left\{ \frac{(-q; q^2)_\infty}{(-q^9; q^{18})_\infty} - \frac{(q; q^2)_\infty}{(q^9; q^{18})_\infty} \right\}.$$

Equating the coefficients of $q^{2N}/2$ on both sides of the equation, we finish the proof. $\square$

Example 3.2. Set $N = 6$ in Theorem 3.1. Then $A(6) = B(6) = 2$, and the relevant partitions are

$$\begin{aligned} 12 &= 6 + 4 + 2; \\ 11 &= 7 + 3 + 1. \end{aligned}$$

Theorem 3.3. Let S denote the set of partitions into distinct parts, none of which are multiples of 25. Let $A(N)$ be the number of partitions of $2N + 2$ into an odd number of even parts in S . Let $B(N)$ be the number of partitions of $2N - 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.

Proof. Consider the multiplier of degree 25 [2, p. 392, Entry 17.3.27]

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left\{\frac{(1-\beta)}{(1-\alpha)}\right\}^{1/8} - \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/8} - 2\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/12}, \quad (3.2)$$

where m is defined by (1.4). Multiplying both sides of (3.2) by $q^{2\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/12}}$, we find that

$$\begin{aligned} &\frac{2^{-1/3}\sqrt{z_1}\{\alpha(1-\alpha)/q\}^{1/12}}{2^{-1/3}\sqrt{z_{25}}\{\beta(1-\beta)/q^{25}\}^{1/12}} \\ &= q^3 \frac{2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}}{2^{1/6}(1-\beta)^{1/12}(\beta/q^{25})^{-1/24}} + \frac{2^{1/3}(1-\beta)^{1/24}(\beta/q^{25})^{-1/12}}{2^{1/3}(1-\alpha)^{1/24}(\alpha/q)^{-1/12}} - q^3 \frac{2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}}{2^{1/6}\{\beta(1-\beta)/q^{25}\}^{-1/24}} - 2q^2, \end{aligned}$$

which can be transformed into

$$\frac{f(-q^2)}{f(-q^{50})} = q^3 \frac{\chi(-q)}{\chi(-q^{25})} + \frac{\chi(-q^{50})}{\chi(-q^2)} - q^3 \frac{\chi(q)}{\chi(q^{25})} - 2q^2,$$

by the use of all the evaluations in Lemma 1.2. Because of (1.5), (1.6), and Euler's identity (1.11), we can rewrite the formula above in the form

$$\frac{(-q^2; q^2)_\infty}{(-q^{50}; q^{50})_\infty} - \frac{(q^2; q^2)_\infty}{(q^{50}; q^{50})_\infty} - 2q^2 = q^3 \left\{ \frac{(-q; q^2)_\infty}{(-q^{25}; q^{50})_\infty} - \frac{(q; q^2)_\infty}{(q^{25}; q^{50})_\infty} \right\}.$$

Extracting the coefficients of $q^{2N+2}/2$ on both sides of the identity, we complete the proof. $\square$

Theorem 3.3 is equivalent to Conjecture 3.24 in Sandon and Zanello's paper [22].

Example 3.4. Let $N = 13$ in Theorem 3.3. Then $A(13) = B(13) = 11$. The relevant partitions are given by

$$\begin{aligned} 28 &= 22 + 4 + 2 = 20 + 6 + 2 = 18 + 8 + 2 = 18 + 6 + 4 = 16 + 10 + 2 \\ &= 16 + 8 + 4 = 14 + 12 + 2 = 14 + 10 + 4 = 14 + 8 + 6 = 12 + 10 + 6; \\ 21 &+ 3 + 1 = 19 + 5 + 1 = 17 + 7 + 1 = 17 + 5 + 3 = 15 + 9 + 1 = 15 + 7 + 3 \\ &= 13 + 11 + 1 = 13 + 9 + 3 = 13 + 7 + 5 = 11 + 9 + 5 = 9 + 7 + 5 + 3 + 1. \end{aligned}$$

4. PARTITIONS WITH AND WITHOUT MULTIPLES OF 3, 5, 7 OR 11

Theorem 4.1. Let S denote the set of partitions with parts in all four colors appearing at most once, and with parts in two of the colors appearing in multiples of 5 and not in multiples of 3. Let $A(N)$ be the number of partitions of $2N + 2$ into an odd number of even parts in S . Let $B(N)$ be the number of partitions of $2N - 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.

Proof. Let α , β , γ , and δ be of the first, third, fifth, and fifteenth degrees, respectively. Then [7, p. 384, Entry 11 (x)],

$$mm' = \left(\frac{\beta\delta}{\alpha\gamma} \right)^{1/4} + \left\{ \frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)} \right\}^{1/4} - \left\{ \frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)} \right\}^{1/4} - 4 \left\{ \frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)} \right\}^{1/6}, \quad (4.1)$$

where $m = z_1/z_3$ and $m' = z_5/z_{15}$. Multiplying both sides of the identity (4.1) by $q^2 \left\{ \frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)} \right\}^{1/6}$, we see that

$$\begin{aligned} & \frac{2^{-4/3} z_1 \{\alpha(1-\alpha)/q\}^{1/6} z_5 \{\gamma(1-\gamma)/q^5\}^{1/6}}{2^{-4/3} z_3 \{\beta(1-\beta)/q^3\}^{1/6} z_{15} \{\delta(1-\delta)/q^{15}\}^{1/6}} \\ &= q^3 \frac{2^{2/3} (1-\alpha)^{1/6} (\alpha/q)^{-1/12} (1-\gamma)^{1/6} (\gamma/q^5)^{-1/12}}{2^{2/3} (1-\beta)^{1/6} (\beta/q^3)^{-1/12} (1-\delta)^{1/6} (\delta/q^{15})^{-1/12}} + \frac{2^{4/3} (1-\beta)^{1/12} (\beta/q^3)^{-1/6} (1-\delta)^{1/12} (\delta/q^{15})^{-1/6}}{2^{4/3} (1-\alpha)^{1/12} (\alpha/q)^{-1/6} (1-\gamma)^{1/12} (\gamma/q^5)^{-1/6}} \\ & \quad - q^3 \frac{2^{2/3} \{\alpha(1-\alpha)/q\}^{-1/12} \{\gamma(1-\gamma)/q^5\}^{-1/12}}{2^{2/3} \{\beta(1-\beta)/q^3\}^{-1/12} \{\delta(1-\delta)/q^{15}\}^{-1/12}} - 4q^2. \end{aligned}$$

Utilizing (1.7)–(1.10) of Lemma 1.2, we can rewrite this equation as

$$\frac{f^2(-q^2)f^2(-q^{10})}{f^2(-q^6)f^2(-q^{30})} = q^3 \frac{\chi^2(-q)\chi^2(-q^5)}{\chi^2(-q^3)\chi^2(-q^{15})} + \frac{\chi^2(-q^6)\chi^2(-q^{30})}{\chi^2(-q^2)\chi^2(-q^{10})} - q^3 \frac{\chi^2(q)\chi^2(q^5)}{\chi^2(q^3)\chi^2(q^{15})} - 4q^2. \quad (4.2)$$

Making use of (1.5), (1.6), and Euler's identity (1.11), and rewriting (4.2), we have

$$\begin{aligned} & \frac{(-q^2; q^2)_\infty^2 (-q^{10}; q^{10})_\infty^2}{(-q^6; q^6)_\infty^2 (-q^{30}; q^{30})_\infty^2} - \frac{(q^2; q^2)_\infty^2 (q^{10}; q^{10})_\infty^2}{(q^6; q^6)_\infty^2 (q^{30}; q^{30})_\infty^2} - 4q^2 \\ &= q^3 \left\{ \frac{(-q; q^2)_\infty^2 (-q^5; q^{10})_\infty^2}{(-q^3; q^6)_\infty^2 (-q^{15}; q^{30})_\infty^2} - \frac{(q; q^2)_\infty^2 (q^5; q^{10})_\infty^2}{(q^3; q^6)_\infty^2 (q^{15}; q^{30})_\infty^2} \right\}. \end{aligned}$$

Equating the coefficients of $q^{2N+2}/2$ on both sides of this last identity, we finish the proof. $\square$

Example 4.2. Set $N = 6$ in Theorem 4.1. Then $A(6) = 2 + 4 + 2 \cdot 2^2 = 14$ and $B(6) = 2 + 2\binom{4}{2} = 14$. Suppose that the green and red parts are multiples of 5 but not multiples of 3. The corresponding partitions are given by

$$14_o = 14_b = 10_o + 2_o + 2_b = \dots = 8_o + 4_o + 2_o = \dots;$$

$$11_o = 11_b = 5_o + 5_b + 1_o = 5_r + 5_g + 1_o = \dots$$

Theorem 4.3. Let S denote the set of partitions whose parts in two colors each appear at most once, and with parts of one color appearing only in multiples of 3 and never in multiples of 7. Let $A(N)$ denote the number of partitions of $2N + 2$ into an odd number of even parts in S . Let $B(N)$ denote the number of partitions of $2N - 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.

Proof. Recall the modular equation

$$\sqrt{\frac{z_1 z_3}{z_7 z_{21}}} = \left(\frac{\gamma \delta}{\alpha \beta} \right)^{1/8} \left\{ \frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)} \right\}^{1/8} - \left\{ \frac{\gamma \delta (1-\gamma)(1-\delta)}{\alpha \beta (1-\alpha)(1-\beta)} \right\}^{1/8} - 2 \left\{ \frac{\gamma \delta (1-\gamma)(1-\delta)}{\alpha \beta (1-\alpha)(1-\beta)} \right\}^{1/12}, \quad (4.3)$$

where β , γ , and δ are of degrees 3, 7, and 21, respectively [7, p. 401, Entry 13 (iii)]. Multiplying both sides of the identity (4.3) by $q^2 \left\{ \frac{\alpha \beta (1-\alpha)(1-\beta)}{\gamma \delta (1-\gamma)(1-\delta)} \right\}^{1/12}$, we arrive at

$$\begin{aligned} & \frac{2^{-2/3} \sqrt{z_1} \{ \alpha(1-\alpha)/q \}^{1/12} \sqrt{z_3} \{ \beta(1-\beta)/q^3 \}^{1/12}}{2^{-2/3} \sqrt{z_7} \{ \gamma(1-\gamma)/q^7 \}^{1/12} \sqrt{z_{21}} \{ \delta(1-\delta)/q^{21} \}^{1/12}} \\ &= q^3 \frac{2^{1/3} (1-\alpha)^{1/12} (\alpha/q)^{-1/24} (1-\beta)^{1/12} (\beta/q^3)^{-1/24}}{2^{1/3} (1-\gamma)^{1/12} (\gamma/q^7)^{-1/24} (1-\delta)^{1/12} (\delta/q^{21})^{-1/24}} + \frac{2^{2/3} (1-\gamma)^{1/24} (\gamma/q^7)^{-1/12} (1-\delta)^{1/24} (\delta/q^{21})^{-1/12}}{2^{2/3} (1-\alpha)^{1/24} (\alpha/q)^{-1/12} (1-\beta)^{1/24} (\beta/q^3)^{-1/12}} \\ & \quad - q^3 \frac{2^{1/3} \{ \alpha(1-\alpha)/q \}^{-1/24} \{ \beta(1-\beta)/q^3 \}^{-1/24}}{2^{1/3} \{ \gamma(1-\gamma)/q^7 \}^{-1/24} \{ \delta(1-\delta)/q^{21} \}^{-1/24}} - 2q^2. \end{aligned}$$

Using (1.7)–(1.10) in Lemma 1.2, we find that

$$\frac{f(-q^2)f(-q^6)}{f(-q^{14})f(-q^{42})} = q^3 \frac{\chi(-q)\chi(-q^3)}{\chi(-q^7)\chi(-q^{21})} + \frac{\chi(-q^{14})\chi(-q^{42})}{\chi(-q^2)\chi(-q^6)} - q^3 \frac{\chi(q)\chi(q^3)}{\chi(q^7)\chi(q^{21})} - 2q^2.$$

Applying the definitions of f in (1.5) and χ in (1.6), and Euler's identity (1.11), we obtain

$$\begin{aligned} & \frac{(-q^2; q^2)_\infty (-q^6; q^6)_\infty}{(-q^{14}; q^{14})_\infty (-q^{42}; q^{42})_\infty} - \frac{(q^2; q^2)_\infty (q^6; q^6)_\infty}{(q^{14}; q^{14})_\infty (q^{42}; q^{42})_\infty} - 2q^2 \\ &= q^3 \left\{ \frac{(-q; q^2)_\infty (-q^3; q^6)_\infty}{(-q^7; q^{14})_\infty (-q^{21}; q^{42})_\infty} - \frac{(q; q^2)_\infty (q^3; q^6)_\infty}{(q^7; q^{14})_\infty (q^{21}; q^{42})_\infty} \right\}. \end{aligned}$$

Extracting the coefficients of $q^{2N+2}/2$ on both sides of the identity above, we can complete the proof. $\square$

Example 4.4. Let $N = 6$ in Theorem 4.3. Assume that the blue parts appear in multiples of 3 but not in multiples of 7. Then $A(6) = B(6) = 2$, and the corresponding partitions are

$$\begin{aligned} 8_o + 4_o + 2_o &= 6_o + 6_b + 2_o; \\ 11_o &= 5_o + 3_o + 3_b. \end{aligned}$$

Theorem 4.5. Let S denote the set of partitions whose parts in two colors appear at most once. Assume furthermore that parts in one of the colors appear only in multiples of 11 but not in multiples of 3. Let $A(N)$ designate the number of partitions of $2N + 2$ into an odd number of even parts in S , and let $B(N)$ denote the number of partitions of $2N - 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.

Proof. Let β , γ , and δ be of the third, eleventh, and thirty-third degrees, respectively. Then [7, p. 408, Entry 14 (i)]

$$\sqrt{mm'} = \left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8} + \left\{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right\}^{1/8} - \left\{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right\}^{1/8} - 2\left\{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right\}^{1/12}, \quad (4.4)$$

where $m = z_1/z_3$ and $m' = z_{11}/z_{33}$. Multiply both sides of (4.4) by $q^2 \left\{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right\}^{1/12}$. Hence,

$$\begin{aligned} & \frac{2^{-2/3} \sqrt{z_1} \{\alpha(1-\alpha)/q\}^{1/12} \sqrt{z_{11}} \{\gamma(1-\gamma)/q^{11}\}^{1/12}}{2^{-2/3} \sqrt{z_3} \{\beta(1-\beta)/q^3\}^{1/12} \sqrt{z_{33}} \{\delta(1-\delta)/q^{33}\}^{1/12}} \\ &= q^3 \frac{2^{1/3} (1-\alpha)^{1/12} (\alpha/q)^{-1/24} (1-\gamma)^{1/12} (\gamma/q^{11})^{-1/24}}{2^{1/3} (1-\beta)^{1/12} (\beta/q^3)^{-1/24} (1-\delta)^{1/12} (\delta/q^{33})^{-1/24}} + \frac{2^{2/3} (1-\beta)^{1/24} (\beta/q^3)^{-1/12} (1-\delta)^{1/24} (\delta/q^{33})^{-1/12}}{2^{2/3} (1-\alpha)^{1/24} (\alpha/q)^{-1/12} (1-\gamma)^{1/24} (\gamma/q^{11})^{-1/12}} \\ & \quad - q^3 \frac{2^{1/3} \{\alpha(1-\alpha)/q\}^{-1/24} \{\gamma(1-\gamma)/q^{11}\}^{-1/24}}{2^{1/3} \{\beta(1-\beta)/q^3\}^{-1/24} \{\delta(1-\delta)/q^{33}\}^{-1/24}} - 2q^2, \end{aligned}$$

which is equivalent to

$$\frac{f(-q^2)f(-q^{22})}{f(-q^6)f(-q^{66})} = q^3 \frac{\chi(-q)\chi(-q^{11})}{\chi(-q^3)\chi(-q^{33})} + \frac{\chi(-q^6)\chi(-q^{66})}{\chi(-q^2)\chi(-q^{22})} - q^3 \frac{\chi(q)\chi(q^{11})}{\chi(q^3)\chi(q^{33})} - 2q^2,$$

by an appeal to all four representations in Lemma 1.2. Utilizing the definitions of f in (1.5) and χ in (1.6), and Euler's identity (1.11), we obtain

$$\begin{aligned} & \frac{(-q^2; q^2)_\infty (-q^{22}; q^{22})_\infty}{(-q^6; q^6)_\infty (-q^{66}; q^{66})_\infty} - \frac{(q^2; q^2)_\infty (q^{22}; q^{22})_\infty}{(q^6; q^6)_\infty (q^{66}; q^{66})_\infty} - 2q^2 \\ &= q^3 \left\{ \frac{(-q; q^2)_\infty (-q^{11}; q^{22})_\infty}{(-q^3; q^6)_\infty (-q^{33}; q^{66})_\infty} - \frac{(q; q^2)_\infty (q^{11}; q^{22})_\infty}{(q^3; q^6)_\infty (q^{33}; q^{66})_\infty} \right\}. \end{aligned}$$

Equate the coefficients of $q^{2N+2}/2$ on both sides of the equation above to finish the proof. $\square$

Example 4.6. Let $N = 6$ in Theorem 4.5. Then $A(6) = B(6) = 2$. Assume that the blue parts appear in multiples of 11 but not in multiples of 3. The corresponding partitions are then given by

$$\begin{aligned} 14_o &= 8_o + 4_o + 2_o; \\ 11_o &= 11_b. \end{aligned}$$

5. PARTITIONS WITHOUT MULTIPLES OF 3, 5, 7 OR 13

Theorem 5.1. *Let S denote the set of partitions into distinct parts, without parts in multiples of 3 or 5. Let $A(N)$ be the number of partitions of $2N$ into an even number of even parts in S . Let $B(N)$ be the number of partitions of $2N + 1$ into odd parts in S . Then, for $N \geq 0$, $A(N) = B(N)$.*

Proof. As written at the beginning of Section 2, we do not provide many details in this proof. Let α, β, γ , and δ be of the first, third, fifth, and fifteenth degrees, respectively. If $m = z_1/z_3$ and $m' = z_5/z_{15}$, then [7, p. 384, Entry 11 (ix)]

$$-\sqrt{\frac{m}{m'}} = \left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left\{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right\}^{1/8} - \left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/8}, \quad (5.1)$$

which has the q -product formulation

$$\begin{aligned} & q \left\{ \frac{(-q^2; q^2)_\infty (-q^{30}; q^{30})_\infty}{(-q^6; q^6)_\infty (-q^{10}; q^{10})_\infty} + \frac{(q^2; q^2)_\infty (q^{30}; q^{30})_\infty}{(q^6; q^6)_\infty (q^{10}; q^{10})_\infty} \right\} \\ &= \frac{(-q; q^2)_\infty (-q^{15}; q^{30})_\infty}{(-q^3; q^6)_\infty (-q^5; q^{10})_\infty} - \frac{(q; q^2)_\infty (q^{15}; q^{30})_\infty}{(q^3; q^6)_\infty (q^5; q^{10})_\infty}. \end{aligned}$$

Equating the coefficients of $q^{2N+1}/2$ on both sides of the identity above, we finish the proof. $\square$

Example 5.2. *Let $N = 9$ in Theorem 5.1. Then $A(9) = B(9) = 2$, and the derived partitions are*

$$\begin{aligned} 16 + 2 &= 14 + 4; \\ 19 &= 11 + 7 + 1. \end{aligned}$$

Theorem 5.3. *Let S denote the set of partitions in two colors, where the parts are distinct and not in multiples of 3 or 7. Let $A(N)$ denote the number of partitions of $2N - 2$ into an odd number of even parts in S . Let $B(N)$ denote the number of partitions of $2N + 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.*

Proof. Suppose that the moduli β, γ , and δ are of degrees 3, 7, and 21, respectively. Then [7, p. 401, Entry 13 (i)]

$$\frac{z_1 z_{21}}{z_3 z_7} = \left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/4} + \left\{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right\}^{1/4} - \left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/4} + 4 \left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/6}. \quad (5.2)$$

Multiplying both sides of the identity (5.2) by $q^{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}^{1/6}$, we find that

$$\begin{aligned} & q^3 \frac{2^{-4/3} z_1 \{\alpha(1-\alpha)/q\}^{1/6} z_{21} \{\delta(1-\delta)/q^{21}\}^{1/6}}{2^{-4/3} z_3 \{\beta(1-\beta)/q^3\}^{1/6} z_7 \{\gamma(1-\gamma)/q^7\}^{1/6}} \\ &= \frac{2^{2/3} (1-\alpha)^{1/6} (\alpha/q)^{-1/12} (1-\delta)^{1/6} (\delta/q^{21})^{-1/12}}{2^{2/3} (1-\beta)^{1/6} (\beta/q^3)^{-1/12} (1-\gamma)^{1/6} (\gamma/q^7)^{-1/12}} + q^3 \frac{2^{4/3} (1-\beta)^{1/12} (\beta/q^3)^{-1/6} (1-\gamma)^{1/12} (\gamma/q^7)^{-1/6}}{2^{4/3} (1-\alpha)^{1/12} (\alpha/q)^{-1/6} (1-\delta)^{1/12} (\delta/q^{21})^{-1/6}} \\ &\quad - \frac{2^{2/3} \{\alpha(1-\alpha)/q\}^{-1/12} \{\delta(1-\delta)/q^{21}\}^{-1/12}}{2^{2/3} \{\beta(1-\beta)/q^3\}^{-1/12} \{\gamma(1-\gamma)/q^7\}^{-1/12}} + 4q, \end{aligned}$$

which can be transformed into

$$q^3 \frac{f^2(-q^2)f^2(-q^{42})}{f^2(-q^6)f^2(-q^{14})} = \frac{\chi^2(-q)\chi^2(-q^{21})}{\chi^2(-q^3)\chi^2(-q^7)} + q^3 \frac{\chi^2(-q^6)\chi^2(-q^{14})}{\chi^2(-q^2)\chi^2(-q^{42})} - \frac{\chi^2(q)\chi^2(q^{21})}{\chi^2(q^3)\chi^2(q^7)} + 4q,$$

by (1.7)–(1.10) in Lemma 1.2. We utilize the definitions of f in (1.5) and χ in (1.6), and Euler's identity (1.11), and then obtain

$$\begin{aligned} & q^3 \left\{ \frac{(-q^2; q^2)_\infty^2 (-q^{42}; q^{42})_\infty^2}{(-q^6; q^6)_\infty^2 (-q^{14}; q^{14})_\infty^2} - \frac{(q^2; q^2)_\infty^2 (q^{42}; q^{42})_\infty^2}{(q^6; q^6)_\infty^2 (q^{14}; q^{14})_\infty^2} \right\} + 4q \\ &= \frac{(-q; q^2)_\infty^2 (-q^{21}; q^{42})_\infty^2}{(-q^3; q^6)_\infty^2 (-q^7; q^{14})_\infty^2} - \frac{(q; q^2)_\infty^2 (q^{21}; q^{42})_\infty^2}{(q^3; q^6)_\infty^2 (q^7; q^{14})_\infty^2}. \end{aligned}$$

Extracting the coefficients of $q^{2N+1}/2$ on both sides of the equation above, we can complete the proof. $\square$

Example 5.4. Set $N = 10$ in Theorem 5.3. Then $A(10) = B(10) = 4$, and the corresponding partitions are

$$\begin{aligned} 10_o + 4_o + 4_b &= 10_b + 4_o + 4_b = 8_o + 8_b + 2_o = 8_o + 8_b + 2_b; \\ 19_o + 1_o + 1_b &= 19_b + 1_o + 1_b = 11_o + 5_o + 5_b = 11_b + 5_o + 5_b. \end{aligned}$$

Theorem 5.5. Let S denote the set of partitions into distinct parts, without parts in multiples of 5 or 7. Let $A(N)$ denote the number of partitions of $2N - 2$ into an even number of even parts in S . Let $B(N)$ be the number of partitions of $2N + 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.

Proof. If β , γ , and δ are of degrees 5, 7, and 35, respectively, then [7, p. 423, Entry 18 (vii)]

$$-\sqrt{\frac{m}{m'}} = \left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left\{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right\}^{1/8} - \left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/8} + 2\left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/12}, \quad (5.3)$$

where $m = z_1/z_5$ and $m' = z_7/z_{35}$. Multiplying both sides of (5.3) by $q^{\left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/12}}$, we deduce that

$$\begin{aligned} & -q^3 \frac{2^{-2/3} \sqrt{z_1} \{\alpha(1-\alpha)/q\}^{1/12} \sqrt{z_{35}} \{\delta(1-\delta)/q^{35}\}^{1/12}}{2^{-2/3} \sqrt{z_5} \{\beta(1-\beta)/q^5\}^{1/12} \sqrt{z_7} \{\gamma(1-\gamma)/q^7\}^{1/12}} \\ &= \frac{2^{1/3} (1-\alpha)^{1/12} (\alpha/q)^{-1/24} (1-\delta)^{1/12} (\delta/q^{35})^{-1/24}}{2^{1/3} (1-\beta)^{1/12} (\beta/q^5)^{-1/24} (1-\gamma)^{1/12} (\gamma/q^7)^{-1/24}} + q^3 \frac{2^{2/3} (1-\beta)^{1/24} (\beta/q^5)^{-1/12} (1-\gamma)^{1/24} (\gamma/q^7)^{-1/12}}{2^{2/3} (1-\alpha)^{1/24} (\alpha/q)^{-1/12} (1-\delta)^{1/24} (\delta/q^{35})^{-1/12}} \\ & \quad - \frac{2^{1/3} \{\alpha(1-\alpha)/q\}^{-1/24} \{\delta(1-\delta)/q^{35}\}^{-1/24}}{2^{1/3} \{\beta(1-\beta)/q^5\}^{-1/24} \{\gamma(1-\gamma)/q^7\}^{-1/24}} + 2q. \end{aligned}$$

Employing each result from Lemma 1.2, we can rewrite the last identity as

$$-q^3 \frac{f(-q^2)f(-q^{70})}{f(-q^{10})f(-q^{14})} = \frac{\chi(-q)\chi(-q^{35})}{\chi(-q^5)\chi(-q^7)} + q^3 \frac{\chi(-q^{10})\chi(-q^{14})}{\chi(-q^2)\chi(-q^{70})} - \frac{\chi(q)\chi(q^{35})}{\chi(q^5)\chi(q^7)} + 2q.$$

Make use of (1.5), (1.6), and Euler's identity (1.11), and simplify to obtain

$$q^3 \left\{ \frac{(-q^2; q^2)_\infty (-q^{70}; q^{70})_\infty}{(-q^{10}; q^{10})_\infty (-q^{14}; q^{14})_\infty} + \frac{(q^2; q^2)_\infty (q^{70}; q^{70})_\infty}{(q^{10}; q^{10})_\infty (q^{14}; q^{14})_\infty} \right\} + 2q$$

$$= \frac{(-q; q^2)_\infty (-q^{35}; q^{70})_\infty}{(-q^5; q^{10})_\infty (-q^7; q^{14})_\infty} - \frac{(q; q^2)_\infty (q^{35}; q^{70})_\infty}{(q^5; q^{10})_\infty (q^7; q^{14})_\infty}.$$

Equating the coefficients of $q^{2N+1}/2$ on both sides of the last identity, we finish the proof. $\square$

Example 5.6. Let $N = 8$ in Theorem 5.5. Then $A(8) = B(8) = 2$. The relevant partitions are given by

$$12 + 2 = 8 + 6;$$

$$17 = 13 + 3 + 1.$$

Theorem 5.7. Let S denote the set of partitions into distinct parts, without parts in multiples of 3 or 13. Let $A(N)$ be the number of partitions of $2N - 2$ into an odd number of even parts in S . Let $B(N)$ be the number of partitions of $2N + 1$ into odd parts in S . Then, for $N \geq 1$, $A(N) = B(N)$.

Proof. Let β , γ , and δ have degrees 3, 13, and 39, respectively. Then [7, p. 426, Entry 19 (iv)]

$$\sqrt{\frac{m}{m'}} = \left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left\{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right\}^{1/8} - \left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/8} + 2\left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/12}, \quad (5.4)$$

where $m = z_1/z_3$ and $m' = z_{13}/z_{39}$. Multiplying both sides of (5.4) by $q^{\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\}^{1/12}}$, we find that

$$\begin{aligned} & q^3 \frac{2^{-2/3} \sqrt{z_1} \{\alpha(1-\alpha)/q\}^{1/12} \sqrt{z_{39}} \{\delta(1-\delta)/q^{39}\}^{1/12}}{2^{-2/3} \sqrt{z_3} \{\beta(1-\beta)/q^3\}^{1/12} \sqrt{z_{13}} \{\gamma(1-\gamma)/q^{13}\}^{1/12}} \\ &= \frac{2^{1/3} (1-\alpha)^{1/12} (\alpha/q)^{-1/24} (1-\delta)^{1/12} (\delta/q^{39})^{-1/24}}{2^{1/3} (1-\beta)^{1/12} (\beta/q^3)^{-1/24} (1-\gamma)^{1/12} (\gamma/q^{13})^{-1/24}} + q^3 \frac{2^{2/3} (1-\beta)^{1/24} (\beta/q^3)^{-1/12} (1-\gamma)^{1/24} (\gamma/q^{13})^{-1/12}}{2^{2/3} (1-\alpha)^{1/24} (\alpha/q)^{-1/12} (1-\delta)^{1/24} (\delta/q^{39})^{-1/12}} \\ &\quad - \frac{2^{1/3} \{\alpha(1-\alpha)/q\}^{-1/24} \{\delta(1-\delta)/q^{39}\}^{-1/24}}{2^{1/3} \{\beta(1-\beta)/q^3\}^{-1/24} \{\gamma(1-\gamma)/q^{13}\}^{-1/24}} + 2q. \end{aligned}$$

Utilizing (1.7)–(1.10) from Lemma 1.2, we can obtain

$$q^3 \frac{f(-q^2)f(-q^{78})}{f(-q^6)f(-q^{26})} = \frac{\chi(-q)\chi(-q^{39})}{\chi(-q^3)\chi(-q^{13})} + q^3 \frac{\chi(-q^6)\chi(-q^{26})}{\chi(-q^2)\chi(-q^{78})} - \frac{\chi(q)\chi(q^{39})}{\chi(q^3)\chi(q^{13})} + 2q.$$

Using the definitions of f and χ in (1.5) and (1.6), respectively, and invoking Euler's identity (1.11), we find that

$$\begin{aligned} & q^3 \left\{ \frac{(-q^2; q^2)_\infty (-q^{78}; q^{78})_\infty}{(-q^6; q^6)_\infty (-q^{26}; q^{26})_\infty} - \frac{(q^2; q^2)_\infty (q^{78}; q^{78})_\infty}{(q^6; q^6)_\infty (q^{26}; q^{26})_\infty} \right\} + 2q \\ &= \frac{(-q; q^2)_\infty (-q^{39}; q^{78})_\infty}{(-q^3; q^6)_\infty (-q^{13}; q^{26})_\infty} - \frac{(q; q^2)_\infty (q^{39}; q^{78})_\infty}{(q^3; q^6)_\infty (q^{13}; q^{26})_\infty}. \end{aligned}$$

Extract the coefficients of $q^{2N+1}/2$ on both sides of the equation above to finish the proof. $\square$

Example 5.8. Set $N = 9$ in Theorem 5.7. Then $A(9) = B(9) = 2$, and the derived partitions are

$$16 = 10 + 4 + 2;$$

$$19 = 11 + 7 + 1.$$

6. MORE RECONDITE PARTITION IDENTITIES

Theorem 6.1. *Let S_1 denote the set of partitions into 2 distinct colors, with parts appearing at most once, and without parts in multiples of 17. Let $A(N)$ denote the number of partitions of $2N + 2$ into an odd number of even parts in S_1 . Let $B(N)$ denote the number of partitions of $2N - 2$ into odd parts in S_1 . Let S_2 denote the set of partitions without parts in multiples of 17. Let $C(N)$ be the number of partitions of $2N$ into odd parts in S_2 . Let $D(N)$ denote the difference between the number of partitions of $2N - 2$ into an even number of even parts in S_2 and the number of partitions of $2N - 2$ into an odd number of even parts in S_2 . Then, for $N \geq 1$, $A(N) + B(N) = 2C(N) + D(N)$.*

Proof. Recall the multiplier for degree 17 given by [2, p. 392, Entry 17.3.26]

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left\{\frac{(1-\beta)}{(1-\alpha)}\right\}^{1/4} + \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/4} - 2\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/8} \left\{1 + \left(\frac{\beta}{\alpha}\right)^{1/8} + \left\{\frac{(1-\beta)}{(1-\alpha)}\right\}^{1/8}\right\}, \quad (6.1)$$

where m is defined by (1.4). Multiplying both sides of the identity (6.1) by $q^{8/3} \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/6}$, we obtain

$$\begin{aligned} & \frac{2^{-2/3} z_1 \{\alpha(1-\alpha)/q\}^{1/6}}{2^{-2/3} z_{17} \{\beta(1-\beta)/q^{17}\}^{1/6}} \\ &= q^4 \frac{2^{1/3} (1-\alpha)^{1/6} (\alpha/q)^{-1/12}}{2^{1/3} (1-\beta)^{1/6} (\beta/q^{17})^{-1/12}} + \frac{2^{2/3} (1-\beta)^{1/12} (\beta/q^{17})^{-1/6}}{2^{2/3} (1-\alpha)^{1/12} (\alpha/q)^{-1/6}} + q^4 \frac{2^{1/3} \{\alpha(1-\alpha)/q\}^{-1/12}}{2^{1/3} \{\beta(1-\beta)/q^{17}\}^{-1/12}} \\ & \quad - 2q^2 \frac{2^{1/6} \{\alpha(1-\alpha)/q\}^{1/24}}{2^{1/6} \{\beta(1-\beta)/q^{17}\}^{1/24}} - 2q^4 \frac{2^{1/3} (1-\alpha)^{1/24} (\alpha/q)^{-1/12}}{2^{1/3} (1-\beta)^{1/24} (\beta/q^{17})^{-1/12}} - 2q^2 \frac{2^{1/6} (1-\beta)^{1/12} (\beta/q^{17})^{-1/24}}{2^{1/6} (1-\alpha)^{1/12} (\alpha/q)^{-1/24}}, \end{aligned}$$

which can be transformed into

$$\begin{aligned} \frac{f^2(-q^2)}{f^2(-q^{34})} &= q^4 \frac{\chi^2(-q)}{\chi^2(-q^{17})} + \frac{\chi^2(-q^{34})}{\chi^2(-q^2)} + q^4 \frac{\chi^2(q)}{\chi^2(q^{17})} \\ & \quad - 2q^2 \frac{\chi(q^{17})}{\chi(q)} - 2q^4 \frac{\chi(-q^2)}{\chi(-q^{34})} - 2q^2 \frac{\chi(-q^{17})}{\chi(-q)}, \end{aligned}$$

by the use of all the identities from Lemma 1.2. Appealing to (1.5), (1.6), and Euler's identity (1.11), and simplifying, we deduce that

$$\begin{aligned} & \frac{(-q^2; q^2)_\infty^2}{(-q^{34}; q^{34})_\infty^2} - \frac{(q^2; q^2)_\infty^2}{(q^{34}; q^{34})_\infty^2} + q^4 \left\{ \frac{(-q; q^2)_\infty^2}{(-q^{17}; q^{34})_\infty^2} + \frac{(q; q^2)_\infty^2}{(q^{17}; q^{34})_\infty^2} \right\} \\ &= 2q^2 \left\{ \frac{(-q^{17}; q^{34})_\infty}{(-q; q^2)_\infty} + \frac{(q^{17}; q^{34})_\infty}{(q; q^2)_\infty} \right\} + 2q^4 \frac{(-q^{34}; q^{34})_\infty}{(-q^2; q^2)_\infty}. \end{aligned}$$

Equate the coefficients of $q^{2N+2}/2$ on both sides of the identity above to complete the proof. $\square$

Example 6.2. *Let $N = 4$ in Theorem 6.1. Then $A(4) = 6$, $B(4) = 5$, $C(4) = 6$, and $D(4) = 1 - 2 = -1$. The relevant partitions are given by*

$$10_o = 10_b = 6_o + 2_o + 2_b = 6_b + 2_o + 2_b = 4_o + 4_b + 2_o = 4_o + 4_b + 2_b;$$

$$5_o + 1_b = 5_o + 1_o = 5_b + 1_o = 5_b + 1_b = 3_o + 3_b;$$

$$7 + 1 = 5 + 3 = 5 + 1 + 1 + 1 = 3 + 3 + 1 + 1 = 3 + 1 + 1 + 1 + 1 + 1 = 1^8;$$

$$6 = 4 + 2 = 2 + 2 + 2.$$

Theorem 6.3. *Let S_1 denote the set of partitions into 4 distinct colors, with all parts appearing at most once and with the parts in two colors appearing in only multiples of 3 but not in multiples of 5. Let $A(N)$ denote the number of partitions of $2N + 2$ into an odd number of even parts in S_1 . Let $B(N)$ be the number of partitions of $2N - 2$ into odd parts in S_1 . Let S_2 denote the set of partitions into 2 distinct colors, with the parts of one color appearing in multiples of 3, but not in multiples of 5. Let $C(N)$ be the number of partitions of $2N$ into odd parts in S_2 . Let $D(N)$ be the difference between the number of partitions of $2N - 2$ into an even number of even parts in S_2 and the number of partitions of $2N - 2$ into an odd number of even parts in S_2 . Then, for $N \geq 1$, $A(N) + B(N) = 2C(N) + D(N)$.*

Proof. Suppose that β , γ , and δ are of degrees 3, 5, and 15, respectively. Then [7, p. 384, Entry 11 (xii)]

$$\begin{aligned} \frac{z_1 z_3}{z_5 z_{15}} &= \left(\frac{\gamma \delta}{\alpha \beta} \right)^{1/4} + \left\{ \frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)} \right\}^{1/4} + \left\{ \frac{\gamma(1-\gamma)\delta(1-\delta)}{\alpha(1-\alpha)\beta(1-\beta)} \right\}^{1/4} \\ &\quad - 2 \left\{ \frac{\gamma(1-\gamma)\delta(1-\delta)}{\alpha(1-\alpha)\beta(1-\beta)} \right\}^{1/8} \left\{ 1 + \left(\frac{\gamma \delta}{\alpha \beta} \right)^{1/8} + \left\{ \frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)} \right\}^{1/8} \right\}. \end{aligned} \quad (6.2)$$

Multiplying both sides of (6.2) by $q^{8/3} \left\{ \frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)} \right\}^{1/6}$, we deduce that

$$\begin{aligned} &\frac{2^{-4/3} z_1 \{ \alpha(1-\alpha)/q \}^{1/6} z_3 \{ \beta(1-\beta)/q^3 \}^{1/6}}{2^{-4/3} z_5 \{ \gamma(1-\gamma)/q^5 \}^{1/6} z_{15} \{ \delta(1-\delta)/q^{15} \}^{1/6}} \\ &= q^4 \frac{2^{2/3} (1-\alpha)^{1/6} (\alpha/q)^{-1/12} (1-\beta)^{1/6} (\beta/q^3)^{-1/12}}{2^{2/3} (1-\gamma)^{1/6} (\gamma/q^5)^{-1/12} (1-\delta)^{1/6} (\delta/q^{15})^{-1/12}} + \frac{2^{4/3} (1-\gamma)^{1/12} (\gamma/q^5)^{-1/6} (1-\delta)^{1/12} (\delta/q^{15})^{-1/6}}{2^{4/3} (1-\alpha)^{1/12} (\alpha/q)^{-1/6} (1-\beta)^{1/12} (\beta/q^3)^{-1/6}} \\ &\quad + q^4 \frac{2^{2/3} \{ \alpha(1-\alpha)/q \}^{-1/12} \{ \beta(1-\beta)/q^3 \}^{-1/12}}{2^{2/3} \{ \gamma(1-\gamma)/q^5 \}^{-1/12} \{ \delta(1-\delta)/q^{15} \}^{-1/12}} - 2q^2 \frac{2^{1/3} \{ \alpha(1-\alpha)/q \}^{1/24} \{ \beta(1-\beta)/q^3 \}^{1/24}}{2^{1/3} \{ \gamma(1-\gamma)/q^5 \}^{1/24} \{ \delta(1-\delta)/q^{15} \}^{1/24}} \\ &\quad - 2q^4 \frac{2^{2/3} (1-\alpha)^{1/24} (\alpha/q)^{-1/12} (1-\beta)^{1/24} (\beta/q^3)^{-1/12}}{2^{2/3} (1-\gamma)^{1/24} (\gamma/q^5)^{-1/12} (1-\delta)^{1/24} (\delta/q^{15})^{-1/12}} \\ &\quad - 2q^2 \frac{2^{1/3} (1-\gamma)^{1/12} (\gamma/q^5)^{-1/24} (1-\delta)^{1/12} (\delta/q^{15})^{-1/24}}{2^{1/3} (1-\alpha)^{1/12} (\alpha/q)^{-1/24} (1-\beta)^{1/12} (\beta/q^3)^{-1/24}}, \end{aligned}$$

which can be transformed into

$$\begin{aligned} \frac{f^2(-q^2)f^2(-q^6)}{f^2(-q^{10})f^2(-q^{30})} &= q^4 \frac{\chi^2(-q)\chi^2(-q^3)}{\chi^2(-q^5)\chi^2(-q^{15})} + \frac{\chi^2(-q^{10})\chi^2(-q^{30})}{\chi^2(-q^2)\chi^2(-q^6)} + q^4 \frac{\chi^2(q)\chi^2(q^3)}{\chi^2(q^5)\chi^2(q^{15})} \\ &\quad - 2q^2 \frac{\chi(q^5)\chi(q^{15})}{\chi(q)\chi(q^3)} - 2q^4 \frac{\chi(-q^2)\chi(-q^6)}{\chi(-q^{10})\chi(-q^{30})} - 2q^2 \frac{\chi(-q^5)\chi(-q^{15})}{\chi(-q)\chi(-q^3)}, \end{aligned}$$

by (1.7)–(1.10) from Lemma 1.2. Using the definitions of f in (1.5) and χ in (1.6), and Euler's identity (1.11), upon simplification, we see that

$$\frac{(-q^2; q^2)_\infty^2 (-q^6; q^6)_\infty^2}{(-q^{10}; q^{10})_\infty^2 (-q^{30}; q^{30})_\infty^2} - \frac{(q^2; q^2)_\infty^2 (q^6; q^6)_\infty^2}{(q^{10}; q^{10})_\infty^2 (q^{30}; q^{30})_\infty^2}$$

$$\begin{aligned}
& + q^4 \left\{ \frac{(-q; q^2)_\infty (-q^3; q^6)_\infty}{(-q^5; q^{10})_\infty (-q^{15}; q^{30})_\infty} + \frac{(q; q^2)_\infty (q^3; q^6)_\infty}{(q^5; q^{10})_\infty (q^{15}; q^{30})_\infty} \right\} \\
& = 2q^2 \left\{ \frac{(-q^5; q^{10})_\infty (-q^{15}; q^{30})_\infty}{(-q; q^2)_\infty (-q^3; q^6)_\infty} + \frac{(q^5; q^{10})_\infty (q^{15}; q^{30})_\infty}{(q; q^2)_\infty (q^3; q^6)_\infty} \right\} \\
& + 2q^4 \frac{(-q^{10}; q^{10})_\infty (-q^{30}; q^{30})_\infty}{(-q^2; q^2)_\infty (-q^6; q^6)_\infty}.
\end{aligned}$$

Equating the coefficients of $q^{2N+2}/2$ on both sides of the previous identity, we finish the proof. $\square$

Example 6.4. Let $N = 5$ in Theorem 6.3. Then $A(5) = 6 + 4 \cdot 2^2 = 22$, $B(5) = 4 + \binom{4}{2} = 10$, $C(5) = 5 + 4 + 3 + 2 + 1 = 15$, and $D(5) = 4 - 2 = 2$. The red, green, and pink parts appear only in multiples of 3 and not in multiples of 5. The corresponding partitions are

$$\begin{aligned}
12_o &= 12_b = 12_r = 12_g = 8_o + 2_o + 2_b = 8_b + 2_o + 2_b = 6_o + 4_o + 2_o = \cdots = 6_r + 4_o + 2_o = \cdots; \\
7_o + 1_o &= \cdots = 3_o + 3_b + 1_o + 1_b = 3_r + 3_g + 1_o + 1_b = \cdots; \\
9_y + 1_y &= 9_p + 1_y = 7_y + 3_y = 7_y + 3_p = 7_y + 1_y + 1_y + 1_y = 3_y + 3_y + 3_p + 1_y = \cdots \\
&= 3_y + 3_p + 1_y + 1_y + 1_y + 1_y = \cdots = 3_p + 1_y + 1_y + 1_y + 1_y + 1_y + 1_y + 1_y = \cdots = 1_y^{10}; \\
8_y &= 6_y + 2_y = 6_p + 2_y = 4_y + 4_y = 4_y + 2_y + 2_y = 2_y + 2_y + 2_y + 2_y.
\end{aligned}$$

Theorem 6.5. Let S_1 denote the set of partitions into 4 distinct colors, with the parts in each color appearing at most once, but with parts in two colors appearing in multiples of 7 but not in multiples of 3. Let $A(N)$ be the number of partitions of $2N + 2$ into an odd number of even parts in S_1 . Let $B(N)$ be the number of partitions of $2N - 2$ into odd parts in S_1 . Let S_2 denote the set of partitions into 2 distinct colors, with the parts in each color appearing at most once, but with the parts in one color appearing in multiples of 7 but not in multiples of 3. Let $C(N)$ be the number of partitions of $2N$ into odd parts in S_2 . Let $D(N)$ be the difference between the number of partitions of $2N - 2$ into an even number of even parts in S_2 and the number of partitions of $2N - 2$ into an odd number of even parts in S_2 . Then, for $N \geq 1$, $A(N) + B(N) = 2C(N) + D(N)$.

Proof. Suppose that β , γ , and δ are of degrees 3, 7, and 21, respectively. Then [7, p. 401, Entry 13 (v)]

$$\begin{aligned}
\frac{z_1 z_7}{z_3 z_{21}} &= \left(\frac{\beta \delta}{\alpha \gamma} \right)^{1/4} + \left\{ \frac{(1 - \beta)(1 - \delta)}{(1 - \alpha)(1 - \gamma)} \right\}^{1/4} + \left\{ \frac{\beta(1 - \beta)\delta(1 - \delta)}{\alpha(1 - \alpha)\gamma(1 - \gamma)} \right\}^{1/4} \\
&\quad - 2 \left\{ \frac{\beta(1 - \beta)\delta(1 - \delta)}{\alpha(1 - \alpha)\gamma(1 - \gamma)} \right\}^{1/8} \left\{ 1 + \left(\frac{\beta \delta}{\alpha \gamma} \right)^{1/8} + \left\{ \frac{(1 - \beta)(1 - \delta)}{(1 - \alpha)(1 - \gamma)} \right\}^{1/8} \right\}. \quad (6.3)
\end{aligned}$$

Multiplying both sides of (6.3) by $q^{8/3} \left\{ \frac{\alpha \gamma (1 - \alpha)(1 - \gamma)}{\beta \delta (1 - \beta)(1 - \delta)} \right\}^{1/6}$, we find that

$$\begin{aligned}
& \frac{2^{-4/3} z_1 \{ \alpha(1 - \alpha)/q \}^{1/6} z_7 \{ \gamma(1 - \gamma)/q^7 \}^{1/6}}{2^{-4/3} z_3 \{ \beta(1 - \beta)/q^3 \}^{1/6} z_{21} \{ \delta(1 - \delta)/q^{21} \}^{1/6}} \\
& = q^4 \frac{2^{2/3} (1 - \alpha)^{1/6} (\alpha/q)^{-1/12} (1 - \gamma)^{1/6} (\gamma/q^7)^{-1/12}}{2^{2/3} (1 - \beta)^{1/6} (\beta/q^3)^{-1/12} (1 - \delta)^{1/6} (\delta/q^{21})^{-1/12}} + \frac{2^{4/3} (1 - \beta)^{1/12} (\beta/q^3)^{-1/6} (1 - \delta)^{1/12} (\delta/q^{21})^{-1/6}}{2^{4/3} (1 - \alpha)^{1/12} (\alpha/q)^{-1/6} (1 - \gamma)^{1/12} (\gamma/q^7)^{-1/6}}
\end{aligned}$$

$$\begin{aligned}
& + q^4 \frac{2^{2/3} \{\alpha(1-\alpha)/q\}^{-1/12} \{\gamma(1-\gamma)/q^7\}^{-1/12}}{2^{2/3} \{\beta(1-\beta)/q^3\}^{-1/12} \{\delta(1-\delta)/q^{21}\}^{-1/12}} - 2q^2 \frac{2^{1/3} \{\alpha(1-\alpha)/q\}^{1/24} \{\gamma(1-\gamma)/q^7\}^{1/24}}{2^{1/3} \{\beta(1-\beta)/q^3\}^{1/24} \{\delta(1-\delta)/q^{21}\}^{1/24}} \\
& - 2q^4 \frac{2^{2/3} (1-\alpha)^{1/24} (\alpha/q)^{-1/12} (1-\gamma)^{1/24} (\gamma/q^7)^{-1/12}}{2^{2/3} (1-\beta)^{1/24} (\beta/q^3)^{-1/12} (1-\delta)^{1/24} (\delta/q^{21})^{-1/12}} \\
& - 2q^2 \frac{2^{1/3} (1-\beta)^{1/12} (\beta/q^3)^{-1/24} (1-\delta)^{1/12} (\delta/q^{21})^{-1/24}}{2^{1/3} (1-\alpha)^{1/12} (\alpha/q)^{-1/24} (1-\gamma)^{1/12} (\gamma/q^7)^{-1/24}},
\end{aligned}$$

which can be transformed into

$$\begin{aligned}
\frac{f^2(-q^2)f^2(-q^{14})}{f^2(-q^6)f^2(-q^{42})} &= q^4 \frac{\chi^2(-q)\chi^2(-q^7)}{\chi^2(-q^3)\chi^2(-q^{21})} + \frac{\chi^2(-q^6)\chi^2(-q^{42})}{\chi^2(-q^2)\chi^2(-q^{14})} + q^4 \frac{\chi^2(q)\chi^2(q^7)}{\chi^2(q^3)\chi^2(q^{21})} \\
&\quad - 2q^2 \frac{\chi(q^3)\chi(q^{21})}{\chi(q)\chi(q^7)} - 2q^4 \frac{\chi(-q^2)\chi(-q^{14})}{\chi(-q^6)\chi(-q^{42})} - 2q^2 \frac{\chi(-q^3)\chi(-q^{21})}{\chi(-q)\chi(-q^7)},
\end{aligned}$$

by the use of (1.7)–(1.10) in Lemma 1.2. Employing the definitions of f and χ in (1.5) and (1.6), respectively, and applying Euler's identity (1.11), we deduce that

$$\begin{aligned}
& \frac{(-q^2; q^2)_\infty^2 (-q^{14}; q^{14})_\infty^2}{(-q^6; q^6)_\infty^2 (-q^{42}; q^{42})_\infty^2} - \frac{(q^2; q^2)_\infty^2 (q^{14}; q^{14})_\infty^2}{(q^6; q^6)_\infty^2 (q^{42}; q^{42})_\infty^2} \\
& + q^4 \left\{ \frac{(-q; q^2)_\infty^2 (-q^7; q^{14})_\infty^2}{(-q^3; q^6)_\infty^2 (-q^{21}; q^{42})_\infty^2} + \frac{(q; q^2)_\infty^2 (q^7; q^{14})_\infty^2}{(q^3; q^6)_\infty^2 (q^{21}; q^{42})_\infty^2} \right\} \\
& = 2q^2 \left\{ \frac{(-q^3; q^6)_\infty (-q^{21}; q^{42})_\infty}{(-q; q^2)_\infty (-q^7; q^{14})_\infty} + \frac{(q^3; q^6)_\infty (q^{21}; q^{42})_\infty}{(q; q^2)_\infty (q^7; q^{14})_\infty} \right\} \\
& + 2q^4 \frac{(-q^6; q^6)_\infty (-q^{42}; q^{42})_\infty}{(-q^2; q^2)_\infty (-q^{14}; q^{14})_\infty}.
\end{aligned}$$

Equate the coefficients of $q^{2N+2}/2$ on both sides of the foregoing identity to finish the proof. $\square$

Example 6.6. Set $N = 5$ in Theorem 6.5. Then $A(5) = 2$, $B(5) = 8$, $C(5) = 5$, $D(5) = 0$. With the red, green, and pink parts appearing in multiples of 7 but not in multiples of 3, the illustrative partitions are given by

$$\begin{aligned}
8_o + 2_o + 2_b &= 8_b + 2_o + 2_b; \\
7_o + 1_o &= 7_b + 1_o = 7_r + 1_o = 7_g + 1_o = 7_o + 1_b = \cdots; \\
7_y + 1_y + 1_y + 1_y &= 7_p + 1_y + 1_y + 1_y = 5_y + 5_y = 5_y + 1_y + 1_y + 1_y + 1_y = 1_y^{10}; \\
8_y &= 4_y + 4_y = 4_y + 2_y + 2_y = 2_y + 2_y + 2_y + 2_y.
\end{aligned}$$

7. CONCLUDING REMARKS AND ACKNOWLEDGMENTS

Sun Kim has informed us that since Theorems 2.1, 2.3, 3.1, and 5.1 are special cases of Warnaar's identity [24, equation (5)], she can provide combinatorial proofs for these four theorems along the same lines of her bijective proofs in [15]. Warnaar's identity is given by

$$\begin{aligned}
& d(-c, -q/c, -ac, -q/ac, -bc, -q/bc, -abcd^2, -q/abcd^2; q)_\infty \\
& - d(c, q/c, ac, q/ac, bc, q/bc, abcd^2, q/abcd^2; q)_\infty
\end{aligned}$$

$$\begin{aligned}
&= c(-d, -q/d, -ad, -q/ad, -bd, -q/bd, -abc^2d, -q/abc^2d; q)_\infty \\
&\quad - c(d, q/d, ad, q/ad, bd, q/bd, abc^2d, q/abc^2d; q)_\infty,
\end{aligned} \tag{7.1}$$

where

$$(a_1, \dots, a_n; q)_\infty := \prod_{k=1}^n (a_k; q)_\infty.$$

For example, equation (7.1) with $q = q^6$, $a = b = 1$, $c = q^2$, and $d = q$ implies Theorem 2.1; equation (7.1) with $q = q^{10}$, $a = b = q^2$, $c = q^2$, and $d = q$ implies Theorem 2.3; and equation (7.1) with $q = q^{18}$, $a = q^2$, $b = q^4$, $c = q^2$, and $d = q$ implies Theorem 3.1.

If we divide both sides of the identity (7.1) by d , we can derive the equation

$$\begin{aligned}
&(-c, -q/c, -ac, -q/ac, -bc, -q/bc, -abcd^2, -q/abcd^2; q)_\infty \\
&\quad - (c, q/c, ac, q/ac, bc, q/bc, abcd^2, q/abcd^2; q)_\infty \\
&= c(-d^{-1}, -qd, -ad, -q/ad, -bd, -q/bd, -abc^2d, -q/abc^2d; q)_\infty \\
&\quad + c(d^{-1}, qd, ad, q/ad, bd, q/bd, abc^2d, q/abc^2d; q)_\infty,
\end{aligned} \tag{7.2}$$

which is equivalent to Theorem 3.1 in Sun Kim's paper [15]. With the use of Warnaar's bijection and (7.2), as communicated to us by Sun Kim, bijective proofs of the aforementioned four theorems can be given. For example, equation (7.2) with $q = q^{30}$, $a = q^6$, $b = q^{10}$, $c = q$, and $d = q^{-2}$ implies Theorem 5.1. However, we are currently unable to follow the lead of Warnaar [24] and Kim [15], that is to say, we neither know how to find more general theta function identities nor to give bijective proofs for the remaining theorems in this paper. Most likely, new methods will need to be developed, and consequently it would be exceedingly interesting and worthwhile both to find general theta function identities and to construct bijective proofs.

Lastly, we remark that the remaining conjectures of Sandon and Zanello have been independently proved by the present authors [10] and by Baruah and B. Boruah [6]. However, none of these proofs are combinatorial; indeed, it is eminently desirable to construct bijective proofs.

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