

AN UNPUBLISHED MANUSCRIPT OF RAMANUJAN ON INFINITE SERIES IDENTITIES

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Abstract. Published with Ramanujan's lost notebook is an incomplete handwritten manuscript on infinite series identities. We examine all theorems stated by Ramanujan in this fragment and discuss Ramanujan's arguments, some of which are not rigorous.

1. INTRODUCTION

Published with Ramanujan's lost notebook [25, pp. 318–321] is a four-page, previously unpublished, handwritten fragment on infinite series. Partial fraction expansions, the Riemann zeta-function $\zeta(s)$, alternating sums over the odd integers, divisor sums, Bernoulli numbers, and Euler numbers are featured in the formulas in this manuscript. The first result has the equation number (18) attached to it. Thus, the manuscript was likely intended to be the completion of either a published paper or another unpublished manuscript. We conjecture that this fragment was originally intended to be a part of Ramanujan's paper, *Some formulae in the analytic theory of numbers*, [22], [23, pp. 133–135]. This paper contains several theorems about $\zeta(s)$ and $\sigma_k(n)$, and so the topics in the unpublished manuscript mesh well with those in the published paper. However, the last tagged equation in [22] is (22), whereas we would expect it to be (17) if our conjecture is correct. However, often Ramanujan would think of additional results and add them to the paper as he was writing it, and so this could easily account for the discrepancy in equation numbers. We remark here that the manuscript does not provide any proofs, but Ramanujan usually gives an indication (in one line) how the formula may be deduced.

Why did not Ramanujan include this unpublished manuscript in his paper [22], for the paper is rather short, and the unpublished manuscript would add at most four pages to the length of the paper? We think that Ramanujan discovered that one of his claims, namely (21), was incorrect and that two of his deductions (albeit both correct) were not corollaries of his (incorrect) formula, as he had previously thought. Moreover, we suspect that he realized that some of his arguments were not rigorous. Since he had abandoned his intention to publish this portion, he did not bother to indicate on the fragment that changes or corrections needed to be made.

Ramanujan loved partial fraction expansions. Chapter 14 in his second notebook [24], [4], in particular, contains several such expansions, and others are scattered throughout

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all three notebooks. See [6, Chap. 30] for these scattered partial fraction decompositions. However, Ramanujan's arguments were not always rigorous. Because of his apparent weakness in complex analysis, he evidently did not have a firm grasp of the Mittag-Leffler Theorem, for claim (21) in his unpublished manuscript arises from an incorrect application of the Mittag-Leffler Theorem, as we detail below. After claim (21), he then asserted several corollaries arising from this (incorrect) partial fraction decomposition. All of the corollaries are indeed correct, but two of them do not follow from this partial fraction expansion. Ramanujan undoubtedly had previously been familiar with all of these corollaries and probably had derived them by other methods as well. Therefore, realizing that certain results were easy deductions from his expansion, he incorrectly assumed that others could also be so derived. It is interesting that the same incorrect partial fraction expansion occurs in Entry 19(i) of Chapter 14 of his second notebook [24], [4, p. 271], where it was derived by a different method, namely a general elementary theorem, Entry 18 of Chapter 14 [24], [4, pp. 267–268]. R. Sitaramachandrarao [29], [4, pp. 271–272] found an alternative version of Ramanujan's partial fraction expansion. After we provide Ramanujan's argument, we show that we can actually use Sitaramachandrarao's result to derive a corrected version of Ramanujan's partial fraction expansion. We shall see that Ramanujan's defective argument missed one expression; all other portions of Ramanujan's formula are correct. One of the two claims which did not follow from Ramanujan's expansion now is a corollary of the corrected version. However, this corrected version still does not allow us to formally deduce the other result.

The most celebrated result in this manuscript is probably claim (28), which is a famous formula for $\zeta(2n + 1)$, where n is a positive integer. There are a large number of proofs of this result and many generalizations as well. References are given after we provide Ramanujan's proof of (28). Ramanujan's argument is rigorous and ironically is independent of whether his formula or the corrected version is used.

In (22), Ramanujan gives another partial fraction expansion, but this one is correct. All of its corollaries claimed by Ramanujan are correct, but not all the deductions can be rigorously established by Ramanujan's methods. These corollaries, like those arising from (19) are all well known, with some having been proved in the literature several times.

In the remainder of the paper, we record all of Ramanujan's formulas, prove them rigorously in some cases, and "prove" them nonrigorously in other cases, i.e., we argue as Ramanujan most likely did. Most of the results appear in Ramanujan's notebooks, and for all theorems we provide references where other proofs can be found. In providing references, we have composed the following rules. For each principal theorem, we locate it in Ramanujan's notebooks, indicate who gave the first proof, and lastly refer to the pages in the author's books, primarily, [4], where references to further proofs can be found. Since the publication of [4], additional proofs have been found in some instances, and so we provide references to those recent proofs of which we are aware.

The residue of a meromorphic function $f(z)$ at a pole z_0 will be denoted by $R(f, z_0) = R(z_0)$.

2. THREE FORMULAS CONTAINING DIVISOR SUMS

Entry 2.1 (p. 318, formula (18)). *Let $\chi(n)$ denote the nonprincipal, primitive character of modulus 4, i.e., $\chi(2n) = 0$ and $\chi(2n+1) = (-1)^n$, for each nonnegative integer n . Let $d(n)$ denote the number of positive divisors of the positive integer n . Then, if $x \neq in$, for each integer n ,*

$$\sum_{n=1}^{\infty} \frac{\chi(n)d(n)n}{n^2 + x^2} = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \operatorname{sech} \left(\frac{\pi x}{2n} \right). \quad (2.1)$$

Proof. Recall the partial fraction expansion [13, p. 44, formula 1.422, no. 1]

$$\operatorname{sech} \left(\frac{\pi x}{2} \right) = \frac{4}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2k-1}{(2k-1)^2 + x^2}.$$

Thus,

$$\begin{aligned} \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \operatorname{sech} \left(\frac{\pi x}{2n} \right) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \sum_{k=1}^{\infty} \frac{\chi(k)k}{k^2 + x^2/n^2} \\ &= \sum_{n,k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2 + x^2} \\ &= \sum_{r=1}^{\infty} \frac{\chi(r)d(r)r}{r^2 + x^2}. \end{aligned}$$

This completes the proof. □

Entry (2.1) is a simple example of a large class of formulas involving sech and arithmetical functions. See papers by Berndt [2, Ex. 3] and P. V. Krishnaiah and R. Sita Rama Chandra Rao [19] for examples.

Entry 2.2 (p. 318, formula (19)). *Let $\sigma_k(n) = \sum_{d|n} d^k$. Then, for $\operatorname{Re} s > 1$ and $\operatorname{Re}(s-r) > 1$,*

$$\zeta(s)\zeta(s-r) = \sum_{n=1}^{\infty} \frac{\sigma_r(n)}{n^s}. \quad (2.2)$$

The formula (2.2) is classical and simple to prove. Ramanujan [22], [23, pp. 133–135] found beautiful extensions of it. See also Titchmarsh's text [30, p. 8].

Entry 2.3 (p. 318, formula (20)). *Let χ be defined as in Entry 2.1, and let $\sigma_k(n)$ be as in Entry 2.2. Then, for $\operatorname{Re} s > 0$ and $\operatorname{Re}(s-r) > 0$,*

$$\sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s-r}} = \sum_{n=1}^{\infty} \frac{\chi(n)\sigma_r(n)}{n^s}.$$

Proof. For $\operatorname{Re} s > 0$ and $\operatorname{Re}(s-r) > 0$,

$$\sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s-r}} = \sum_{m,n=1}^{\infty} \frac{\chi(mn)n^r}{(mn)^s} = \sum_{k=1}^{\infty} \frac{\chi(k)\sigma_r(k)}{k^s},$$

which completes the proof. \square

There are many results in the literature generalizing or extending the last two results. The two most extensive papers in this direction are perhaps those by S. Chowla [9], [10], [12, pp. 92–115, 120–130].

3. RAMANUJAN'S INCORRECT PARTIAL FRACTION EXPANSION AND RAMANUJAN'S CELEBRATED FORMULA FOR $\zeta(2n+1)$

Prior to this next claim, Ramanujan writes, “By the theory of residues it can be shown that”. We think that Ramanujan implied that he used the calculus of residues to calculate the partial fraction decomposition which followed. His formal calculations should depend upon an application of the Mittag–Leffler Theorem, which cannot be applied in this situation. We first state the incorrect expansion, indicate Ramanujan's probable approach, and then offer a correct version. Ramanujan used n to denote a complex variable; we replace it with the more natural notation $w = z^2$.

Entry 3.1 (p. 318, formula (21)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, then*

$$\frac{1}{2w} + \sum_{m=1}^{\infty} \left\{ \frac{m\alpha \coth(m\alpha)}{w + m^2\alpha} + \frac{m\beta \coth(m\beta)}{w - m^2\beta} \right\} = \frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}). \quad (3.1)$$

Proof. (incorrect) Consider

$$f(z) := \frac{\pi}{2} \cot(z\sqrt{\alpha}) \coth(z\sqrt{\beta}),$$

which has simple poles at $z = m\pi/\sqrt{\alpha}$, $-\infty < m < \infty$, $m \neq 0$, with residues

$$R(m\pi/\sqrt{\alpha}) = \frac{\pi}{2\sqrt{\alpha}} \coth(m\beta), \quad (3.2)$$

and simple poles at $z = m\pi i/\sqrt{\beta}$, $-\infty < m < \infty$, $m \neq 0$, with residues

$$R(m\pi i/\sqrt{\beta}) = -\frac{\pi i}{2\sqrt{\beta}} \coth(m\alpha), \quad (3.3)$$

where we used the fact $\alpha\beta = \pi^2$ in our calculations. Clearly also $f(z)$ has a double pole at $z = 0$. Using (3.2) and once again the relation $\alpha\beta = \pi^2$, we find that the contributions of the poles $z = m\pi/\sqrt{\alpha}$ and $z = -m\pi/\sqrt{\alpha}$, $1 \leq m < \infty$, to the partial fraction expansion of $f(z)$ are

$$\frac{\pi}{2\sqrt{\alpha}} \left(\frac{\coth(m\beta)}{z - m\pi/\sqrt{\alpha}} + \frac{\coth(-m\beta)}{z + m\pi/\sqrt{\alpha}} \right) = \frac{m\beta \coth(m\beta)}{z^2 - m^2\beta}. \quad (3.4)$$

Using (3.3) and once again the relation $\alpha\beta = \pi^2$, we find that the sum of the contributions of the poles $z = m\pi i/\sqrt{\beta}$ and $z = -m\pi i/\sqrt{\beta}$, $1 \leq m < \infty$, to the partial fraction decomposition of $f(z)$ equals

$$-\frac{\pi i}{2\sqrt{\beta}} \left(\frac{\coth(m\alpha)}{z - m\pi i/\sqrt{\beta}} - \frac{\coth(m\alpha)}{z + m\pi i/\sqrt{\beta}} \right) = \frac{m\alpha \coth(m\alpha)}{z^2 + m^2\alpha}. \quad (3.5)$$

The part of the partial fraction decomposition arising from the double pole at $z = 0$ clearly equals

$$\frac{\pi}{2\sqrt{\alpha\beta}z^2} = \frac{1}{2z^2}, \quad (3.6)$$

upon again using the relation $\alpha\beta = \pi^2$. Employing (3.4)–(3.6) and formally applying the Mittag–Leffler Theorem, we find that there exists an entire function $g(z)$ such that

$$\frac{\pi}{2} \cot(z\sqrt{\alpha}) \coth(z\sqrt{\beta}) = \frac{1}{2z^2} + \sum_{m=1}^{\infty} \left\{ \frac{m\alpha \coth(m\alpha)}{z^2 + m^2\alpha} + \frac{m\beta \coth(m\beta)}{z^2 - m^2\beta} \right\} + g(z). \quad (3.7)$$

Here Ramanujan probably assumed that $g(z) \equiv 0$ and so completed his “proof” of (3.1). \square

Normally, in applications of the Mittag-Leffler theorem, one lets $z \rightarrow \infty$ to conclude that $g(z) \equiv 0$. However, this argument is invalid here, because $\cot(z\sqrt{\alpha}) \coth(z\sqrt{\beta})$ oscillates and does not have a limit as $z \rightarrow \infty$. Moreover, one cannot justify taking the limit as $z \rightarrow \infty$ under the summation sign in (3.7).

In attempting to find a corrected version of (3.1), Sitaramachandrarao [29], [4, pp. 271–272] proved that

$$\begin{aligned} \pi^2 xy \cot(\pi x) \coth(\pi y) = & 1 + \frac{\pi^2}{3}(y^2 - x^2) \\ & - 2\pi xy \sum_{m=1}^{\infty} \left(\frac{y^2 \coth(\pi mx/y)}{m(m^2 + y^2)} + \frac{x^2 \coth(\pi my/x)}{m(m^2 - x^2)} \right). \end{aligned} \quad (3.8)$$

Using the elementary identities

$$\frac{y^2}{m(m^2 + y^2)} = -\frac{m}{m^2 + y^2} + \frac{1}{m}$$

and

$$\frac{x^2}{m(m^2 - x^2)} = \frac{m}{m^2 - x^2} - \frac{1}{m},$$

we find that (3.8) can be rewritten in the form

$$\begin{aligned} \pi^2 xy \cot(\pi x) \coth(\pi y) = & 1 + \frac{\pi^2}{3}(y^2 - x^2) \\ & + 2\pi xy \sum_{m=1}^{\infty} \left(\frac{m \coth(\pi mx/y)}{m^2 + y^2} - \frac{m \coth(\pi my/x)}{m^2 - x^2} \right) \\ & - 2\pi xy \sum_{m=1}^{\infty} \frac{1}{m} (\coth(\pi mx/y) - \coth(\pi my/x)) \\ = & 1 + \frac{\pi^2}{3}(y^2 - x^2) \\ & + 2\pi xy \sum_{m=1}^{\infty} \left(\frac{m \coth(\pi mx/y)}{m^2 + y^2} - \frac{m \coth(\pi my/x)}{m^2 - x^2} \right) \end{aligned}$$

$$-4\pi xy \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{e^{2\pi mx/y} - 1} - \frac{1}{e^{2\pi my/x} - 1} \right), \quad (3.9)$$

where we used the elementary identity

$$\coth x = 1 + \frac{2}{e^{2x} - 1}. \quad (3.10)$$

We are now in a position to make simple changes of variables in (3.9) to derive a corrected version of (3.1).

Entry 3.2 (Corrected Version of (21)). *Under the hypotheses of Entry 3.1,*

$$\frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}) = \frac{1}{2w} + \frac{1}{2} \log \frac{\beta}{\alpha} + \sum_{m=1}^{\infty} \left\{ \frac{m\alpha \coth(m\alpha)}{w + m^2\alpha} + \frac{m\beta \coth(m\beta)}{w - m^2\beta} \right\}. \quad (3.11)$$

Proof. Let $\pi x = \sqrt{w\alpha}$ and $\pi y = \sqrt{w\beta}$ in (3.9) to deduce that

$$\begin{aligned} \frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}) &= \frac{1}{2w} + \frac{1}{6}(\beta - \alpha) + \sum_{m=1}^{\infty} \left(\frac{m\alpha \coth(m\alpha)}{m^2\alpha + w} - \frac{m\beta \coth(m\beta)}{\beta m^2 - w} \right) \\ &\quad - 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{e^{2m\alpha} - 1} - \frac{1}{e^{2m\beta} - 1} \right) \\ &= \frac{1}{2w} + \frac{1}{6}(\beta - \alpha) + \sum_{m=1}^{\infty} \left(\frac{m\alpha \coth(m\alpha)}{m^2\alpha + w} - \frac{m\beta \coth(m\beta)}{\beta m^2 - w} \right) \\ &\quad - 2 \left(\frac{1}{4} \log \alpha - \frac{\alpha}{12} - \frac{1}{4} \log \beta + \frac{\beta}{12} \right) \\ &= \frac{1}{2w} + \frac{1}{2} \log \frac{\beta}{\alpha} + \sum_{m=1}^{\infty} \left\{ \frac{m\alpha \coth(m\alpha)}{w + m^2\alpha} + \frac{m\beta \coth(m\beta)}{w - m^2\beta} \right\}, \end{aligned}$$

where we have used an equivalent formulation for the transformation of the Dedekind eta-function, namely [7],

$$\sum_{m=1}^{\infty} \frac{1}{m(e^{2m\alpha} - 1)} - \frac{1}{4} \log \alpha + \frac{\alpha}{12} = \sum_{m=1}^{\infty} \frac{1}{m(e^{2m\beta} - 1)} - \frac{1}{4} \log \beta + \frac{\beta}{12}, \quad (3.12)$$

under the condition $\alpha\beta = \pi^2$. This completes the proof of (3.11). \square

Thus, Ramanujan's claim (21) was correct except for the missing term $\frac{1}{2} \log \frac{\beta}{\alpha}$.

We now proceed to examine the four deductions Ramanujan made from (3.1). We first examine the claim that cannot be formally deduced from either (3.1) or the corrected version (3.11) and provide Ramanujan's argument. Ramanujan asserts that "Equating the coefficients of $1/n$ ($1/w$ in our notation) in both sides in (21) we have"

Entry 3.3 (p. 318, formula (23)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, then*

$$\alpha \sum_{m=1}^{\infty} \frac{m}{e^{2m\alpha} - 1} + \beta \sum_{m=1}^{\infty} \frac{m}{e^{2m\beta} - 1} = \frac{\alpha + \beta}{24} - \frac{1}{4}. \quad (3.13)$$

Proof. (incorrect) Following Ramanujan, we equate coefficients of $1/w$ on both sides of (3.11). Observe from the Laurent expansion of $\cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta})$ about $w = 0$ that the coefficient of $1/w$ equals $\frac{1}{2}$ on the left side of (3.11). Note also the term $1/(2w)$ on the right side of (3.11). Hence, the only contribution of $1/w$ that remains must come from

$$\sum_{m=1}^{\infty} \left\{ \frac{m\alpha}{w + m^2\alpha} \left(1 + \frac{2}{e^{2m\alpha} - 1} \right) + \frac{m\beta}{w - m^2\beta} \left(1 + \frac{2}{e^{2m\beta} - 1} \right) \right\}, \quad (3.14)$$

upon the use of (3.10), and this contribution must equal 0.

Proceeding formally, we have

$$\frac{m\alpha}{w + m^2\alpha} = \frac{m\alpha}{w} \sum_{r=0}^{\infty} \left(-\frac{m^2\alpha}{w} \right)^r \quad \text{and} \quad \frac{m\beta}{w - m^2\beta} = \frac{m\beta}{w} \sum_{r=0}^{\infty} \left(\frac{m^2\beta}{w} \right)^r.$$

Thus, from (3.14) we find that a contribution to the coefficient of $1/w$ equals

$$2\alpha \sum_{m=1}^{\infty} \frac{m}{e^{2m\alpha} - 1} + 2\beta \sum_{m=1}^{\infty} \frac{m}{e^{2m\beta} - 1}. \quad (3.15)$$

The remaining contribution to the coefficient of $1/w$ in (3.14) is given by

$$(\alpha + \beta) \sum_{m=1}^{\infty} m = (\alpha + \beta)\zeta(-1) = -\frac{\alpha + \beta}{12}. \quad (3.16)$$

Of course, this argument is not rigorous. The value $\zeta(-1) = -\frac{1}{12}$ can be found in Titchmarsh's book [30, p. 19, eq. (2.4.3)], for example. Alternatively, the "constant" for the series $\sum_{m=1}^{\infty} m$ in Ramanujan's terminology is equal to $-\frac{1}{12}$ [3, p. 135, Ex. 2]. Remembering that the contributions of the coefficients of $1/w$ in (3.14) must equal 0, we find from (3.15) and (3.16) that

$$\alpha \sum_{m=1}^{\infty} \frac{m}{e^{2m\alpha} - 1} + \beta \sum_{m=1}^{\infty} \frac{m}{e^{2m\beta} - 1} = \frac{\alpha + \beta}{24}. \quad (3.17)$$

In comparing (3.17) with (3.13), we find that the term $-\frac{1}{4}$ in (3.13) does not appear in (3.17). This concludes what we think must have been Ramanujan's argument. \square

Entry 3.4 (pp. 318–319, formula (24)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if $\sigma(m) = \sum_{d|m} d$, then*

$$\alpha \sum_{m=1}^{\infty} \sigma(m)e^{-2m\alpha} + \beta \sum_{m=1}^{\infty} \sigma(m)e^{-2m\beta} = \frac{\alpha + \beta}{24} - \frac{1}{4}. \quad (3.18)$$

Proof. Entry 3.4 is simply another version of Entry 3.3. To that end, expand the summands of (3.13) into geometric series and collect the coefficients of $e^{-2m\alpha}$ and $e^{-2m\beta}$ to complete the proof. \square

Ramanujan offered Entry 3.3 as Corollary (i) in Section 8 of Chapter 14 in his second notebook [24], [4, p. 255]. To the best of our knowledge, Entry 3.3 was first proved by O. Schlömilch [27], [28] in 1877. There now exist many proofs; see [4, p. 256] for references to several proofs. Entry 3.3 is equivalent to the transformation formula for Ramanujan's Eisenstein series $P(q)$.

Entry 3.5 (p. 320, formula (29)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if $\sigma_k(m) = \sum_{d|m} d^k$, then*

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{m(e^{2m\alpha} - 1)} - \sum_{m=1}^{\infty} \frac{1}{m(e^{2m\beta} - 1)} &= \sum_{m=1}^{\infty} \sigma_{-1}(m)e^{-2m\alpha} - \sum_{m=1}^{\infty} \sigma_{-1}(m)e^{-2m\beta} \\ &= \frac{1}{4} \log \frac{\alpha}{\beta} - \frac{\alpha - \beta}{12}. \end{aligned} \quad (3.19)$$

Proof. Following but altering Ramanujan's directions, we equate the terms independent of w in (3.11), not (3.1), and use (3.10) to deduce that

$$\frac{\pi}{2} \left(-\frac{\sqrt{\alpha}}{3\sqrt{\beta}} + \frac{\sqrt{\beta}}{3\sqrt{\alpha}} \right) = \frac{1}{2} \log \frac{\beta}{\alpha} + \sum_{m=1}^{\infty} \left\{ \frac{1}{m} \left(1 + \frac{2}{e^{2m\alpha} - 1} \right) - \frac{1}{m} \left(1 + \frac{2}{e^{2m\beta} - 1} \right) \right\}.$$

The desired result (3.19) now follows upon simplification, with the use of the identity $\alpha\beta = \pi^2$. \square

Entry 3.5 is stated by Ramanujan as Corollary (ii) in Section 8 of Chapter 14 in his second notebook [24], [4, p. 256] and as Entry 27(iii) in Chapter 16 of his second notebook [24], [5, p. 43]. It is equivalent to the transformation formula for the Dedekind eta-function. Note that we already used (3.19) in the equivalent form (3.12) in order to obtain a corrected version of Entry 3.1.

The Bernoulli numbers B_m , $m \geq 0$, are defined by

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^m, \quad |z| < 2\pi.$$

This convention for Bernoulli numbers is not the same as that used by Ramanujan in his unpublished manuscript.

Entry 3.6 (p. 319, formula (25)). *Let α and β be positive numbers such that $\alpha\beta = \pi^2$, and let B_m , $m \geq 0$, denote the m th Bernoulli number. Then, if r is a positive integer with $r \geq 2$,*

$$\alpha^r \left(\sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\alpha} - 1} - \frac{B_{2r}}{4r} \right) = (-\beta)^r \left(\sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\beta} - 1} - \frac{B_{2r}}{4r} \right). \quad (3.20)$$

Proof. (nonrigorous) Return to (3.11), use (3.10), and formally expand the summands into geometric series to arrive at

$$\begin{aligned} \frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}) &= \frac{1}{2w} + \frac{1}{2} \log \frac{\beta}{\alpha} \\ + \sum_{m=1}^{\infty} \left\{ \frac{m\alpha}{w} \sum_{k=0}^{\infty} \left(-\frac{m^2\alpha}{w} \right)^k \left(1 + \frac{2}{e^{2m\alpha} - 1} \right) + \frac{m\beta}{w} \sum_{k=0}^{\infty} \left(\frac{m^2\beta}{w} \right)^k \left(1 + \frac{2}{e^{2m\beta} - 1} \right) \right\}. \end{aligned} \quad (3.21)$$

Following Ramanujan's directions, we equate coefficients of $1/w^r$, $r \geq 2$, on both sides of (3.21) to formally deduce that

$$\begin{aligned} 0 &= (-1)^{r-1} \alpha^r \zeta(1-2r) + 2(-1)^{r-1} \alpha^r \sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\alpha} - 1} \\ &\quad + \beta^r \zeta(1-2r) + 2\beta^r \sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\beta} - 1}. \end{aligned}$$

Using the relation [30, p. 19, eq. (2.4.3)]

$$\zeta(1-2r) = -\frac{B_{2r}}{2r}, \quad r \geq 1,$$

dividing both sides by $2(-1)^r$, and simplifying, we deduce (3.20). \square

Entry 3.6 is identical to Entry 13 in Chapter 14 of Ramanujan's second notebook [24], [4, p. 261]. To the best of our knowledge, the first published proof of Entry 3.6 was given by M. B. Rao and M. V. Ayyar [26] in 1923. There exist many proofs of Entry 3.6, and even more proofs for the special case $\alpha = \beta = \pi$; see [4, pp. 261–262] for references.

Expanding the summands in geometric series, we deduce, like we did in previous entries, the following corollary.

Entry 3.7 (p. 319, formula (26)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if r is a positive integer with $r \geq 2$, then*

$$\alpha^r \left(\sum_{m=1}^{\infty} \sigma_{2r-1}(m) e^{-2m\alpha} - \frac{B_{2r}}{4r} \right) = (-\beta)^r \left(\sum_{m=1}^{\infty} \sigma_{2r-1}(m) e^{-2m\beta} - \frac{B_{2r}}{4r} \right).$$

Entry 3.8 (p. 319, formula (27)). *We have*

$$\sum_{m=1}^{\infty} \sigma_5(m) e^{-2\pi m} = \frac{1}{504}.$$

Proof. Entry 3.8 follows immediately from Entry 3.7 by setting $r = 3$ and $\alpha = \beta = \pi$, and then using the fact that $B_6 = \frac{1}{42}$. \square

Entry 3.9 (pp. 319–320, formula (28)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if r is a positive integer, then*

$$(4\alpha)^{-r} \left(\frac{1}{2} \zeta(2r+1) + \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\alpha} - 1)} \right)$$

$$\begin{aligned}
& -(-4\beta)^{-r} \left(\frac{1}{2} \zeta(2r+1) + \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\beta} - 1)} \right) \\
& = (4\alpha)^{-r} \left(\frac{1}{2} \zeta(2r+1) + \sum_{m=1}^{\infty} \sigma_{-1-2r}(m) e^{-2m\alpha} \right) \\
& \quad - (-4\beta)^{-r} \left(\frac{1}{2} \zeta(2r+1) + \sum_{m=1}^{\infty} \sigma_{-1-2r}(m) e^{-2m\beta} \right) \\
& = - \sum_{k=0}^{r+1} \frac{(-1)^k B_{2k} B_{2r+2-2k} \alpha^{r+1-k} \beta^k}{(2k)!(2r+2-2k)!}. \tag{3.22}
\end{aligned}$$

Proof. Return to (3.11), use (3.10), and expand the summands into geometric series to arrive at

$$\begin{aligned}
\frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}) &= \frac{1}{2w} + \frac{1}{2} \log \frac{\beta}{\alpha} \tag{3.23} \\
& + \sum_{m=1}^{\infty} \left\{ \frac{1}{m} \sum_{k=0}^{\infty} \left(-\frac{w}{m^2\alpha} \right)^k \left(1 + \frac{2}{e^{2m\alpha} - 1} \right) - \frac{1}{m} \sum_{k=0}^{\infty} \left(\frac{w}{m^2\beta} \right)^k \left(1 + \frac{2}{e^{2m\beta} - 1} \right) \right\}.
\end{aligned}$$

Following Ramanujan's advice, we equate coefficients of w^r , $r \geq 1$, on both sides of (3.23). On the right side the coefficient of w^r equals

$$\begin{aligned}
& (-\alpha)^{-r} \zeta(2r+1) + 2(-\alpha)^{-r} \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\alpha} - 1)} \\
& - \beta^{-r} \zeta(2r+1) + 2\beta^{-r} \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\beta} - 1)}. \tag{3.24}
\end{aligned}$$

Using the Laurent expansions for $\cot z$ and $\coth z$ about $z = 0$, we find that on the left side of (3.23)

$$\frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}) = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} (w\alpha)^{k-1/2} \cdot \sum_{j=0}^{\infty} \frac{2^{2j} B_{2j}}{(2j)!} (w\beta)^{j-1/2}. \tag{3.25}$$

The coefficient of w^r in (3.25) is easily seen to be equal to

$$2^{2r+1} \sum_{k=0}^{r+1} \frac{(-1)^k B_{2k} B_{2r+2-2k}}{(2k)!(2r+2-2k)!} \alpha^k \beta^{r+1-k}, \tag{3.26}$$

where we used the equality $\alpha\beta = \pi^2$. Now equate the expressions in (3.24) and (3.26), then multiply both sides by $(-1)^r 2^{-2r-1}$, and lastly replace k by $r+1-k$ in the finite sum. We then have shown the equality of the first and third expressions in (3.22). The first equality of (3.22) follows as before by expanding the summands on the left side into geometric series. \square

Entry 3.9 is the same as Entry 21(i) in Chapter 14 of Ramanujan's second notebook [24], [4, pp. 275–276]. An extensive generalization of Entry 3.9 can be found in Entry 20 of Chapter 16 in Ramanujan's first notebook [24], [6, pp. 429–432]. The special

case $\alpha = \beta = \pi$ of Entry 3.9 was first established by M. Lerch [20] in 1901, but the general theorem was not proved in print until S. L. Malurkar [21] did so in 1925. Inspired by two papers by E. Grosswald [14], [15], the author established a proof of Entry 3.9, the first claim from Ramanujan's notebooks which the author examined; his first paper on Ramanujan's work was the survey paper [1] on Ramanujan's formula for $\zeta(2n+1)$. That paper and the author's book [4, p. 276] contain a plethora of references for the many proofs and generalizations of Entry 3.9. Sitaramachandrarao [29] gave a proof of Entry 3.9 based on his partial fraction decomposition (3.8), and so his proof is similar to that of Ramanujan. Further proofs and generalizations have been given by D. Bradley [8] and S. Kanemitsu, Y. Tanigawa, and M. Yoshimoto [16], [17]. An especially interesting proof, arising out of a very general asymptotic formula, has been devised by M. Katsurada [18].

4. A CORRECT PARTIAL FRACTION DECOMPOSITION AND SECH SUMS

As in the previous section, we alter Ramanujan's notation by setting $n = w = z^2$.

Entry 4.1 (p. 318, formula (22)). *If α and β are positive numbers such that $\alpha\beta = \pi^2/4$, and if $w \neq (2m+1)^2\alpha, (2m+1)^2\beta, 0 \leq m < \infty$, then*

$$\frac{\pi}{4} \sec(\sqrt{w\alpha}) \operatorname{sech}(\sqrt{w\beta}) = \sum_{m=0}^{\infty} (-1)^m \left\{ \frac{(2m+1)\alpha \operatorname{sech}(2m+1)\alpha}{w + (2m+1)^2\alpha} - \frac{(2m+1)\beta \operatorname{sech}(2m+1)\beta}{w - (2m+1)^2\beta} \right\}. \quad (4.1)$$

Proof. We apply the Mittag-Leffler Theorem to

$$f(z) := \frac{\pi}{4} \sec(z\sqrt{\alpha}) \operatorname{sech}(z\sqrt{\beta}),$$

which has simple poles at $z = (2m+1)\pi/(2\sqrt{\alpha})$ and $z = (2m+1)\pi i/(2\sqrt{\beta})$, for each integer m . The residues are easily calculated to be

$$R((2m+1)\pi/(2\sqrt{\alpha})) = -\frac{(-1)^m \pi}{4\sqrt{\alpha}} \operatorname{sech}(2m+1)\beta \quad (4.2)$$

and

$$R((2m+1)\pi i/(2\sqrt{\beta})) = \frac{(-1)^m \pi}{4i\sqrt{\beta}} \operatorname{sech}(2m+1)\alpha, \quad (4.3)$$

where we used the relation $\alpha\beta = \pi^2/4$. By (4.2), the contributions from the poles $z = (2m+1)\pi/(2\sqrt{\alpha})$ and $z = -(2m+1)\pi/(2\sqrt{\alpha})$, $m \geq 0$, to the partial fraction decomposition of $f(z)$ are

$$\begin{aligned} & \frac{(-1)^m \pi}{4\sqrt{\alpha}} \left(-\frac{\operatorname{sech}(2m+1)\beta}{z - (2m+1)\pi/(2\sqrt{\alpha})} + \frac{\operatorname{sech}(2m+1)\beta}{z + (2m+1)\pi/(2\sqrt{\alpha})} \right) \\ &= -\frac{(-1)^m (2m+1)\beta \operatorname{sech}(2m+1)\beta}{z^2 - (2m+1)^2\beta}, \end{aligned} \quad (4.4)$$

where we used the equality $\alpha\beta = \pi^2/4$. Next, by (4.3), the contributions of the poles $z = (2m+1)\pi i/(2\sqrt{\beta})$ and $z = -(2m+1)\pi i/(2\sqrt{\beta})$, $m \geq 0$, to the partial fraction decomposition of $f(z)$ are

$$\begin{aligned} & \frac{(-1)^m \pi}{4i\sqrt{\beta}} \left(\frac{\operatorname{sech}(2m+1)\alpha}{z - (2m+1)\pi i/(2\sqrt{\beta})} - \frac{\operatorname{sech}(2m+1)\alpha}{z + (2m+1)\pi i/(2\sqrt{\beta})} \right) \\ &= \frac{(-1)^m (2m+1)\alpha \operatorname{sech}(2m+1)\alpha}{z^2 + (2m+1)^2\alpha}, \end{aligned} \quad (4.5)$$

upon using the equality $\alpha\beta = \pi^2/4$. Thus, applying the Mittag-Leffler Theorem and using (4.4) and (4.5), we find that there exists an entire function $g(z)$ such that

$$\begin{aligned} \frac{\pi}{4} \sec(z\sqrt{\alpha}) \operatorname{sech}(z\sqrt{\beta}) &= \sum_{m=0}^{\infty} (-1)^m \left\{ \frac{(2m+1)\alpha \operatorname{sech}(2m+1)\alpha}{z^2 + (2m+1)^2\alpha} \right. \\ &\quad \left. - \frac{(2m+1)\beta \operatorname{sech}(2m+1)\beta}{z^2 - (2m+1)^2\beta} \right\} + g(z). \end{aligned} \quad (4.6)$$

Letting $z \rightarrow \infty$, we find that $\lim_{z \rightarrow \infty} g(z) = 0$. Hence, $g(z) \equiv 0$, and thus (4.1) follows to complete the proof. \square

An equivalent formulation of Entry 4.1 is found as Entry 19(iv) in Chapter 14 of Ramanujan's second notebook [24], [4, p. 273], where a different kind of proof was indicated by Ramanujan.

Entry 4.2 (p. 320, formula (30)). *If $\alpha\beta = \pi^2/4$, where α and β are positive numbers, and if r is any positive integer, then*

$$\alpha^r \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)^{2r-1}}{\cosh(2m+1)\alpha} + (-\beta)^r \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)^{2r-1}}{\cosh(2m+1)\beta} = 0. \quad (4.7)$$

Proof. (nonrigorous) Return to (4.1) and formally expand the summands on the right side into geometric series to deduce that

$$\begin{aligned} \frac{\pi}{4} \sec(\sqrt{w\alpha}) \operatorname{sech}(\sqrt{w\beta}) &= \sum_{m=0}^{\infty} (-1)^m \left\{ \frac{(2m+1)\alpha}{w} \operatorname{sech}(2m+1)\alpha \sum_{k=0}^{\infty} \left(-\frac{(2m+1)^2\alpha}{w} \right)^k \right. \\ &\quad \left. - \frac{(2m+1)\beta}{w} \operatorname{sech}(2m+1)\beta \sum_{k=0}^{\infty} \left(\frac{(2m+1)^2\beta}{w} \right)^k \right\}. \end{aligned} \quad (4.8)$$

Equating coefficients of $1/w^r$, $r \geq 1$, on both sides of (4.8), we find that

$$0 = \sum_{m=0}^{\infty} (-1)^{m+r-1} (2m+1)^{2r-1} \alpha^r \operatorname{sech}(2m+1)\alpha - \sum_{m=0}^{\infty} (-1)^m (2m+1)^{2r-1} \beta^r \operatorname{sech}(2m+1)\beta,$$

which is easily seen to be equivalent to (4.7). \square

Entry 4.2 is Entry 14 of Chapter 14 in Ramanujan's second notebook [24], [4, p. 262], and the first proof known to us was given by Malurkar [21]. See [4, p. 262] for further references and comments.

As with previous theorems, Ramanujan provides an alternative version of Entry 4.2 in terms of divisor sums. The details are similar to those above, and so we do not give them, but we remark that care must be given to the signs of the summands.

Entry 4.3 (p. 321, formula (31)). *If α and β are positive numbers such that $\alpha\beta = \pi^2/4$, and if r is any positive integer, then*

$$\alpha^r \sum_{m=0}^{\infty} (-1)^m \sigma_{2r-1}(m) e^{-(2m+1)\alpha} + (-\beta)^r \sum_{m=0}^{\infty} (-1)^m \sigma_{2r-1}(m) e^{-(2m+1)\beta} = 0.$$

Recall that the Euler numbers E_{2k} , $k \geq 0$, are defined by [13, p. 42, formula 1.411, no. 10]

$$\operatorname{sech} z = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} z^{2k}, \quad |z| < \pi/2. \quad (4.9)$$

Entry 4.4 (p. 321, formula (32)). *If α and β are positive numbers such that $\alpha\beta = \pi^2/4$, if r is any positive integer, and if χ denotes the nonprincipal, primitive character of modulus 4, as in Section 2, then*

$$\begin{aligned} & 2\alpha^{1-r} \sum_{m=1}^{\infty} \frac{\chi(m)m^{1-2r}}{\cosh(m\alpha)} + 2(-\beta)^{1-r} \sum_{m=1}^{\infty} \frac{\chi(m)m^{1-2r}}{\cosh(m\beta)} \\ &= 4\alpha^{1-r} \sum_{m=1}^{\infty} \chi(m)\sigma_{1-2r}(m)e^{-m\alpha} + 2(-\beta)^{1-r} \sum_{m=1}^{\infty} \chi(m)\sigma_{1-2r}(m)e^{-m\beta} \\ &= \frac{\pi}{2} \sum_{k=0}^{r-1} (-1)^k \frac{E_{2k}E_{2r-2-2k}}{(2k)!(2r-2-2k)!} \alpha^{r-1-k} \beta^k. \end{aligned} \quad (4.10)$$

Proof. Return to (4.1) and expand both sides in Taylor series about 0. Using (4.9), we find that

$$\begin{aligned} & \frac{\pi}{4} \sum_{j=0}^{\infty} (-1)^j \frac{E_{2j}}{(2j)!} (w\alpha)^j \cdot \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} (w\beta)^k \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \operatorname{sech}(2m+1)\alpha \sum_{r=0}^{\infty} (-1)^r \left(\frac{w}{(2m+1)^2\alpha} \right)^r \\ & \quad + \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \operatorname{sech}(2m+1)\beta \sum_{r=0}^{\infty} (-1)^r \left(\frac{w}{(2m+1)^2\beta} \right)^r. \end{aligned} \quad (4.11)$$

In (4.11) we equate coefficients of w^{r-1} , $r \geq 1$, on both sides to deduce that

$$\begin{aligned} \frac{\pi}{4} \sum_{j=0}^{r-1} (-1)^j \frac{E_{2j}E_{2r-2j-2}}{(2j)!(2r-2j-2)!} \alpha^j \beta^{r-j-1} &= \alpha^{1-r} \sum_{m=0}^{\infty} \frac{(-1)^{m+1-r} \operatorname{sech}(2m+1)\alpha}{(2m+1)^{2r-1}} \\ & \quad + \beta^{1-r} \sum_{m=0}^{\infty} \frac{(-1)^m \operatorname{sech}(2m+1)\beta}{(2m+1)^{2r-1}}. \end{aligned} \quad (4.12)$$

Now set $j = r - 1 - k$ in the sum on the left side of (4.12) and multiply both sides of (4.12) by $2(-1)^{r-1}$. We then readily deduce the equality of the first and third expressions in (4.10). The first equality of (4.10) follows as usual from expanding the summands on the left side into geometric series. \square

Entry 4.4 appears in two formulations, Entries 21(ii), (iii), in Chapter 14 of Ramanujan's second notebook [24], [4, pp. 276–277]. The first proofs of Entry 4.4 were found by Malurkar [21] and Chowla [11], [12, pp. 143–170] and further references can be found in [4, p. 277].

Entry 4.5 (p. 321, formula (33)). *We have*

$$4 \sum_{m=0}^{\infty} (-1)^m \sigma_{-1}(m) e^{-(2m+1)\alpha} + 4 \sum_{m=0}^{\infty} (-1)^m \sigma_{-1}(m) e^{-(2m+1)\beta} = \frac{\pi}{2}.$$

Proof. Set $r = 1$ in Entry 4.4. \square

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