RAMANUJAN'S CLASS INVARIANTS WITH APPLICATIONS TO THE VALUES OF q–CONTINUED FRACTIONS AND THETA FUNCTIONS

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1. Introduction

As usual, set

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1,$$

and, following Ramanujan, let

$$\chi(q) = (-q;q^2)_\infty.$$

If n is any postitive rational number and $q=\exp(-\pi\sqrt{n})$, the two *class invariants* G_n and g_n are defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \qquad \text{and} \qquad g_n := 2^{-1/4}q^{-1/24}\chi(-q). \tag{1.1}$$

In the notation of H. Weber [43], $G_n =: 2^{-1/4}f(\sqrt{-n})$ and $g_n =: 2^{-1/4}f_1(\sqrt{-n})$. The term "invariant" is due to Weber. If $\mathbb{Q}(\omega)$ is the algebraic number field generated by the complex quadratic integer ω, such that $\{1,\omega\}$ is a basis for the algebraic integral domain, which is also called the maximal order of $\mathbb{Q}(\omega)$, then the absolute class field of $\mathbb{Q}(\omega)$ is generated by the modular invariant $j(\omega)$, which Weber calls a class invariant. More generally, if $\mathbb{Z}[\omega]$ is any order of $\mathbb{Q}(\omega)$, then the ring field of $\mathbb{Z}[\omega]$ is generated by $j(\omega)$. Often $f(\omega)$, or $f_1(\omega)$, is in $\mathbb{Q}(\omega, j(\omega))$, and in such cases, Weber calls $f(\omega)$, or $f_1(\omega)$, an invariant as well. As G.N. Watson [37] remarked, "For reasons which had commended themselves to Weber and Ramanujan independently, it is customary to determine G_n for odd values of n, and g_n for even values of n."

If, as usual in the theory of elliptic functions, $k=k(q)$ denotes the modulus, then the singular modulus k_n is defined by $k_n = k(e^{-\pi\sqrt{n}})$, where n is a positive rational integer. Following Ramanujan, set $\alpha = k^2$ and $\alpha_n = k_n^2$. Since [3, p. 124]

$$\chi(q) = 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24} \qquad \text{and} \qquad \chi(-q) = 2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24},$$

it follows from (1.1) that

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24} \qquad \text{and} \qquad g_n = 2^{-1/12}(1-\alpha_n)^{1/12}\alpha_n^{-1/24}. \tag{1.2}$$

It is well–known that G_n and g_n are algebraic; for example, see Cox's book [18, p. 214, Theorem 10.23; p. 257, Theorem 12.17]. It follows trivially from (1.2) that α_n is algebraic. However, much more is known [17].

Theorem 1.1.
(a) If $n \equiv 1 \pmod 4$, *then* G_n *and* $2\alpha_n$ *are units.*
(b) If $n \equiv 3 \pmod 8$, *then* $2^{-1/12}G_n$ *and* $2^2\alpha_n$ *are units.*
(c) If $n \equiv 7 \pmod 8$, *then* $2^{-1/4}G_n$ *and* $2^4\alpha_n$ *are units.*
(d) If $n \equiv 2 \pmod 4$,, *then* g_n *and* α_n *are units.*

Section 2 will be devoted to describing Ramanujan's determinations of G_n and g_n and recent efforts to establish Ramanujan's asserted values. We also offer a few remarks about the determination of α_n.

The celebrated Rogers–Ramanujan continued fractions $R(q)$ and $S(q)$ are defined by

$$R(q) = \frac{q^{1/5}}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \cdots, \qquad |q| < 1, \tag{1.3}$$

and

$$S(q) = -R(-q).$$

As intimated, (1.3) was first introduced by L.J. Rogers [30] in 1894. However, most of our knowledge about $R(q)$ arises from Ramanujan who communicated several of its properties in his first and second letters to Hardy [28, pp. xxvii, xxviii]. Several of Ramanujan's theorems on $R(q)$ and $S(q)$ can also be found in his notebooks [27] and lost notebook [29].

Ramanujan studied several other q–continued fractions. In particular, in his second letter to Hardy [28, p. xxviii], Ramanujan offered two theorems on the "cubic" continued fraction

$$G(q) := \frac{q^{1/3}}{1} \, \underset{+}{} \, \frac{q+q^2}{1} \, \underset{+}{} \, \frac{q^2+q^4}{1} \, \underset{+}{} \, \frac{q^3+q^6}{1} \, \underset{+}{} \cdots. \tag{1.4}$$

In Section 3, we focus on explicit values for $R(q)$ and $G(q)$. Class invariants and modular equations are central to our determinations.

At the heart of the theorems on class invariants and q–continued fractions are theta functions. In Ramanujan's notation, set, for $|q| < 1$,

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}, \tag{1.5}$$

and

$$f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.6}$$

In more standard notation, $\varphi(q) = \vartheta_3(0,z), 2q^{1/8}\psi(q) = \vartheta_2(0,z/2)$, and $q^{1/24}f(-q) = \eta(z/2)$, where $q = \exp(\pi i z)$, ϑ_2 and ϑ_3 are classical theta functions [44, p. 464], and $\eta(z)$ denotes the Dedekind eta–function.

Section 4 is devoted to explicit determinations of $\varphi(q)$ and certain products of the functions φ and ψ. The determination of $\varphi(q)$ is equivalent to evaluating the complete elliptic integral of the first kind $K(k)$ and the ordinary hypergeometric series

${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$, as seen in the next paragraph. The theta products of Ramanujan which we examine have remarkable properties.

Because modular equations will be frequently mentioned in the sequel, we conclude this introduction by giving a precise definition of a modular equation, as understood by Ramanujan. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, ℓ, and $\ell' := \sqrt{1-\ell^2}$, respectively, where $0 < k, \ell < 1$. Suppose that

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.7}$$

for some positive rational integer n. A relation between k and ℓ induced by (1.7) is called a *modular equation of degree* n. Following Ramanujan, set

$$\alpha = k^2 \qquad \text{and} \qquad \beta = \ell^2.$$

We often say that β has degree n over α. If

$$q = \exp(-\pi K'/K), \tag{1.8}$$

one of the most fundamental relations in the theory of elliptic functions is given by the formula [3, pp. 101, 102]

$$\varphi^2(q) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) = \frac{2}{\pi}\int_0^{\pi/2} \frac{\theta}{\sqrt{1-k^2\sin^2\theta}} =: \frac{2}{\pi}K(k). \tag{1.9}$$

The first equality in (1.9) and elementary theta function identities make it possible to write each modular equation as a theta function identity [3, pp. 122–124]. (The second equality in (1.9) arises from expanding the integrand in a binomial series and integrating termwise.) Lastly, the multiplier m of degree n is defined by

$$m = \frac{\varphi^2(q)}{\varphi^2(q^n)}. \tag{1.10}$$

2. Class Invariants

Weber primarily was motivated to calculate class invariants so that he could construct Hilbert class fields. The table at the end of Weber's book [43, pp. 721–726] contains the values of 105 class invariants, although in some cases he only gives the irreducible polynomials satisfied by G_n or g_n.

Ramanujan apparently had no knowledge of class field theory and independently calculated class invariants for different reasons. Scattered throughout his first notebook [27] are the values of 106 class invariants. On pages 294–299 in his second notebook [27], Ramanujan gives a table of 77 class invariants, three of which are not found in the first notebook. Since the second notebook is an enlarged edition of the first, it is unclear why Ramanujan failed to record 32 class invariants that he offered in the first notebook. While in England, Ramanujan learned of Weber's work, and so his table of 46 class invariants in his paper [26], [28, pp. 23–39] does not contain any that are found in Weber's book [43]. Except for G_{325} and G_{363}, all of the remaining values are found in Ramanujan's notebooks.

The value of G_{1353} was communicated by Ramanujan [28, p. xxix, eq. (23)] in his second letter to Hardy and was first proved by Watson [36]. In a letter of 1 October 1930 written to B.M. Wilson [13, pp. 237, 238], Watson confided, " ... but 23 which deals with the singular modulus associated with 1353 is included; I was pleased at getting this out, because the bulk of the singular moduli in the Notebooks can be obtained in the same way; I have not yet written out any of the latter properly, but have worked at many of them enough to make the formal writing out an easy matter. You will be interested to hear how Ramanujan got no. 23, particularly when you look at the length of the answer. I am absolutely convinced that he guessed it; I get out the others mentioned above by the same process of guessing."

In the first [37] of two papers devoted to proving Ramanujan's class invariants, Watson employed this "empirical process" to calculate fourteen of Ramanujan's invariants and opined, "I believe that fourteen were obtained by Ramanujan by means of the empirical process which I described in the discussion of G_{1353}." We emphasize that Watson's process is not rigorous. Watson numerically calculated the values of G_n and certain related invariants. He then numerically calculated certain polynomials of products and quotients of these invariants. Next, he found integers in appropriate real quadratic fields that numerically agreed with these polynomials. Then Watson *assumed* that, in fact, they are equal and so solved for G_n. It seems to us that determining these polynomials, the appropriate real quadratic field, and the algebraic integers that approximate them is extremely difficult without knowing beforehand, as Watson did, the requisite number fields and the appropriate algebraic integers. Thus, we are convinced, as much as Watson was to the contrary, that his "empirical process" was definitely not the method employed by Ramanujan in determining the values of class invariants.

In his second paper [38] devoted to calculating Ramanujan's class invariants, Watson utilized modular equations. Here Watson's methods are likely to be close to those of Ramanujan. However, the method is applicable only when n is a square, a small prime times a square, or twice a certain prime, and so can be applied in only a limited number of cases.

Watson wrote four further papers [39]–[42] on the calculation of class invariants. Among the dozens of invariants calculated by Watson in these papers were three previously unproved invariants found in Ramanujan's paper [26], namely, for $n =$ 81, 147, 289, as well as eleven invariants of Ramanujan that had been previously verified.

Thus, after Watson's papers, a total of eighteen invariants of Ramanujan remained to be verified. In the introduction to [37], Watson remarked, "It is intended to publish the calculations involved in the construction of the set N + Q (the invariants appearing in both Ramanujan's paper [26] and the second notebook) as part of the commentary on the note–books by Dr. B.M. Wilson and myself." Although Watson and Wilson's efforts to edit Ramanujan's notebooks have been preserved in the library at Trinity College, Cambridge, Watson's work on these eighteen invariants cannot be found there. Thus, if Watson actually calculated these invariants, it appears that his work has been lost.

Proofs of Ramanujan's remaining eighteen class invariants recently have been given for the first time by the authors [8], [9]. This work will be briefly described.

Five of the eighteen values of n are divisible by 9, namely, $n = 117$, 153, 441, 90, and 198. Our starting point is a relation connecting g_n and g_{9n} found on page 318 of Ramanujan's first notebook [27], but not in his second notebook. K.G. Ramanathan [23], [24] noticed this relation, but apparently he never gave a proof. We [8] have proved this formula as well as a companion formula relating G_n and G_{9n}. Variants of these two formulas were previously found by J.M. and P.B. Borwein [14, pp. 145, 149]. They also derived formulas connecting G_{81n} with G_n and G_{9n}, and g_{81n} with g_n and g_{9n}.

We state below our theorem relating G_n and G_{9n}.

Theorem 2.1. *Let*

$$p = G_n^4 + G_{-n}^4. \tag{2.1}$$

Then

$$\begin{aligned} G_{9n} =& G_n \left(p + \sqrt{p^2 - 1}\right)^{1/6} \\ & \times \left\{ \sqrt{\frac{p^2 - 2 + \sqrt{(p^2-1)(p^2-4)}}{2}} + \sqrt{\frac{p^2 - 4 + \sqrt{(p^2-1)(p^2-4)}}{2}} \right\}^{1/3}. \end{aligned} \tag{2.2}$$

Our proof relies on a modular equation of degree 3 found as Entry 5(xii) in Chapter 19 of Ramanujan's second notebook [3, p. 231].

Lemma 2.2 (Modular Equation of degree 3). *Let*

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \qquad \text{and} \qquad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}. \tag{2.3}$$

Then

$$Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0. \tag{2.4}$$

Lemma 2.2 is useful in the theory of class invariants, because, by (1.2), the expressions P and Q in (2.3) can be expressed in terms of class invariants.

As an illustration, we briefly indicate how to prove that

$$G_{117} = \frac{1}{2}\left(\frac{3+\sqrt{13}}{2}\right)^{1/4} (2\sqrt{3} + \sqrt{13})^{1/6} \left(3^{1/4} + \sqrt{4 + \sqrt{3}}\right).$$

From Weber's treatise [43, p. 721], or Ramanujan's paper [26],

$$G_{13} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4}.$$

It follows easily from (2.1) that $p = \sqrt{13}$. Thus, by Theorem 2.1,

$$G_{117} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4} (2\sqrt{3}+\sqrt{13})^{1/6} \left\{\sqrt{\frac{11+6\sqrt{3}}{2}} + \sqrt{\frac{9+6\sqrt{3}}{2}}\right\}^{1/3}.$$

It therefore remains to show that

$$\left\{\sqrt{\frac{11+6\sqrt{3}}{2}} + \sqrt{\frac{9+6\sqrt{3}}{2}}\right\}^{1/3} = \frac{1}{2}\left(3^{1/4} + \sqrt{4+\sqrt{3}}\right),$$

and we refer to [8] for details.

We remark that Watson [37] computed G_{333}, G_{765}, g_{522}, and g_{630} by his "empirical process," but easier proofs can be constructed by using Theorem 2.1 and its analogue for g_n.

The remaining thirteen values of G_n are for $n = 65$, 69, 77, 141, 145, 205, 213, 217, 265, 301, 445, 505, and 553. Note that each value of n has the form pq, where p is a "small" prime (3, 5, or 7), and q is a "large" prime. Quite astonishingly, the class number for each of these thirteen imaginary quadratic fields $\mathbb{Q}(\sqrt{-n})$ equals 8. Moreover, there are precisely two classes per genus in each case. It was considerably more difficult to prove these thirteen values for G_n than it was for the five previously described values.

We devised three approaches [9]. The first employs Kronecker's limit formula and extends a method due to C.L. Siegel and Ramanathan [20], [24]. This method is perhaps the most successful, since it can be utilized to establish all thirteen invariants. Furthermore, Ramanathan had used a simpler theorem based on Kronecker's limit formula to calculate several invariants, and Zhang [45] has developed further theorems of this type to give rigorous proofs of some of the class invariants unrigorously established by Watson [37]. Ramanujan was unfamiliar with Kronecker's limit formula and the additionally needed concepts of ideal classes and their characters, L–series of an algebraic number field, genus theory and characters, class numbers, and units, and so clearly this approach is not close to any of the methods that Ramanujan might have used.

The second method employs class field theory to make Watson's empirical method rigorous in the cases at hand. This procedure can also be used to calculate all thirteen invariants. Besides being rigorous, our version is superior to that of Watson because there is no need to know the value of the invariant in advance. In fact, a couple new invariants were calculated by using this method. However, Ramanujan was unfamiliar with class field theory, and so this was clearly not the mode employed by Ramanujan.

Our third method employs modular equations. However, Watson's [38] methods using modular equations are inapplicable, because the values of n under consideration here are not of the forms described above, and so a new approach had to be devised. In particular, we used the following general principle. Let p and r denote coprime, positive integers. Set

$$q = \exp(-\pi\sqrt{p/r}) \qquad \text{and} \qquad q' = \exp(-\pi\sqrt{pr}), \tag{2.5}$$

and let β have degree r over α. Then, by (1.2),

$$G_{p/r} = \{4\alpha(1-\alpha)\}^{-1/24} \qquad \text{and} \qquad G_{pr} = \{4\beta(1-\beta)\}^{-1/24}. \tag{2.6}$$

Then, by using (1.7), (1.8), and (2.5), we can conclude that

(2.7) *If β has degree r over α, then β has degree p over $1-\alpha$.*

(See [9] for more details.) We emphasize that (2.7) is not a statement about modular equations in general; (2.7) is only valid when q and q' have the special values given in (2.5). Unfortunately, this third approach can only be utilized to establish Ramanujan's class invariants for $n = 65$, 69, 77, 141, 145, and 213. To establish the remaining seven invariants by employing modular equations, we would need modular equations of degrees equal to the larger of the two prime factors of n in each case, namely, for degrees 31, 41, 43, 53, 79, 89, and 101. Apparently, only for degree 31 did Ramanujan derive a modular equation, for he recorded no modular equations for the remaining six degrees in his notebooks. An attempt to calculate G_{217} using this modular equation of degree 31 generated horrible equations. Thus, Ramanujan's methods remain elusive, since he somehow must have circumvented the need for modular equations of high degree in at least these seven instances.

To give readers some idea of how modular equations are employed in determining class invariants, we give a brief sketch of a proof of Ramanujan's value

$$G_{69} = \left(\frac{5+\sqrt{23}}{\sqrt{2}}\right)^{1/12} \left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{1/8} \left(\sqrt{\frac{6+3\sqrt{3}}{4}} + \sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{1/2}. \tag{2.8}$$

We need two modular equations of degree 3. The first is given in Lemma 2.2; the second is given in Entry 5(ix) of Chapter 19 in the second notebook [3, p. 231], namely,

$$\{\alpha(1-\beta)\}^{1/2} + \{(1-\alpha)\beta\}^{1/2} = 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}. \tag{2.9}$$

We also require two modular equations of degree 23 [3, p. 411, Entry 15(i),(ii)]. If β has degree 23 over α, then

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1 \tag{2.10}$$

and

$$\begin{aligned} 1+&(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2^{4/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \\ &= \left\{2\left(1+(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}\right)\right\}^{1/2}. \end{aligned} \tag{2.11}$$

We now apply (2.7) with $r = 3$ and $p = 23$. Thus, β has degree 23 over $1-\alpha$, and so we replace α by $1-\alpha$ in (2.10) and (2.11). By judiciously combining these new equations with (2.9), we eventually find that

$$2u^6 - 8\sqrt{2}u^5 + 24u^4 - 22\sqrt{2}u^3 + 24u^2 - 8\sqrt{2}u + 2 = 0, \tag{2.12}$$

where $u = G_{69}G_{23/3}$. The form of (2.12) suggests the substitution $x = u+1/u$, and so (2.12) takes the shape

$$x^3 - 4\sqrt{2}x^2 + 9x - 3\sqrt{2} = 0.$$

The root $\sqrt{2}$ can be eliminated by numerical considerations, and we find that $x = (3+\sqrt{3})/\sqrt{2}$ and

$$\frac{1}{G_{69}G_{23/3}} = \frac{1}{u} = \sqrt{\frac{6+3\sqrt{3}}{4}} + \sqrt{\frac{2+3\sqrt{3}}{4}}, \tag{2.13}$$

since $u < 1$. Now observe that, from (2.6) with $p = 23$ and $r = 3$, Lemma 2.2 yields an equation in the two variables u and $v = G_{69}/G_{23/3}$. Using (2.13) in Lemma 2.2, we eventually find that

$$v = \left(\sqrt{748+432\sqrt{3}} + \sqrt{747+432\sqrt{3}}\right)^{1/6}. \tag{2.14}$$

Combining (2.13) and (2.14), we deduce that

$$G_{69} = \left(\sqrt{748+432\sqrt{3}} + \sqrt{747+432\sqrt{3}}\right)^{1/12} \left(\sqrt{\frac{6+3\sqrt{3}}{4}} + \sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{1/2}. \tag{2.15}$$

Comparing (2.15) with (2.8), we see that we must establish a certain equality between radicals, which is easily shown. See [9] for complete details of the proof sketched here.

We conclude this section with a few comments about the calculation of singular moduli α_n. If G_n or g_n is known, then α_n is easily found from (1.2) by solving a quadratic equation. However, the representation that is obtained normally does not evince the fact (from Theorem 1.1) that α_n can be expressed in terms of units. In his first notebook, Ramanujan offered 21 values of α_n for even n and twelve values for odd n. For even n, Ramanujan recorded in his first notebook a remarkable formula to facilitate the calculation of α_n. This formula was noticed by Watson [35] who proved it in order to verify Ramanujan's value for α_{210},

$$\begin{aligned}\alpha_{210} =& (\sqrt{2}-1)^4(2-\sqrt{3})^2(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2 \\ &\times (\sqrt{10}-3)^2(4-\sqrt{15})^4(\sqrt{15}-\sqrt{14})^2(6-\sqrt{35})^2.\end{aligned}$$

communicated by Ramanujan [28, p. xxix] in his second letter to Hardy. For odd n, the calculation of α_n is somewhat more difficult. Proofs for all 33 values of α_n claimed by Ramanujan can be found in our paper [12].

3. q–Continued Fractions

In both his first and second letters to Hardy, Ramanujan [28, pp. xxvii, xxviii] communicated theorems about $R(q)$ and $S(q)$. In particular, in his first letter, he asserted that

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} \tag{3.1}$$

and

$$S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}. \tag{3.2}$$

The evaluation (3.1) follows easily from a reciprocity theorem for $R(q)$, which Ramanujan stated in his second letter, and which was first proved by Watson [34]. The evaluation (3.2) follows from a similar reciprocity theorem for $S(q)$, which apparently Ramanujan did not communicate to Hardy, but which is found in his notebooks [27, p. 204], [3, p. 83] and first proved by Ramanathan [21]. However, (3.2) was first proved by Watson [33] in a different manner.

In both his first [27] and lost notebooks [29], Ramanujan recorded several other evaluations of $R(q)$ and $S(q)$. In fact, page 210 in his lost notebook comprises a list of evaluations and (mostly) intended evaluations. The authors [6], [10] have recently proved all the aforementioned evaluations claimed by Ramanujan in his notebooks. In particular, we have proved four general theorems providing explicit formulas for $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ in terms of class invariants. The similarity evinced in (3.1) and (3.2) is manifest in these general theorems, which depend upon two formulas for $R(q)$ found as Entry 11(iii) in Chapter 19 in Ramanujan's second notebook [3, pp. 265, 266]. If $f(-q)$ is given by (1.6), then

$$R^{-1}(q) - 1 - R(q) = q^{-1/5}\frac{f(-q^{1/5})}{f(-q^5)} \tag{3.3}$$

and

$$R^{-5}(q) - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{3.4}$$

Both (3.3) and (3.4) were first proved by Watson [33]; proofs can also be found in [3, p. 267]. We discuss here only the evaluation of $R(e^{-2\pi\sqrt{n}})$; companion theorems can be established for $S(e^{-\pi\sqrt{n}})$.

First note that, by (3.3) and (3.4), it suffices to evaluate either

$$A := e^{2\pi\sqrt{n}/5}\frac{f(-e^{-2\pi\sqrt{n}/5})}{f(-e^{-10\pi\sqrt{n}})} \qquad \text{or} \qquad A_1 := e^{2\pi\sqrt{n}/6}\frac{f(-e^{-2\pi\sqrt{n}})}{f(-e^{-10\pi\sqrt{n}})}, \tag{3.5}$$

for then $R(e^{-2\pi\sqrt{n}})$ can be found by merely solving a quadratic equation.

Theorem 3.1. *Let A be defined by (3.5), and let*

$$U := \sqrt{\frac{g_{25n}}{g_{n/25}}} \qquad \textit{and} \qquad V := \sqrt{\frac{G_{25n}}{G_{n/25}}}.$$

Then

$$\frac{A}{\sqrt{5}U} + \frac{\sqrt{5}U}{A} = (U+U^{-1})^2\left(\frac{U+U^{-1}}{\sqrt{5}} - \frac{\sqrt{5}}{U+U^{-1}}\right)$$

and

$$\frac{A}{\sqrt{5}V} - \frac{\sqrt{5}V}{A} = (V-V^{-1})^2\left(\frac{V-V^{-1}}{\sqrt{5}} + \frac{\sqrt{5}}{V-V^{-1}}\right).$$

Theorem 3.2. *Let A_1 be defined by (3.5), and let*

$$U_1 := \frac{g_{25n}}{g_n} \qquad \text{and} \qquad V_1 := \frac{G_{25n}}{G_n}.$$

Then

$$\frac{A_1^2}{\sqrt{5}U_1} + \frac{\sqrt{5}U_1}{A_1^2} = \frac{1}{\sqrt{5}}\left(U_1^3 + U_1^{-3}\right)$$

and

$$\frac{A_1^2}{\sqrt{5}V_1} - \frac{\sqrt{5}V_1}{A_1^2} = \frac{1}{\sqrt{5}}\left(V_1^3 - V_1^{-3}\right).$$

The proof of Theorem 3.1 depends upon (3.3) and two modular equations of Ramanujan of degree 25 [3, p. 291, Entry 15 (i), (ii)]. The proof of Theorem 3.2 rests upon (3.4) and two modular equations of degree 5 [3, pp. 281, 282, Entry 13 (xii)].

To illustrate Theorem 3.1, let $n = 9$. Then after determining G_{225} and $G_{9/25}$ and manipulating some radicals, we deduce the following corollary [10], which we formulate in the same manner as Ramanujan [27, p. 311]. Another proof of Corollary 3.3 can be found in [6].

Corollary 3.3. *Let $a = 60^{1/4}$ and $b = 2 - \sqrt{3} + \sqrt{5}$. If $2c = \frac{a+b}{a-b}\sqrt{5} + 1$, then*

$$F(e^{-6\pi}) = \sqrt{c^2+1} - c.$$

To illustrate Theorem 3.2, let $n = 1/5$. Then $V_1 = G_5/G_{1/5} = 1$, since $G_n = G_{1/n}$. It follows trivially from Theorem 3.2 that $A_1^2 = \sqrt{5}$. We then conclude from (3.4) that

$$R^5(e^{-2\pi/\sqrt{5}}) = \sqrt{\left(\frac{5\sqrt{5}+11}{2}\right)^2 + 1} - \frac{5\sqrt{5}+11}{2}.$$

K.G. Ramanathan devoted several papers, in particular, [21], [22], and [23], to evaluating special cases of the Rogers–Ramanujan continued fraction.

In his first letter to Hardy, Ramanujan [28, p. xxvii] made the very strong assertion,

(3.6) $R(e^{-\pi\sqrt{n}})$ can be exactly found if n be any positive rational quantity.

Watson [33] vaguely discussed (3.6) and claimed that $R(e^{-\pi\sqrt{n}})$ is algebraic. Using a lemma which can be found in Stark's paper [32], we [10] have proved the stronger assertion that $R(e^{-\pi\sqrt{n}})$ is a unit for each postive rational number n.

Ramanujan found many other properties of the Rogers–Ramanujan continued fraction which he recorded at scattered places in the unorganized portion of his second notebook, his third notebook, and his lost notebook. For proofs of some of these theorems, see [2] and [1]. In particular, Ramanujan found beautiful equations relating $R(q)$ with $R(-q)$, $R(q^2)$, $R(q^3)$, $R^(q^4)$, or $R(q^5)$. For example [2], if $u = R(q)$ and $v = R(q^2)$, then

$$\frac{v-u^2}{v+u^2} = uv^2. \tag{3.7}$$

These equations can also be used to find certain values of the Rogers–Ramanujan continued fraction, but generally the form of the evaluation that one so obtains is not attractive.

In a fragment published with his lost notebook [29, pp. 363–366], Ramanujan gives a list of fourteen theorems that he established for the Rogers–Ramanujan continued fraction. This is followed by five theorems for his cubic continued fraction $G(q)$, defined in (1.4). In fact, Ramanujan writes, "I have also found empirically the following result ... and many results analogous to the previous continued fraction." Motivated by this claim, Chan [15], [16] has developed much of the theory for $G(q)$ and likely has found several of the "many results" not divulged by Ramanujan.

First, we offer an exact formula for $G(e^{-\pi\sqrt{n}})$ that was proved in [8]. This is analogous to the formulas for $R(e^{-2\pi\sqrt{n}})$ that arise from Theorems 3.1 and 3.2; a similar result can be proved for $G(-e^{-\pi\sqrt{n}})$.

Theorem 3.4. *If $G(q)$ is defined by (1.4) and p is given by (2.1), then, for every positive rational number n,*

$$G(e^{-\pi\sqrt{n}})$$
$$=\sqrt{\frac{\sqrt{p^2+1}-p}{\sqrt{p^2+4}+p}}\left(\sqrt{\frac{p^2+4+\sqrt{(p^2+1)(p^2+4)}}{2}}-\sqrt{\frac{p^2+2+\sqrt{(p^2+1)(p^2+4)}}{2}}\right).$$

For the first example, let $n = 2$. Then [43, p. 721], [26], $g_2 = 1$. Hence, $p = 0$, and, by Theorem 3.4,

$$G(e^{-\pi\sqrt{2}}) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{4+2}{2}}-\sqrt{\frac{2+2}{2}}\right) = \frac{\sqrt{6}-2}{2}. \tag{3.8}$$

For the second example, let $n = 10$. Then [43, p. 721], $g_{10} = \sqrt{(\sqrt{5}+1)/2}$. Thus, $p = \sqrt{5}$, and, by Theorem 3.4,

$$G(e^{-\pi\sqrt{10}}) = \frac{\sqrt{9+3\sqrt{6}}-\sqrt{7+3\sqrt{6}}}{(1+\sqrt{5})\sqrt{\sqrt{6}+\sqrt{5}}}. \tag{3.9}$$

The value (3.9) was given by Ramanujan on page 366 in the publication of the lost notebook [29]. Ramanathan [21] also proved (3.9) by another method.

Further values of $G(q)$ can also be computed from reciprocity theorems satisfied by $G(q)$. For example, using primarily modular equations of degree 3 found in Entry 1 of Chapter 20 of Ramanujan's second notebook [3, pp. 345, 346], Chan [15], [16, Section 6.3] proved the elegant reciprocity theorems

$$\left(1-2G(-e^{-\pi\alpha})\right)\left(1-2G(-e^{-\pi\beta})\right) = 3$$

and

$$\left(1-2G(-e^{-\sqrt{2}\pi\alpha})\right)\left(1+\frac{1}{G(-e^{-\sqrt{2}\pi\beta})}\right)=3,$$

where $\alpha,\beta>0$ and $\alpha\beta=1$. In particular, if we set $\alpha=\beta=1$ in the latter equality, we again deduce (3.8).

Like $R(q)$, $G(q)$ satisfies some beautiful modular equations. For example, if $u=G(q)$ and $v=G(q^2)$, then

$$u^2+2uv^2=v,$$

which is an analogue of (3.7).

Ramanujan studied several other q–continued fractions possessing product representations. For example, in Entry 1(ii) of Chapter 19 of his second notebook [3, p. 221], Ramanujan claimed that

$$\begin{aligned} H(q) :=& \frac{\sqrt{q}}{1+q}\,\underset{+}{}\,\frac{q^2}{1+q^3}\,\underset{+}{}\,\frac{q^4}{1+q^5}\,\underset{+}{}\,\frac{q^6}{1+q^7}\,\underset{+}{}\,\cdots, \qquad |q|<1, \\ =& \sqrt{q}\,\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}, \end{aligned} \tag{3.10}$$

which was first proved by B. Gordon [19] in 1965. Another representation for $H(q)$ was given by Ramanujan later in his second notebook [3, p. 290], [2, p. 23, Entry 13]. The continued fraction $H(q)$ also enjoys many beautiful properties that have been discovered by Chan and S.–S. Huang [17].

4. The Values of Theta Functions

Recall that $\varphi(q)$ is defined in (1.5). Observe from (1.9) that an evaluation of any of the functions φ, ${}_2F_1$, or K yields an evaluation of the other two functions. However, such evaluations may not be explicit. For example, if $K(k)$ is known for a certain value of k, it may be difficult (or impossible) to explicitly determine $K^{'}=K(k^{'})$, and so q would then not be explicitly determined. Conversely, it may be possible to evaluate $\varphi(q)$ for a certain value of q, but it may be impossible to determine the corresponding value of k. Recall that k is given by [3, p. 102]

$$k=\sqrt{1-\frac{\varphi^4(-q)}{\varphi^4(q)}}.$$

In the literature more attention has been devoted to determining ${}_2F_1$ and K, and we refer to our paper [7] for several references. In particular, concentration has focused on the instances when $K^{'}/K=\sqrt{n}$, where n is a positive integer.

At scattered places in his notebooks, particularly in his first notebook, Ramanujan recorded several values of $\varphi(q)$, some of which have not been heretofore proved in print. For example,

$$\frac{\varphi(e^{-3\pi})}{\varphi(e^{-\pi})}=\frac{1}{\sqrt[4]{6\sqrt{3}-9}}, \tag{4.1}$$

$$\frac{\varphi(e^{-5\pi})}{\varphi(e^{-\pi})}=\frac{1}{\sqrt{5\sqrt{5}-10}}, \tag{4.2}$$

and

$$\frac{\varphi^2(e^{-7\pi})}{\varphi^2(e^{-\pi})} = \frac{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}{14}(28)^{1/8}. \tag{4.3}$$

The value

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(\frac{3}{4})}$$

is well known [44, p. 525], [3, p. 103], and so (4.1)–(4.3) provide explicit values for $\varphi(e^{-3\pi})$, $\varphi(e^{-5\pi})$, and $\varphi(e^{-7\pi})$, respectively.

As indicated above, as corollaries, values of ${}_2F_1$ can be deduced. For example,

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{3\sqrt{3}-5}{4}\right) = \frac{\sqrt{\pi}}{(12)^{1/8}\sqrt{\sqrt{3}-1}\Gamma^2(\frac{3}{4})}.$$

We observe that the values of $\varphi(e^{-n\pi})/\varphi(e^{-\pi})$ given above are algebraic. Indeed, the authors [11] have shown that $\varphi(e^{-n\pi})/\varphi(e^{-\pi})$ is algebraic for every positive integer n.

J.M. and P.B. Borwein [14, p. 145] first observed that class invariants could be used to calculate certain values of $\varphi(e^{-n\pi})$. Indeed, we have verified all of Ramanujan's values for $\varphi(e^{-n\pi})$ and some new ones as well by combining Ramanujan's class invariants with his modular equations in judicious manners. We illustrate these ideas by giving a proof of (4.2).

Recall that the multiplier m is defined by (1.10). If β has degree 5 and m is the multiplier for degree 5, then from Chapter 19 of Ramanujan's second notebook [3, Entry 13(xii), pp. 281, 282]

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \tag{4.4}$$

and

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}. \tag{4.5}$$

Set $\alpha = \frac{1}{2}$, so that, by (1.8), $q = e^{-\pi}$. From (4.4) we find that

$$(2\beta)^{1/4} + (2(1-\beta))^{1/4} = m + (4\beta(1-\beta))^{1/4}, \tag{4.6}$$

and, from (4.5) and (4.6), we find that

$$\begin{aligned}\frac{5}{m} &= \frac{(2(1-\beta))^{1/4} + (2\beta)^{1/4} - 1}{(4\beta(1-\beta))^{1/4}} \\ &= \frac{m + (4\beta(1-\beta))^{1/4} - 1}{(4\beta(1-\beta))^{1/4}} \\ &= \frac{m + G_{25}^{-6} - 1}{G_{25}^{-6}}, \qquad (4.7)\end{aligned}$$

by (1.2), with α_n replaced by β_n. From Ramanujan's paper [26], [28, p. 26] or Weber's tables [43, p. 722],

$$G := G_{25} = \frac{1+\sqrt{5}}{2}. \tag{4.8}$$

Hence, from (4.7) and (4.8), since $G^3 = 2+\sqrt{5}$,

$$G^3m - \frac{5}{G^3m} = G^3 - G^{-3} = 4,$$

from which we deduce that $G^3m = 5$, or $m = 5(\sqrt{5}-2)$. The evaluation (4.2) now follows from (1.10).

On page 338 in his first notebook, Ramanujan defines

$$a_{m,n} := ne^{-(\pi/4)(n-1)\sqrt{m/n}}\frac{\psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{m/n}})\varphi^2(-e^{-2\pi\sqrt{m/n}})},$$

where m and n evidently are intended to be positive integers. Then, on pages 338 and 339, he offers a list of 18 particular values, which we present in the following table.

m,n	$a_{m,n}$	m,n	$a_{m,n}$
3,3	$\frac{1}{\sqrt{3}}$	$3,13$	$\left(\sqrt{\frac{5+\sqrt{13}}{8}}-\sqrt{\frac{\sqrt{13}-3}{8}}\right)^8$
3,9	$\frac{1}{(2^{1/3}+1)^2}$	$3,23$	$\left(\sqrt{\frac{7+4\sqrt{3}}{2}}-\sqrt{\frac{5+4\sqrt{3}}{2}}\right)^2$
3,15	$\frac{2-\sqrt{3}}{3}$	$3,71$	$\left(\sqrt{\frac{175+100\sqrt{3}}{2}}-\sqrt{\frac{173+100\sqrt{3}}{2}}\right)^2$
3,5	$\frac{3-\sqrt{5}}{2}$	$5,9$	$(2-\sqrt{3})^2$
3,7	$2-\sqrt{3}$	$5,11$	$\left(\sqrt{\frac{7+\sqrt{5}}{8}}-\sqrt{\frac{\sqrt{5}-1}{8}}\right)^8$
3,11	$2\sqrt{3}-\sqrt{11}$	$5,13$	$\left(\sqrt{\frac{9+\sqrt{65}}{2}}-\sqrt{\frac{7+\sqrt{65}}{2}}\right)^2$
3,19	$2\sqrt{19}-5\sqrt{3}$	$5,17$	$(\sqrt{17}-4)^2$
3,31	$(2-\sqrt{3})^3$	$5,29$	$\left(\sqrt{49+4\sqrt{145}}-\sqrt{48+4\sqrt{145}}\right)^2$
3,59	$102\sqrt{3}-23\sqrt{59}$	$7,9$	$\left(\sqrt{\frac{5+\sqrt{21}}{8}}-\sqrt{\frac{\sqrt{21}-5}{8}}\right)^8$

Upon examining this table, we see that when $(m,n)=1$ each value is a unit in some algebraic number field. In fact, we have proved some general theorems which ensure that $a_{m,n}$ is a unit for large classes of pairs m, n that include the examples above [11]. Several approaches were devised to calculate $a_{m,n}$.

First, we derived formulas for $a_{m,n}$, when $m = 3$, 5, and 7, in terms of class invariants. For example,

$$a_{3,n} = \frac{1}{3}\left(\frac{G_{n/3}^6}{G_{3n}^6} + \frac{2\sqrt{2}}{G_{n/3}^3 G_{3n}^3}\right). \tag{4.9}$$

In each case G_{3n} is known. To find $G_{n/3}$ we use Lemma 2.2 in the form

$$\left(\frac{G_{n/3}}{G_{3n}}\right)^6 + \left(\frac{G_{3n}}{G_{n/3}}\right)^6 + 2\sqrt{2}\left(\frac{1}{G_{n/3}^3 G_{3n}^3} + G_{n/3}^3 G_{3n}^3\right) = 0. \tag{4.10}$$

Analogues of (4.9) and (4.10) exist for $m = 5$ and 7 as well.

Secondly, we were able to find formulas for $a_{m,n}$, in each case $m = 3$, 5, and 7, that are in the spirit of Theorems 3.1 and 3.2. For example, if $V_n = G_{n/3}/G_{3n}$, then

$$a_{3,n} - \frac{1}{a_{3,n}} = \frac{1}{3}\left(V_n^6 - V_n^{-6}\right). \tag{4.11}$$

Equation (4.11) is used in conjunction with (4.10), as in the first approach. Analogues of (4.11) have also been established for $a_{5,n}$ and $a_{7,n}$.

Thirdly, general formulas for $a_{m,n}$ were derived by using Kronecker's limit formula. These formulas involve class numbers and fundamental units in real quadratic fields, and so this approach would have been unknown to Ramanujan.

We give one simple example to illustrate the first approach. Let $n = 5$. Then, from Weber's book [43, p. 721], or from Ramanujan's notebooks [27, vol. 1, p. 289; vol. 2, p. 294],

$$G_{15} = 2^{-1/12}(\sqrt{5}+1)^{1/3}.$$

From (4.9), we easily verify that

$$G_{5/3} = 2^{-1/12}(\sqrt{5}-1)^{1/3}.$$

Using these values in (4.10) with $n = 5$, we readily find that

$$a_{3,5} = \frac{3-\sqrt{5}}{2}.$$

Another interesting product of theta functions is examined by Ramanujan in his lost notebook, and these numbers were studied by Ramanathan [24], [25].

5. Concluding Remarks

Ramanujan recorded several hundred modular equations in his notebooks, and proofs of most of these can be found in [3], [4], and [5]. However, many of our proofs are not those found by Ramanujan, and more enlightening proofs are greatly desired. Perhaps Ramanujan merely enjoyed deriving modular equations. However, since some of them have now been employed in the computation of class invariants and values of theta functions, it is very likely that Ramanujan had applications in mind for many of his modular equations.

It also could be surmised that Ramanujan enjoyed computing class invariants because they are lovely numbers. This is likely true, but we have seen from their applications to resolving values of continued fractions and theta functions that Ramanujan had a broader agenda in mind.

The contemporary stages for much of the work described here are "modern" algebra and algebraic number theory. Ramanujan evidently knew nothing of these subjects. However, he must have discerned the special nature of the arithmetic and number theory in these situations and so had a deep understanding that he would have conveyed in a different language. His calculations of some invariants are particularly enigmatic. Our understanding of these subjects would be greatly enhanced if we could discover some of Ramanujan's thinking.

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