EXPLICIT EVALUATIONS AND RECIPROCITY THEOREMS FOR FINITE TRIGONOMETRIC SUMS

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1. Introduction

The mathematical literature contains many evaluations of finite trigonometric sums of the sort

$$\sum_{j=1}^{k-1} \cot^2\left(\frac{\pi j}{k}\right) = \frac{(k-1)(k-2)}{3}. \tag{1.1}$$

This evaluation can be found in standard tables of series, such as those of E. R. Hansen [44, p. 262, eq. (30.1.2)], L. B. W. Jolly [56, pp. 102–103, eq. (352)], and A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev [70, p. 646, eq. 6]. We do not know who first proved (1.1), but several proofs exist. If we replace the power 2 on the left side by an arbitrary positive even power, finding an explicit evaluation becomes more difficult. Often, in applications, one does not need an explicit evaluation in closed form but only an asymptotic formula; for example, see T. M. Apostol's paper [4].

The earliest evaluation known to us of a sum of the type (1.1) is by M. Stern [79, p. 155], who in 1861 proved that, for any positive odd integer k,

$$\sum_{j=1}^{k-1} \tan^2\left(\frac{\pi j}{k}\right) = k(k-1), \tag{1.2}$$

which can also be found in standard tables, for example, [44, p. 258, eq. (21.1.2)] and [70, p. 646, eq. 5].

Many finite trigonometric sums do not evidently have evaluations in closed form. However, they may possess beautiful reciprocity theorems. The earliest such reciprocity theorem of which we are aware was stated as an exercise by G. Eisenstein in 1844 [28], [30, pp. 108–110]. Let

$$f(h,k) := \sum_{j=1}^{k-1} \frac{\tan(hj\pi/k)}{\tan(2j\pi/k)}.$$

Then, if h and k are odd, coprime, positive integers, then

$$h\,f(h,k) + k\,f(k,h) = -\frac{(h-k)^2}{2}. \tag{1.3}$$

(In [28], (1.3) is stated with a misprint, which was corrected in [29, p. 35] and in [30, p. 109].) The first published proof of (1.3) is due to Stern [79, p. 160] in 1861.

The most famous reciprocity theorem for trigonometric sums is undoubtedly that which is equivalent to the reciprocity theorem for Dedekind sums. Let

$$((x)) := \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

The classical Dedekind sum $s(h,k)$ is defined by

$$s(h,k) := \sum_{j \pmod{k}} ((hj/k))((j/k)). \tag{1.4}$$

Then the well-known reciprocity theorem for $s(h,k)$ is given by [73, p. 4]

$$s(h,k) + s(k,h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk}\right), \tag{1.5}$$

where h and k are coprime positive integers. Now [43], [73, p. 18],

$$s(h,k) = \frac{1}{4k}\sum_{j=1}^{k-1} \cot\left(\frac{\pi hj}{k}\right)\cot\left(\frac{\pi j}{k}\right) =: \frac{1}{4k} r(h,k). \tag{1.6}$$

Thus, from (1.5),

$$h\ r(h,k) + k\ r(k,h) = -hk + \tfrac{1}{3}\left(h^2 + k^2 + 1\right). \tag{1.7}$$

The primary objective of this paper is to establish some general theorems on explicit evaluations and reciprocity theorems for trigonometric sums. All sums considered here are linear in the summation variable. Those sums quadratic in the summation variable would likely have connections with Gauss sums. All of our proofs involve contour integration. We are uncertain who first used contour integration for this purpose; to the best of our knowledge, the earliest such reference is H. Rademacher's paper [71] in 1933. The first author [13] extensively used contour integration to derive reciprocity formulas for various generalizations and analogues of Dedekinds sums. We also establish some asymptotic formulas. A second goal is to give a history of evaluations and reciprocity theorems for trigonometric sums. This history is, not surprisingly, sporadic, and consequently authors often publish results without being aware that their theorems had previously been published elsewhere. It would be arrogant (and incorrect) of us to claim that we have referenced all relevant literature. However, we have taken this task seriously and will be grateful for further references from readers.

The most comprehensive paper on evaluating trigonometric sums is undoubtedly by W. Chu and A. Marini [24], who employed generating functions to systematically evaluate in closed form 24 different classes of trigonometric sums. We only examine some of these sums in this paper. However, we emphasize that the methods used here can be applied to evaluate all the sums considered by Chu and Marini [24]. Generally, in our approach we consider more general sums in each instance, and so often derive reciprocity theorems, a subject not examined by Chu and Marini.

Different approaches to the explicit evaluations of finite trigonometric sums often yield distinctly different types of closed form evaluations. For example, the approach used by

Chu and Marini [24] yields evaluations in terms of multiple sums of binomial coefficients, while our approach yields sums of Bernoulli numbers. It does not appear that one type of evaluation has any particular advantage over another, and, in any case, as the powers of the trigonometric functions increase, explicit examples become increasingly tedious to work out. Although any two different correct evaluations of the same sum are obviously equivalent, we leave all such exercises to readers.

In Section 2, we establish a general theorem about cotangent sums, which includes as special cases the evaluations and reciprocity theorems cited above. By almost identically the same argument, we can also prove similar general theorems for tangent, secant, and cosecant sums. Alternatively, by using elementary trigonometric identities, we can often evaluate the desired tangent, secant, or cosecant sum by converting it to a cotangent sum. Thus, to avoid the repetition of arguments, we confine our attention to only cotangent sums.

In Section 3, a general theorem on alternating cosecant sums is proved. Again, similar ideas can be employed to examine other alternating trigonometric sums.

In that same paper of Eisenstein [28] mentioned above, he also posed the identity

$$\sum_{j=1}^{k-1} \sin\left(\frac{2\pi aj}{k}\right)\cot\left(\frac{\pi j}{k}\right) = k - 2a \tag{1.8}$$

as an exercise, where a and k are integers such that $0 < a < k$. This was also first proved in print by Stern [79, p. 152]. K. S. Williams and N.–Y. Zhang [84] generalized (1.8) by replacing cot in the sum by an arbitrary positive power of cot. In Section 4, we give another formulation of their theorem with a short proof by the same method of contour integration used in the two previous sections.

Interesting cotangent and cosecant sums arise in the work of B. M. McCoy and W. P. Orrick [61] on the chiral Potts model in statisical mechanics. In Section 5, we explicitly evaluate a class of such sums.

In the last section of this paper, we briefly discuss other finite trigonometric sums of the sort considered in earlier sections and whose evaluations or reciprocity theorems can be effected by the same methods used in this paper.

We complete Section 1 with a few words about notation. A simple closed curve is denoted by C, with its interior by $I(C)$. The residue of a meromorphic function f at a pole z_0 is denoted by $\text{Res}(f, z_0)$.

2. Cotangent Sums

Let m be a nonnegative integer, h and k be positive integers, and $0 \leq a, b < 1$. The general sum which we examine in this section is defined by

$$s_m(h,k;a,b) := \sum_{0<r+b<k} \cot^m\left(\pi\frac{r+b}{k}\right)\cot\left(\pi\left(\frac{h(r+b)}{k} - a\right)\right). \tag{2.1}$$

We need the well-known Laurent expansion

$$z\ \cot z := \sum_{j=0}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!}(2z)^{2j}, \qquad |z| < \pi, \tag{2.2}$$

where $B_n, 0 \leq n < \infty$, is the nth Bernoulli number. Also define, for integers $j_1, j_2, \dots, j_n \geq 0$,

$$C(j_1, j_2, \dots, j_n) := \prod_{r=1}^{n} (-1)^{j_r} 2^{2j_r} \frac{B_{2j_r}}{(2j_r)!}. \tag{2.3}$$

If a is not an integer, $\cot(\pi(hz - a))$ is analytic at $z = 0$. Now by induction on n, it is easily shown that, for each positive integer n,

$$\frac{d^n}{dz^n}(\cot(cz)) = c^n\ T_{n+1}\left(\cot(cz)\right),$$

where $T_m(x)$ is a polynomial in x of degree m, with its coefficients independent of c. It readily follows that, for $\pi|hz - a| < 1$,

$$\cot\left(\pi(hz - a)\right) = \sum_{j=0}^{\infty} P_{j+1}(\cot(\pi a))(\pi hz)^j, \tag{2.4}$$

where $P_m(x)$ is a polynomial in x of degree m. The first four polynomials are

$$P_1(x) = -x, \quad P_2(x) = -(x^2 + 1), \quad P_3(x) = -x(x^2 + 1), \quad P_4(x) = -\tfrac{1}{3}\left(3x^4 + 4x^2 + 1\right).$$

Lastly, define

$$\delta_m := \frac{1}{2} i^{m+1}(1 + (-1)^{m+1}). \tag{2.5}$$

We are now ready to state the primary theorem of this section.

Theorem 2.1. *Let $0 \leq a, b < 1$, and assume that h and k are positive integers. Suppose that $(j+b)/k \neq (r+a)/h$, where j and r are nonnegative integers, such that $0 < j + b < k$ and $0 < r + a < h$. Then*

$$\begin{aligned} &\frac{1}{k} s_m(h, k; a, b) + \frac{1}{h} s_m(k, h; b, a) - \delta_m \\ &\quad = \begin{cases} -\sum_1 h^{\mu} k^{\nu} C(j_1, \dots, j_m) P_{\mu+1}(\cot(\pi a)) P_{\nu+1}(\cot(\pi b)), & \text{if } a, b > 0, \\ -\sum_2 h^{2\mu-1} k^{\nu} C(j_1, \dots, j_m, \mu) P_{\nu+1}(\cot(\pi b)), & \text{if } a = 0, b > 0, \\ -\sum_3 h^{2\mu-1} k^{2\nu-1} C(j_1, \dots, j_m, \mu, \nu), & \text{if } a = 0, b = 0, \end{cases} \end{aligned} \tag{2.6}$$

where the polynomials P_m are defined by (2.4), and δ_m is defined by (2.5). Furthermore, the sums $\sum_1$, $\sum_2$, and $\sum_3$ are over all $(m+2)$-tuples $(j_1, \dots, j_m, \mu, \nu)$ such that, respectively, $2(j_1 + \cdots + j_m) + \mu + \nu = m - 1$, $2(j_1 + \cdots + j_m) + 2\mu + \nu = m$, and $2(j_1 + \cdots + j_m) + 2\mu + 2\nu = m + 1$.

Proof. Let $C = C_R$ denote the positively oriented indented rectangle with vertices at $\pm iR$ and $1 \pm iR$, with $R > \epsilon$, and with semicircular indentations of radius $\epsilon < \min\{(h-1+a)/h, (k-1+b)/k\}$ to the left of both 0 and 1. Let

$$f(z) := \cot^m(\pi z)\cot(\pi(hz-a))\cot(\pi(kz-b))$$

and consider

$$\frac{1}{2\pi i}\int_C f(z)\ dz. \tag{2.7}$$

On $I(C)$, $f(z)$ has simple poles when $\pi(hz-a) = \pi j$ and $\pi(kz-b) = \pi r$, i. e., when $z = (j+a)/h$ and $z = (r+b)/k$, where j and r are nonnegative integers such that $0 < j+a < h$ and $0 < r+b < k$, respectively. Straightforward calculations show that

$$\operatorname{Res}(f,(j+a)/h) = \frac{1}{h\pi}\cot^m\left(\pi\frac{j+a}{h}\right)\cot\left(\pi\left(\frac{k(j+a)}{h} - b\right)\right), \tag{2.8}$$

$$\operatorname{Res}(f,(r+b)/k) = \frac{1}{k\pi}\cot^m\left(\pi\frac{r+b}{k}\right)\cot\left(\pi\left(\frac{h(r+b)}{k} - a\right)\right). \tag{2.9}$$

On $I(C)$, $f(z)$ also has a pole at $z = 0$. We distinguish three cases.

Case 1. $a, b > 0$. Then f has a pole of order m at $z = 0$. From (2.2) and (2.4), we have

$$f(z) = \left(\sum_{j=0}^{\infty}(-1)^j\frac{B_{2j}}{(2j)!}2^{2j}(\pi z)^{2j-1}\right)^m \sum_{\mu=0}^{\infty}P_{\mu+1}(\cot(\pi a))(\pi hz)^{\mu} \times \sum_{\nu=0}^{\infty}P_{\nu+1}(\cot(\pi b))(\pi kz)^{\nu}.$$

Thus,

$$\operatorname{Res}(f,0) = \frac{1}{\pi}\sum\nolimits_1 h^{\mu}k^{\nu}C(j_1,\dots,j_m)P_{\mu+1}(\cot(\pi a))P_{\nu+1}(\cot(\pi b)), \tag{2.10}$$

where $C(j_1,\dots,j_m)$ is defined by (2.3).

Case 2. $a = 0, b > 0$. Now f has a pole of order $m+1$ at $z = 0$. Again, from (2.2) and (2.4), we have

$$f(z) = \left(\sum_{j=0}^{\infty}(-1)^j\frac{B_{2j}}{(2j)!}2^{2j}(\pi z)^{2j-1}\right)^m \sum_{\mu=0}^{\infty}(-1)^{\mu}\frac{B_{2\mu}}{(2\mu)!}2^{2\mu}(\pi hz)^{2\mu-1} \times \sum_{\nu=0}^{\infty}P_{\nu+1}(\cot(\pi b))(\pi kz)^{\nu}.$$

Thus,

$$\operatorname{Res}(f,0) = \frac{1}{\pi}\sum\nolimits_2 h^{2\mu-1}k^{\nu}C(j_1,\dots,j_m,\mu)P_{\nu+1}(\cot(\pi b)). \tag{2.11}$$

Case 3. $a = b = 0$. In this case f has a pole of order $m+2$ at $z=0$. By (2.2) and (2.4),

$$f(z) = \left(\sum_{j=0}^{\infty}(-1)^j \frac{B_{2j}}{(2j)!}2^{2j}(\pi z)^{2j-1}\right)^m \sum_{\mu=0}^{\infty}(-1)^{\mu}\frac{B_{2\mu}}{(2\mu)!}2^{2\mu}(\pi h z)^{2\mu-1}$$
$$\times \sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{B_{2\nu}}{(2\nu)!}2^{2\nu}(\pi k z)^{2\nu-1}.$$

Hence,

$$\text{Res}(f,0) = \frac{1}{\pi}\sum\nolimits_3 h^{2\mu-1}k^{2\nu-1}C(j_1,\dots,j_m,\mu,\nu). \tag{2.12}$$

We now evaluate directly the integral in (2.7). Since $f(z)$ has period 1, the integrals over the indented vertical sides of C cancel. Let $z = x+iy$, with x and y real. From the definition of $\cot(cz+d)$, for $c>0$ and d real, we easily find that

$$\lim_{y\to\infty}\cot(cz+d) = \begin{cases} -i, & \text{if } y>0,\\ i, & \text{if } y<0. \end{cases} \tag{2.13}$$

Hence,

$$\begin{aligned}\frac{1}{2\pi i}\int_C f(z)\,dz &= \frac{1}{2\pi i}\int_1^0 (-i)^{m+2}dx + \frac{1}{2\pi i}\int_0^1 i^{m+2}dx\\ &= \frac{i^{m+1}}{2\pi}\left(1-(-1)^m\right).\end{aligned} \tag{2.14}$$

Hence, applying the residue theorem, using (2.8), (2.9), and (2.14) in (2.7), and recalling the definition (2.1), we deduce that

$$\frac{i^{m+1}}{2\pi}\left(1-(-1)^m\right) = \frac{1}{h\pi}s_m(k,h;b,a) + \frac{1}{k\pi}s_m(h,k;a,b) + \text{Res}(f,0), \tag{2.15}$$

where $\text{Res}(f,0)$ is given by (2.10)–(2.12). Using these three values of $\text{Res}(f,0)$ and (2.5) in (2.15) and employing very moderate simplification, we readily deduce (2.6). □

We now offer several consequences of Theorem 2.1.

Corollary 2.2. *Let n be any positive integer. Then*

$$\frac{1}{k}\sum_{j=1}^{k-1}\cot^{2n}\left(\frac{\pi j}{k}\right) = (-1)^n - (-1)^n 2^{2n}\sum_{\substack{j_0,j_1,\dots,j_{2n}\ge 0\\ j_0+j_1+\cdots+j_{2n}=n}} k^{2j_0-1}\prod_{r=0}^{2n}\frac{B_{2j_r}}{(2j_r)!}, \tag{2.16}$$

where B_j, $j\ge 0$, denotes the jth Bernoulli number.

Proof. In Theorem 2.1, set $m = 2n-1$, $h=1$, and $a=b=0$. The corollary now easily follows after a slight renaming of the parameters. □

Different proofs and different formulations of Corollary 2.2 have been given by Chu and Marini [24, p. 137] and D. Cvijović and J. Klinowski [25].

Of course, if the even power $2n$ on the left side of (2.16) is replaced by an odd positive power, the sum is then trivially equal to 0.

Corollary 2.3. *We have*

$$\sum_{j=1}^{k-1} \cot^2\left(\frac{\pi j}{k}\right) = \frac{(k-1)(k-2)}{3}.$$

Proof. Set $n = 1$ in Corollary 2.2. Recalling that $B_2 = 1/6$, we easily complete the proof. □

As indicated in the opening paragraph of this paper, Corollary 2.3 (or (1.1)) can be found in several tables. For either all positive k, or for odd k, one can also find Corollary 2.3 as a problem in the texts of T. J. Ia. Bromwich [17, p. 224], I. Niven, H. S. Zuckerman, and H. L. Montgomery [67, p. 492, Exer. 3], A. W. Siddons and R. T. Hughes [75, p. 385, Exer. 10], K. R. Stromberg [80, p. 234], and A. M. Yaglom and I. M. Yaglom [82, p. 23]. Again, for odd k, Corollary 2.3 is also proved in papers of M. Bencze [10], F. Holme [52], and I. Papadimitriou [69]. P. S. Bruckman [19] posed an equivalent version of Corollary 2.3 as a problem. For $n = 1, 2, 3$, the equivalent sums with cot replaced by csc, and various other trigonometric sums as well, are evaluated in a problem by M. Henkel [48] in the *SIAM Review*. Furthermore, a variation with cot replaced by csc appeared in the following *Monthly* problem [18].

Corollary 2.4. *We have*

$$\sum_{j=0}^{k-1} \csc^2\left(\frac{\pi(2j+1)}{2k}\right) = k^2.$$

Proof. By two applications of Corollary 2.2,

$$\begin{aligned}\sum_{j=0}^{k-1} \csc^2\left(\frac{\pi(2j+1)}{2k}\right) &= \sum_{j=0}^{k-1}\left\{1+\cot^2\left(\frac{\pi(2j+1)}{2k}\right)\right\}\\ &= k + \sum_{j=1}^{2k-1}\cot^2\left(\frac{\pi j}{2k}\right) - \sum_{j=1}^{k-1}\cot^2\left(\frac{\pi j}{k}\right) = k + k^2 - k = k^2.\end{aligned}$$

□

The Editorial Notes in [18] cite further references. In particular, K. S. Williams informed readers that Corollary 2.3 for odd k appeared as a question in the Higher Certificate Mathematics, Oxford and Cambridge Schools Examination Board, Mathematics Group III (Paper 5) (1945), Question 9.

G. J. Byrne and S. J. Smith [20] evaluated two large classes of cotangent and cosecent sums. They were motivated by the fact that the value of each sum in the classes they examined is an integer. One of the simplest examples is given by

$$\sum_{j=1}^{k} \cot^2\left(\frac{\pi(2j-1)}{4k}\right) = 2k^2 - k,$$

which follows from two applications of Corollary 2.3.

The solution of another *Monthly* problem [47] also follows from Corollary 2.3.

Corollary 2.5. *Let* $\omega_j = e^{2\pi ij/k}$, $0 \le j \le k-1$. *Then*

$$\sum_{j\neq m} |\omega_j - \omega_m|^{-2} = \tfrac{1}{12}k(k^2-1).$$

Proof. Observe that, by Corollary 2.3,

$$\begin{aligned}
\sum_{j\neq m} |\omega_j - \omega_m|^{-2} &= \sum_{j\neq m} \frac{1}{4\sin^2\{\pi(j-m)/k\}} \\
&= \frac{k}{4}\sum_{j=1}^{k-1} \csc^2\left(\frac{\pi j}{k}\right) \\
&= \frac{k}{4}\sum_{j=1}^{k-1} \left(1 + \cot^2\left(\frac{\pi j}{k}\right)\right) \\
&= \frac{k}{4}\left(k - 1 + \frac{(k-1)(k-2)}{3}\right) = \frac{k(k^2-1)}{12}.
\end{aligned}$$

□

R. K. Stanley [77] and I. Gessel [41] established a generating function for the sum

$$S_n(k) := \sum_{\substack{\zeta^k=1 \\ \zeta\neq 1}} |1-\zeta|^{-2n},$$

which is clearly related in the case $n = 1$ to the sum of Corollary 2.5.

Corollary 2.6. *We have*

$$\sum_{j=1}^{k-1} \cot^4\left(\frac{\pi j}{k}\right) = \frac{(k-1)(k-2)(k^2+3k-13)}{45}.$$

Proof. Set $n = 2$ in Corollary 2.2 and use the values $B_2 = 1/6$ and $B_4 = -1/30$. □

Corollary 2.6 can be found in the aforementioned tables of Hansen [44, p. 262], Jolley [56, pp. 102–103, eq. (353)], and (for odd k) Prudnikov, Brychkov, and Marichev [70, p. 646, eq. 10], as well as in the book by the Yagloms [82, p. 132].

A solution of the following problem in the *SIAM Review* [57] follows from Corollaries 2.3 and 2.6.

Corollary 2.7. *We have*

$$\frac{1}{2}\sum_{\substack{j=1\\ j\ odd}}^{k-1}\cot^2\left(\frac{\pi j}{2k}\right)\csc^2\left(\frac{\pi j}{2k}\right)=\frac{k^2(k^2-1)}{12}.$$

Proof. Let S denote the sum on the left side above. Then, by two applications of each of Corollaries 2.3 and 2.6,

$$\begin{aligned}4S&=\sum_{\substack{j=1\\ j\text{ odd}}}^{2k-1}\cot^2\left(\frac{\pi j}{2k}\right)+\sum_{\substack{j=1\\ j\text{ odd}}}^{2k-1}\cot^4\left(\frac{\pi j}{2k}\right)\\&=\sum_{j=1}^{2k-1}\cot^2\left(\frac{\pi j}{2k}\right)+\sum_{j=1}^{2k-1}\cot^4\left(\frac{\pi j}{2k}\right)\\&\quad-\sum_{j=1}^{k-1}\cot^2\left(\frac{\pi j}{k}\right)-\sum_{j=1}^{k-1}\cot^4\left(\frac{\pi j}{k}\right)\\&=\frac{k^4-k^2}{3},\end{aligned}$$

after elementary algebra. This completes the proof. □

The following *Monthly* problem by A. J. Duran [27] is very closely related to the sums evaluated above. If n is a given positive integer, prove that we can find a polynomial g_n with rational coefficients and degree at most n such that

$$\sum_{j=1}^{k-1}(e^{2\pi ij/k}-1)^{-n}=g_n(k)$$

for any positive integer k. For example, $g_1(k)=-(k-1)/2$, $g_2(k)=-(k-1)(k-5)/12$, and $g_3(k)=(k-1)(k-3)/8$.

Corollary 2.8. *As* $k\to\infty$,

$$\sum_{j=1}^{k-1}\cot^{2n}\left(\frac{\pi j}{k}\right)\sim(-1)^{n+1}(2k)^{2n}\frac{B_{2n}}{(2n)!},$$

where $B_j, j\geq 0$, *denotes the* j*th Bernoulli number.*

Proof. This asymptotic formula follows immediately from Corollary 2.2. □

Corollary 2.8 has been previously established by Apostol [4], L. A. Gardner, Jr. [38], I. Skau and E. Selmer [76], and K. S. Williams [83].

Corollary 2.9. *Recall that* $r(h,k)$ *is defined in* (1.6). *Then, if* h *and* k *are coprime positive integers,*

$$h\ r(h,k)+k\ r(k,h)=-hk+\tfrac{1}{3}\left(h^2+k^2+1\right). \tag{2.17}$$

Proof. In Theorem 2.1, set $a = b = 0$ and $m = 1$. The desired result immediately follows. □

As noted in the Introduction, (2.17), or (1.7), is equivalent to the reciprocity theorem for the classical Dedekind sums $s(h,k)$. Proofs by contour integration of the reciprocity theorem for $s(h,k)$ have been given by Rademacher [71], [72, pp. 26–36], K. Iseki [53], and E. Grosswald [43]. See also [73, pp. 21–22]. G. H. Hardy [45], [46, pp. 362–392] employed a different method of contour integration to prove the reciprocity theorem for $s(h,k)$. In this connection, see also Berndt's paper [12].

Corollary 2.10. *Let* $0 < a < 1$. *Then, for positive integers* h *and* k,

$$\frac{1}{k}\sum_{j=1}^{k-1}\cot\left(\frac{\pi j}{k}\right)\cot\left(\pi\left(\frac{hj}{k}-a\right)\right)$$
$$+\frac{1}{h}\sum_{j=0}^{h-1}\cot\left(\pi\frac{j+a}{h}\right)\cot\left(\pi\frac{k(j+a)}{h}\right)=-1+\frac{h}{k}\csc^2(\pi a).$$

Proof. In Theorem 2.1 set $m = 1$ and $b = 0$. The desired result then easily follows. □

Corollary 2.11. *Let* $0 < a, b < 1$. *Then, under the hypotheses of Theorem 2.1,*

$$\frac{1}{k}\sum_{j=0}^{k-1}\cot\left(\pi\frac{j+b}{k}\right)\cot\left(\pi\left(\frac{h(j+b)}{k}-a\right)\right)$$
$$+\frac{1}{h}\sum_{j=0}^{h-1}\cot\left(\pi\frac{j+a}{h}\right)\cot\left(\pi\left(\frac{k(j+a)}{h}-b\right)\right)=-1-\cot(\pi a)\cot(\pi b).$$

Proof. Let $m = 1$ in Theorem 2.1, and the result easily follows. □

The preceeding two corollaries are originally due to W. Meyer and R. Sczech [63] and independently (and slightly later) to U. Dieter [26].

Corollary 2.12. *If* $0 < b < 1$ *and* k *is a positive integer, then*

$$\sum_{j=0}^{k-1}\cot^2\left(\pi\frac{j+b}{k}\right)=-k+k^2\csc^2(\pi b).$$

Proof. Set $m = 1$, $h = 1$, and $a = 0$ in Theorem 2.1. The desired result now readily follows. □

Corollary 2.12 can be found in the tables of Hansen [44, p. 262, eq. (30.1.1)] and Prudnikov, Brychkov, and Marichev [70, p. 646, eq. 4], as well as in papers of Berndt [11] and E. H. Neville [66]. An equivalent formulation with cot replaced by csc was posed as a problem in *Mathematics Magazine* [31].

We next show that Stern's result (1.2) is a special case of Corollary 2.12.

Corollary 2.13. *If k is an odd positive integer, then*

$$\sum_{j=1}^{k-1} \tan^2\left(\frac{\pi j}{k}\right) = k(k-1). \tag{2.18}$$

Proof. Applying Corollary 2.12 with $b = k/2$, we see that

$$\begin{aligned}\sum_{j=1}^{k-1} \tan^2\left(\frac{\pi j}{k}\right) &= \sum_{j=1}^{k-1} \cot^2\left(\frac{\pi j}{k} + \frac{\pi}{2}\right)\\ &= -k + k^2 \csc^2\left(\frac{\pi k}{2}\right) - \cot^2\left(\frac{\pi k}{2}\right)\\ &= -k + k^2.\end{aligned}$$

□

For odd k, Corollary 2.13 was posed as a problem in 1937 [74]. Corollary 2.13 was also given by H.–J. Seiffert [15] as a consequence of his explicit evaluation of the more general sum

$$\sum_{j=1}^{k-1} \frac{\sin^2(\pi j/k)}{(1 + a^2 - 2a\cos(\pi j/k))^2}.$$

This problem and another evaluation of a trigonometric sum in [5] are connected with Chebyshev polynomials.

The evaluation of another tangent sum,

$$\sum_{j=0}^{k-1} (-1)^j \tan\left(\frac{\pi(2j+1)}{4k}\right),$$

was posed as a problem and solved by contour integration [1].

Before concluding this section by proving the reciprocity formula of Eisenstein [28] given by (1.3) or in Corollary 2.15 below, we prove a reciprocity theorem which generalizes Corollary 2.9. Another special case is needed, in fact, to prove Corollary 2.15. We are grateful to Ron Evans for communicating this beautiful reciprocity theorem to us. We first make a definition. Let h and k be coprime positive integers, and set $h + k = \mu c$, where μ and c are positive integers. Define

$$r_\mu(h,k) := \sum_{j=1}^{k-1} \cot\left(\frac{\pi \mu j}{k}\right) \cot\left(\frac{\pi h j}{k}\right). \tag{2.19}$$

Note that when $\mu = 1$, $r_1(h,k) = r(h,k)$, which is defined by (1.6).

Theorem 2.14. *If h, k, μ, c, and r_μ are defined as above, then*

$$h r_\mu(h,k) + k r_\mu(k,h) = \frac{1}{3\mu}\left(h^2 + k^2 + \mu^2\right) + \frac{hk}{3\mu}\left(\mu^2 - 6\mu + 2\right). \tag{2.20}$$

Observe that when $\mu = 1$, (2.20) reduces to (2.17).

Proof. The proof is similar to that of Theorem 2.1. Let $C = C_R$ denote the same positively oriented indented rectangle as in the proof of Theorem 2.1, except that now $a = b = 0$. Set

$$f(z) := \cot(\pi hz)\cot(\pi kz)\cot(\pi\mu z)$$

and consider

$$\frac{1}{2\pi i}\int_C f(z)\,dz. \tag{2.21}$$

On $I(C)$, $f(z)$ has simple poles at $z = j/h, 1 \le j < h$, $z = j/k, 1 \le j < k$, and $z = j/\mu, 1 \le j < \mu$, and a triple pole at $z = 0$. The condition $(h,k) = 1$ ensures that no two of these poles coalesce. Straightforward calculations show that

$$\begin{aligned}
\operatorname{Res}(f, j/h) &= \frac{1}{h\pi}\cot\left(\frac{\pi kj}{h}\right)\cot\left(\frac{\pi\mu j}{h}\right),\\
\operatorname{Res}(f, j/k) &= \frac{1}{k\pi}\cot\left(\frac{\pi hj}{k}\right)\cot\left(\frac{\pi\mu j}{k}\right),\\
\operatorname{Res}(f, j/\mu) &= \frac{1}{\mu\pi}\cot\left(\frac{\pi hj}{\mu}\right)\cot\left(\frac{\pi kj}{\mu}\right),
\end{aligned}$$

and, by (2.2),

$$\operatorname{Res}(f, 0) = -\frac{1}{3\pi}\left(\frac{k}{h\mu}+\frac{h}{k\mu}+\frac{\mu}{hk}\right).$$

Moreover, by the same sort of calculation that gave (2.14),

$$\frac{1}{2\pi i}\int_C f(z)\,dz = -\frac{1}{\pi}.$$

Applying the residue theorem and using all the calculations above, we deduce that

$$\begin{aligned}
-\frac{1}{\pi} =& \frac{1}{h\pi}\sum_{j=1}^{h-1}\cot\left(\frac{\pi\mu j}{h}\right)\cot\left(\frac{\pi kj}{h}\right) + \frac{1}{k\pi}\sum_{j=1}^{k-1}\cot\left(\frac{\pi\mu j}{k}\right)\cot\left(\frac{\pi hj}{k}\right)\\
&+\frac{1}{\mu\pi}\sum_{j=1}^{\mu-1}\cot\left(\frac{\pi hj}{\mu}\right)\cot\left(\frac{\pi kj}{\mu}\right) - \frac{1}{3\pi}\left(\frac{k}{h\mu}+\frac{h}{k\mu}+\frac{\mu}{hk}\right).
\end{aligned} \tag{2.22}$$

Multiplying both sides of (2.22) by πhk and using the definition (2.19), we find that

$$\begin{aligned}
hr_\mu(h,k) + kr_\mu(k,h) =& -hk + \frac{hk}{3}\left(\frac{k}{h\mu}+\frac{h}{k\mu}+\frac{\mu}{hk}\right)\\
&-\frac{hk}{\mu}\sum_{j=1}^{\mu-1}\cot\left(\frac{\pi hj}{\mu}\right)\cot\left(\frac{\pi kj}{\mu}\right).
\end{aligned} \tag{2.23}$$

Now recall that $h+k=\mu c$. Thus, by the fact that h and k are coprime and (1.1),

$$\begin{aligned}-\frac{hk}{\mu}\sum_{j=1}^{\mu-1}\cot\left(\frac{\pi hj}{\mu}\right)\cot\left(\frac{\pi kj}{\mu}\right) =&\frac{hk}{\mu}\sum_{j=1}^{\mu-1}\cot^2\left(\frac{\pi hj}{\mu}\right)\\ =&\frac{hk}{\mu}\sum_{j=1}^{\mu-1}\cot^2\left(\frac{\pi j}{\mu}\right)=\frac{hk(\mu-1)(\mu-2)}{3\mu}.\end{aligned}\tag{2.24}$$

Hence, by (2.23) and (2.24), we conclude that

$$hr_\mu(h,k)+kr_\mu(k,h)=-hk+\frac{hk}{3}\left(\frac{k}{h\mu}+\frac{h}{k\mu}+\frac{\mu}{hk}\right)+\frac{hk(\mu-1)(\mu-2)}{3\mu},$$

which upon simplification yields (2.20) □

Corollary 2.15. *Let h and k denote coprime, positive odd integers. Set*

$$f(h,k):=\sum_{j=1}^{k-1}\frac{\tan(\pi hj/k)}{\tan(2\pi j/k)}.\tag{2.25}$$

Then

$$h\,f(h,k)+k\,f(k,h)=-\frac{(h-k)^2}{2}.\tag{2.26}$$

Proof. Recall that $r(h,k)$ is defined by (1.6) and that $r_2(2,k)$ is defined by (2.19). Then, since $2\cot(2x)=\cot x-\tan x$, we find that

$$\begin{aligned}&h\,f(h,k)+k\,f(k,h)\\ =&-2h\sum_{j=1}^{k-1}\cot\left(\frac{2\pi hj}{k}\right)\cot\left(\frac{2\pi j}{k}\right)+h\sum_{j=1}^{k-1}\cot\left(\frac{\pi hj}{k}\right)\cot\left(\frac{2\pi j}{k}\right)\\ &-2k\sum_{j=1}^{h-1}\cot\left(\frac{2\pi kj}{h}\right)\cot\left(\frac{2\pi j}{h}\right)+k\sum_{j=1}^{h-1}\cot\left(\frac{\pi kj}{h}\right)\cot\left(\frac{2\pi j}{h}\right)\\ =&-2h\,r(h,k)-2k\,r(k,h)+h\,r_2(h,k)+kr_2(k,h).\end{aligned}\tag{2.27}$$

Hence, by (2.27), (2.17), and (2.20) with $\mu=2$,

$$h\,f(h,k)+k\,f(k,h)=2hk-\tfrac{2}{3}\left(h^2+k^2+1\right)+\tfrac{1}{6}\left(h^2+k^2+4\right)-hk=-\tfrac{1}{2}(h-k)^2.$$

□

3. Alternating Cosecant Sums

In this section we establish a general theorem about alternating cosecant sums. Since the proof is very similar to that for Theorem 2.1, we shall not provide all the details. Most of the corollaries of Section 2 have analogues here, and since the proofs are similar, we state only two of the corollaries.

Let m be a nonnegative integer, h and k be positive integers, and $0 \leq a, b < 1$. The general sum to be examined is defined by

$$c_m(h,k;a,b) := \sum_{0<r+b<k} (-1)^j \csc^m\left(\pi\frac{r+b}{k}\right)\csc\left(\pi\left(\frac{h(r+b)}{k}-a\right)\right). \tag{3.1}$$

We shall need the well-known Laurent expansion

$$z \csc z = \sum_{j=0}^{\infty}(-1)^{j-1}2(2^{2j-1}-1)\frac{B_{2j}}{(2j)!}z^{2j}, \qquad |z| < \pi, \tag{3.2}$$

where $B_n, 0 \leq n < \infty$, is the nth Bernoulli number. Define, for integers $j_1, j_2, \dots, j_n \geq 0$,

$$D(j_1, j_2, \dots, j_n) := \prod_{r=1}^{n}(-1)^{j_r-1}2(2^{2j_r-1}-1)\frac{B_{2j_r}}{(2j_r)!}. \tag{3.3}$$

If a is not an integer, $\csc(\pi(hz-a))$ is analytic at $z=0$. Now by induction on n, it is easily shown that, for each positive integer n,

$$\frac{d^n}{dz^n}(\csc(cz)) = c^n \csc(cz) U_n\left(\cot(cz)\right),$$

where $U_m(x)$ is a polynomial in x of degree m, with its coefficients independent of c. It readily follows that, for $\pi|hz-a| < 1$,

$$\csc\left(\pi(hz-a)\right) = \csc(\pi a)\sum_{j=0}^{\infty} Q_j(\cot(\pi a))(\pi hz)^j, \tag{3.4}$$

where $Q_m(x)$ is a polynomial in x of degree m. The first four polynomials are

$$Q_0(x) = -1, \quad Q_1(x) = -x, \quad Q_2(x) = -(x^2+\tfrac{1}{2}), \quad Q_3(x) = -(x^3+\tfrac{5}{6}x).$$

We are now ready to state the primary theorem of this section.

Theorem 3.1. *Let $0 \leq a, b < 1$. Let h, k, and m denote positive integers such that $m+h+k$ is even. Suppose that $(j+b)/k \neq (r+a)/h$, where j and r are nonnegative integers, such that $0 < j+b < k$ and $0 < r+a < h$. Then*

$$\frac{1}{k}c_m(h,k;a,b) + \frac{1}{h}c_m(k,h;b,a) \tag{3.5}$$
$$= \begin{cases} -\csc(\pi a)\csc(\pi b)\sum_1 h^{\mu}k^{\nu}D(j_1,\dots,j_m)Q_{\mu}(\cot(\pi a))Q_{\nu}(\cot(\pi b)), & \text{if } a,b>0,\\ -\csc(\pi b)\sum_2 h^{2\mu-1}k^{\nu}D(j_1,\dots,j_m,\mu)Q_{\nu}(\cot(\pi b)), & \text{if } a=0, b>0,\\ -\sum_3 h^{2\mu-1}k^{2\nu-1}D(j_1,\dots,j_m,\mu,\nu), & \text{if } a=0, b=0,\end{cases}$$

where the polynomials Q_m are defined by (3.4). The sums $\sum_1$, $\sum_2$, and $\sum_3$ are over all $(m+2)$-tuples $(j_1,\dots,j_m,\mu,\nu)$ such that, respectively, $2(j_1+\cdots+j_m)+\mu+\nu = m-1$, $2(j_1+\cdots+j_m)+2\mu+\nu = m$, and $2(j_1+\cdots+j_m)+2\mu+2\nu = m+1$.

Proof. Let $C = C_R$ denote the same closed contour as in the proof of Theorem 2.1. Let

$$f(z) := \csc^m(\pi z)\csc(\pi(hz-a))\csc(\pi(kz-b))$$

and consider

$$\frac{1}{2\pi i}\int_C f(z)\,dz. \tag{3.6}$$

On $I(C)$, $f(z)$ has simple poles at $z = (j+a)/h$ and $z = (r+b)/k$, where j and r are nonnegative integers such that $0 < j+a < h$ and $0 < r+b < k$, respectively. Straightforward calculations show that

$$\operatorname{Res}(f,(j+a)/h) = \frac{(-1)^j}{h\pi}\csc^m\left(\pi\frac{j+a}{h}\right)\csc\left(\pi\left(\frac{k(j+a)}{h}-b\right)\right), \tag{3.7}$$

$$\operatorname{Res}(f,(r+b)/k) = \frac{(-1)^r}{k\pi}\csc^m\left(\pi\frac{r+b}{k}\right)\csc\left(\pi\left(\frac{h(r+b)}{k}-a\right)\right). \tag{3.8}$$

On $I(C)$, $f(z)$ also has a pole at $z = 0$. We consider three cases.

Case 1. $a, b > 0$. Then f has a pole of order m at $z = 0$. From (3.2) and (3.4), we have

$$f(z) = \left(\sum_{j=0}^{\infty}(-1)^{j-1}2(2^{2j-1}-1)\frac{B_{2j}}{(2j)!}(\pi z)^{2j-1}\right)^m \csc(\pi a)\sum_{\mu=0}^{\infty}Q_\mu(\cot(\pi a))(\pi hz)^\mu$$
$$\times \csc(\pi b)\sum_{\nu=0}^{\infty}Q_\nu(\cot(\pi b))(\pi kz)^\nu.$$

Thus,

$$\operatorname{Res}(f,0) = \frac{1}{\pi}\csc(\pi a)\csc(\pi b)\sum\nolimits_1 h^\mu k^\nu D(j_1,\dots,j_m)Q_\mu(\cot(\pi a))Q_\nu(\cot(\pi b)), \tag{3.9}$$

where $D(j_1,\dots,j_m)$ is defined by (3.3).

Case 2. $a = 0, b > 0$. In the second case, f has a pole of order $m+1$ at $z = 0$. By the same type of calculation as above,

$$\operatorname{Res}(f,0) = \frac{1}{\pi}\csc(\pi b)\sum\nolimits_2 h^{2\mu-1}k^\nu D(j_1,\dots,j_m,\mu)Q_\nu(\cot(\pi b)). \tag{3.10}$$

Case 3. $a = b = 0$. Now f has a pole of order $m+2$ at $z = 0$. By a calculation similar to those above,

$$\operatorname{Res}(f,0) = \frac{1}{\pi}\sum\nolimits_3 h^{2\mu-1}k^{2\nu-1}D(j_1,\dots,j_m,\mu,\nu). \tag{3.11}$$

We now evaluate the integral in (3.6) directly. Note that

$$f(z+1) = (-1)^{m+h+k}f(z) = f(z),$$

since $m+h+k$ is even. Thus, the integrals over the indented vertical sides of C cancel. Since $f(z)$ tends to 0 uniformly on $0 \le x \le 1$ as $|y| \to \infty$, we conclude that

$$\frac{1}{2\pi i}\int_C f(z)\,dz = 0. \tag{3.12}$$

Hence, applying the residue theorem and utilizing (3.7) and (3.8) in (3.12), we find that

$$0=\frac{1}{h\pi}c_m(k,h;b,a)+\frac{1}{k\pi}c_m(h,k;a,b)+\operatorname{Res}(f,0), \tag{3.13}$$

where $\operatorname{Res}(f,0)$ is given by (3.9)–(3.11). Using these three values in (3.13) and simplifying, we complete the proof. □

Corollary 3.2. *Let n be any positive integer. Then*

$$\frac{1}{k}\sum_{j=1}^{k-1}(-1)^j\csc^{2n}\left(\frac{\pi j}{k}\right)=(-1)^n 2^{2n+1}\sum_{\substack{j_0,j_1,\dots,j_{2n}\geq 0\\ j_0+j_1+\cdots+j_{2n}=n}}k^{2j_0-1}\prod_{r=0}^{2n}(2^{2j_r-1}-1)\frac{B_{2j_r}}{(2j_r)!}, \tag{3.14}$$

where $B_j, j\geq 0$, denotes the jth Bernoulli number.

Proof. In Theorem 3.1, set $m=2n-1$, $h=1$, and $a=b=0$. The corollary now easily follows. □

Chu and Marini [24, p. 149] have established an alternative version of Corollary 3.2. The next corollary is an analogue of Corollary 2.9.

Corollary 3.3. *Let h and k denote coprime positive integers of opposite parity. Then*

$$\frac{1}{h}\sum_{j=1}^{h-1}(-1)^j\csc\left(\frac{\pi j}{h}\right)\csc\left(\frac{\pi kj}{h}\right)+\frac{1}{k}\sum_{j=1}^{k-1}(-1)^j\csc\left(\frac{\pi j}{k}\right)\csc\left(\frac{\pi hj}{k}\right)=\frac{1}{6}\left(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}\right).$$

Proof. We apply Theorem 3.1 with $a=b=0$ and $m=1$. Observe that

$$\sum\nolimits_3 h^{2\mu-1}k^{2\nu-1}D(j,\mu,\nu)=-\frac{1}{6}\left(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}\right),$$

where the sum is over all nonnegative integers j,μ,ν such that $j+\mu+\nu=1$. The desired result now follows. □

4. Sums of Eisenstein, Williams, and Zhang

In the Introduction, we mentioned a beautiful result (1.8) of Eisenstein first proved by Stern. There are several proofs of (1.8) in the literature; see, for example, a paper by F. Calogero and A. M. Perelomov [21] and a problem of Maier and Götze [60]. Williams and Zhang generalized (1.8) by proving general formulas for

$$e_n(k,a):=\sum_{j=1}^{k-1}\sin\left(\frac{2\pi aj}{k}\right)\cot^n\left(\frac{\pi j}{k}\right) \tag{4.1}$$

and

$$\sum_{j=1}^{k-1}\cos\left(\frac{2\pi aj}{k}\right)\cot^n\left(\frac{\pi j}{k}\right), \tag{4.2}$$

where k, n, and a are positive integers with $a < k$. A completely different kind of generalization of (1.8) is due to K. Wang [81]. Further proofs of Wang's result have been given by T. Okada [68] and M. Ishibashi [54].

Our goal in this section is to show how the methods of the two previous sections can be utilized to evaluate (4.1) and (4.2). We confine ourselves to (4.1) only, since the treatments are almost identical. Our methods yield evaluations in terms of Bernoulli numbers, while that of Williams and Zhang [84] yields formulas in terms of values of Bernoulli polynomials. Since $e_n(k,a) = 0$ trivially when n is even, we assume in the sequel that n is odd.

Recall the generating function for the Bernoulli numbers $B_j, 0 \leq j < \infty$,

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} z^j, \qquad |z| < 2\pi. \tag{4.3}$$

Define, for integers $j_1, j_2, \dots, j_n \geq 0$,

$$E(j_1, j_2, \dots, j_n) := \prod_{r=1}^{n} \frac{B_{2j_r}}{(2j_r)!}. \tag{4.4}$$

Theorem 4.1. *Let a, k, and m denote positive integers with $a < k$. Recall that $e_n(k,a)$ is defined in (4.1) and that $E(j_1, \dots, j_n)$ is defined in (4.4). Then*

$$e_{2m-1}(k,a) = -2^{2m-1} \sum (-1)^{(\mu+\nu-1)/2} a^\mu k^\nu \frac{1}{\mu!} \frac{B_\nu}{\nu!} E(j_1, \dots, j_{2m-1}), \tag{4.5}$$

where the sum on the right side is over all nonnegative integers $j_1, \dots, j_{2m-1}, \mu, \nu$ such that $2j_1 + \dots + 2j_{2m-1} + \mu + \nu = 2m-1$ and such that (necessarily) $\mu + \nu - 1$ is even.

Proof. Let

$$f(z) := \frac{e^{2\pi i a z} \cot^{2m-1}(\pi z)}{e^{2\pi i k z} - 1} + \frac{e^{-2\pi i a z} \cot^{2m-1}(\pi z)}{e^{-2\pi i k z} - 1}$$

and consider

$$\frac{1}{2\pi i} \int_C f(z)\, dz,$$

where $C = C_R$ is the same indented rectangle as in the proofs of Theorems 2.1 and 3.1. Because $f(z)$ has period 1, the integrals along the indented vertical sides of C cancel. Since $0 < a < k$, a brief calculation shows that $f(z)$ tends to 0 uniformly for $0 \leq x \leq 1$ as $|y| \to \infty$. Hence,

$$\frac{1}{2\pi i} \int_C f(z)\, dz = 0. \tag{4.6}$$

On $I(C)$, f has a simple pole at $z = j/k, 1 \leq j \leq k-1$, with

$$\begin{aligned} \operatorname{Res}(f, j/k) &= \frac{e^{2\pi i a j/k} \cot^{2m-1}(\pi j/k)}{2\pi i k} - \frac{e^{-2\pi i a j/k} \cot^{2m-1}(\pi j/k)}{2\pi i k} \\ &= \frac{1}{\pi k} \sin\left(\frac{2\pi a j}{k}\right) \cot^{2m-1}\left(\frac{\pi j}{k}\right). \end{aligned} \tag{4.7}$$

On $I(C)$, f also has a pole of order $2m$ at $z=0$. Now, by (2.2) and (4.3),

$$\begin{aligned} f(z) = &\sum_{\mu=0}^{\infty}\frac{(2\pi iaz)^{\mu}}{\mu!}\left(\sum_{j=0}^{\infty}(-1)^j\frac{B_{2j}}{(2j)!}2^{2j}(\pi z)^{2j-1}\right)^{2m-1}\\ &\times\sum_{\nu=0}^{\infty}\frac{B_\nu}{\nu!}(2\pi ikz)^{\nu-1}\\ &-\sum_{\mu=0}^{\infty}\frac{(-2\pi iaz)^{\mu}}{\mu!}\left(\sum_{j=0}^{\infty}(-1)^j\frac{B_{2j}}{(2j)!}2^{2j}(\pi z)^{2j-1}\right)^{2m-1}\\ &\times\sum_{\nu=0}^{\infty}\frac{B_\nu}{\nu!}(-2\pi ikz)^{\nu-1}. \end{aligned} \tag{4.8}$$

Then, using (4.8), we find, by a straightforward calculation, that

$$\operatorname{Res}(f,0) = (-1)^{(\mu+\nu-1)/2}\frac{2^{2m-1}}{\pi}\sum a^{\mu}k^{\nu-1}\frac{1}{\mu!}\frac{B_\nu}{\nu!}E(j_1,\dots,j_{2m-1}), \tag{4.9}$$

where $E(j_1,\dots,j_n)$ is defined by (4.4), and the sum is over all nonnegative integers $j_1,\dots,j_{2m-1},\mu,\nu$ such that $2j_1+\cdots+2j_{2m-1}+\mu+\nu=2m-1$ and such that $\mu+\nu-1$ is even.

Applying the residue theorem and using the calculations (4.7) and (4.9) in (4.6), we complete the proof. □

We now establish Eisenstein's result [28] from 1844.

Corollary 4.2. *Let a and k be integers with $0<a<k$. Then*

$$e_1(k,a) = k-2a.$$

Proof. Set $m=1$ in Theorem 4.1. Using the value $B_1=-1/2$, we complete the proof. □

Corollary 4.3. *Let a and k be integers with $0<a<k$. Then*

$$e_3(k,a) = \tfrac{4}{3}a^3-2a^2k+\tfrac{2}{3}ak^2+2a-k.$$

Proof. Apply Theorem 4.1 with $m=2$. Note that in the sum on the right side of (4.5) we need to calculate ten terms, although only six are distinct. We need the values, $B_1=-1/2, B_2=1/6$, and $B_3=0$. The desired result now follows. □

Both of these corollaries were also calculated by Williams and Zhang [84] from their general theorem.

We close by remarking that we can use the methods of this section to also determine the values of

$$\sum_{j=1}^{k-1}\sin^m\left(\frac{2\pi aj}{k}\right)\cot^n\left(\frac{\pi j}{k}\right) \qquad \text{and} \qquad \sum_{j=1}^{k-1}\cos^m\left(\frac{2\pi aj}{k}\right)\cot^n\left(\frac{\pi j}{k}\right),$$

but we need to strengthen the hypotheses by requiring that $0<ma<k$.

5. Trigonometric Sums Arising in the Chiral Potts Model

In their study of the chiral Potts model and of certain integrable chiral quantum chains, McCoy and Orrick [61] showed that the high- and low-temperature expansions of the free energy serve as generating functions for certain classes of trigonometric sums. Several of these sums were subsequently studied in detail by A. Gervois and M. L. Mehta [39], [40]. In this section, we explicitly evaluate perhaps the most elegant, interesting class.

In a different notation, Gervois and Mehta [39], [40] studied the sum

$$t_n(k,a) := \sum_{j=1}^{k-1} \frac{\sin^2(\pi a j/k)}{\sin^{2n}(\pi j/k)}, \tag{5.1}$$

where n, k, and a are positive integers with $a < k$. In particular, they derived a recurrence relation in the index n for $t_n(k,a)$. We shall use the methods of the previous sections to derive an explicit evaluation for $t_n(k,a)$.

Since $\sin^2 x = \frac{1}{2}(1-\cos(2x))$, we have

$$\begin{aligned} t_n(k,a) &= \frac{1}{2}\sum_{j=1}^{k-1} \frac{1}{\sin^{2n}(\pi j/k)} - \frac{1}{2}\sum_{j=1}^{k-1} \frac{\cos(2\pi a j/k)}{\sin^{2n}(\pi j/k)} \\ &=: \tfrac{1}{2} t_{n,1}(k) - \tfrac{1}{2} t_{n,2}(k,a). \end{aligned} \tag{5.2}$$

Thus, it suffices to evaluate $t_{n,1}(k)$ and $t_{n,2}(k,a)$. Since $\csc^2 x = 1+\cot^2 x$, the evaluation of $t_{n,1}(k)$ can be effected by the use of Corollary 2.2. However, for completeness, we evaluate both $t_{n,1}(k)$ and $t_{n,2}(k,a)$ below.

Define, for nonnegative integers $j_1, j_2, \ldots, j_n$,

$$M(j_1, j_2, \ldots, j_n) = \prod_{r=1}^{n} 2(2^{2j_r-1}-1)\frac{B_{2j_r}}{(2j_r)!}, \tag{5.3}$$

where $B_j, j \geq 0$, denotes the jth Bernoulli number.

Theorem 5.1. *If k and n are positive integers, then*

$$t_{n,1}(k) = -(-1)^n \sum (2k)^{2\mu} \frac{B_{2\mu}}{(2\mu)!} M(j_1, j_2, \ldots, j_{2n}), \tag{5.4}$$

where the sum is over all nonnegative integers $j_1, j_2, \ldots, j_{2n}, \mu$ such that $j_1 + j_2 + \cdots + j_{2n} + \mu = n$.

Proof. Let

$$f(z) := \csc^{2n}(\pi z)\cot(\pi k z).$$

Integrate f over the same contour C as in the proof of Theorem 2.1. On $I(C)$, f has simple poles at $z = j/k, 1 \leq j \leq k-1$, with

$$\operatorname{Res}(f, j/k) = \frac{1}{\pi k}\csc^{2n}\left(\frac{\pi j}{k}\right). \tag{5.5}$$

At $z = 0$, f has a pole of order $2n+1$. Using (2.2) and (3.2), we find, after a straightforward calculation, that

$$\operatorname{Res}(f, 0) = (-1)^n \frac{1}{\pi} \sum 2^{2\mu} \frac{B_{2\mu}}{(2\mu)!} k^{2\mu-1} M(j_1, j_2, \dots, j_{2n}), \tag{5.6}$$

where the sum is over all nonnegative integers $j_1, j_2, \dots, j_{2n}, \mu$ such that $j_1 + j_2 + \cdots + j_{2n} + \mu = n$.

Since $f(z)$ has period 1, and since $f(z)$ tends to 0 uniformly on $0 \le x \le 1$ as $|y| \to \infty$, we find that

$$\frac{1}{2\pi i} \int_C f(z)\, dz = 0. \tag{5.7}$$

Applying the residue theorem and using (5.5) and (5.6) in (5.7), we readily deduce (5.4). □

Chu and Marini [24] and Gervois and Mehta [39] also established a version of Theorem 5.1.

The sum

$$\sum_{j=1}^{k-1} \csc^{2n}\left(\frac{j\pi + \delta}{k}\right)$$

was evaluated by contour integration in a problem appearing in the *Siam Review* [38].

We now state two special cases of Theorem 5.1.

Corollary 5.2. *We have*

$$\begin{aligned} t_{1,1}(k) =& \frac{k^2 - 1}{3}, \\ t_{2,1}(k) =& \frac{k^4 + 10k^2 - 11}{45}. \end{aligned}$$

Proof. Apply Theorem 5.1 with $n = 1, 2$, respectively. We need the values $B_2 = 1/6$ and $B_4 = -1/30$. The calculations are straightforward. □

Theorem 5.3. *Let k, n, and a be positive integers with $a < k$. Then*

$$t_{n,2}(k, a) = -\sum (-1)^{(\mu+\nu)/2} 2^{\mu+\nu} \frac{a^\mu}{\mu!} \frac{B_\nu}{\nu!} k^\nu D(j_1, j_2, \dots, j_{2n}), \tag{5.8}$$

where $D(j_1, j_2, \dots, j_{2n})$ is defined by (3.3), and where the sum is over all nonnegative integers $j_1, j_2, \dots, j_{2n}, \mu, \nu$ such that $2j_1 + 2j_2 + \cdots + 2j_{2n} + \mu + \nu = 2n$.

Proof. Let

$$f(z) := \frac{e^{2\pi i a z} \csc^{2n}(\pi z)}{e^{2\pi i k z} - 1} - \frac{e^{-2\pi i a z} \csc^{2n}(\pi z)}{e^{-2\pi i k z} - 1}.$$

Integrate f over the same contour C as in the proofs above. On $I(C)$, f has simple poles at $z = j/k, 1 \le j \le k - 1$, with

$$\operatorname{Res}(f, j/k) = \frac{\cos(2\pi a j/k) \csc^{2n}(\pi j/k)}{\pi i k}. \tag{5.9}$$

On $I(C)$, f also has a pole of order $2n+1$ at $z=0$. Calculating its residue with the help of (3.2) and (4.3), we find that

$$\operatorname{Res}(f,0)=\frac{1}{\pi i}\sum(-1)^{(\mu+\nu)/2}2^{\mu+\nu}\frac{a^\mu}{\mu!}\frac{B_\nu}{\nu!}k^{\nu-1}D(j_1,j_2,\ldots,j_{2n}), \tag{5.10}$$

where the sum is over all nonnegative integers $j_1, j_2, \ldots, j_{2n}, \mu, \nu$ such that $2j_1+2j_2+\cdots+2j_{2n}+\mu+\nu=2n$, and where we used the fact that $\mu+\nu$ is necessarily even.

Observe that $f(z)$ has period 1, and so the integrals over the indented vertical sides of C cancel. Because $0<a<k$, we see that $f(z)$ tends to 0 uniformly on $0\le x\le 1$ as $|y|\to\infty$. It follows that

$$\frac{1}{2\pi i}\int_C f(z)\,dz=0. \tag{5.11}$$

Applying the residue theorem and using (5.9) and (5.10) in (5.11), we complete the proof of (5.8). □

Corollary 5.4. *For* $0<a<k$,

$$\begin{aligned} t_{1,2}(k,a)=&\frac{6a^2+k^2-6ak-1}{3},\\ t_{2,2}(k,a)=&\frac{k^4-30a^4+10k^2+60a^2-30a^2k^2+60a^3k-60ak-11}{45}. \end{aligned}$$

Proof. Set $n=1,2$ respectively, in Theorem 5.3. We need the values $B_1=-1/2, B_2=1/6, B_3=0$, and $B_4=-1/30$ to complete the calculations. □

Corollary 5.5. *For* $0<a<k$,

$$\begin{aligned} t_1(k,a)=&-a^2+ak,\\ t_2(k,a)=&\frac{a^4-2a^2+a^2k^2-2a^3k+2ak}{3}. \end{aligned}$$

Proof. These equalities follow immediately from (5.2) and Corollaries 5.2 and 5.4. □

6. Further Trigonometric Sums

Several analogues and generalizations of the classical Dedekind sum $s(h,k)$, and certain Franel integrals related to Dedekind sums, have representations in terms of trigonometric sums and may possess reciprocity theorems. See, for example, papers by G. Almkvist [2], Apostol [3], M. Beck [7], Berndt and L. A. Goldberg [14], Dieter [26], G. Greaves, R. R. Hall, M. N. Huxley, and J. C. Wilson [42], R. McIntosh [62], and M. Mikolás [64].

Certain multiple Dedekind sums, which have representations as sums of products of several cotangent functions, appear to have been first studied by L. Carlitz [22], [23], and later by Berndt [13]. They arise in certain topological problems and consequently have also been studied by E. Breiskorn [16], F. Hirzebruch [49], [50], D. Zagier [85], [86], Hirzebruch and Zagier [51], and S. Fukuhara [35]. In the latter paper, these sums are generalized to complex parameters. An entirely different kind of multiple cotangent sum has been introduced by M. Ishibashi [55]. The more elementary multiple cotangent sum evaluation

$$\sum_{1\le j_1\le j_2\le\cdots\le j_k\le n}\prod_{r=1}^{k}\cot^2\left(\frac{j_r\pi}{2n+1}\right)=\frac{(2n)!}{(2k+1)!(2n-2k)!},\qquad 1\le k\le n,$$

appears as a problem [9] in *Elemente der Mathematik*. See also a paper by Skau and Selmer [76].

Other trigometric sums similar to those studied in this paper have appeared in a variety of contexts. For example, cotangent sums, in particular, Dedekind sums, arise in the work of M. Beck [6] and Beck and S. Robins [8] in the theory of Ehrhart polynomials. In the latter paper, there are also applications to partitions. Fukuhara [33] has established reciprocity theorems for sums involving one cotangent and one cosecant, which can be proved by the same technique used here. Arising in the theory of Jacobi forms are cotangent sums involving complex parameters for which Fukuhara [34] has established a reciprocity law. T. Lawson [59] has studied a certain sum of cotangents and sines which appears in gauge theory. Y. Fukumoto [36] established a reciprocity theorem for a certain sum of cotangents and cosecants arising in the theory of the Dirac operator on weighted projective spaces. For an account of many of the ways Dedekind sums and generalized Dedekind symbols occur in geometry and topology, see Fukuhara's paper [32].

Finite trigonometric sums occasionally arise in the theory of determinants, permanents, and matrices. In particular, see papers by R. Kittappa [58], H. Minc [65], J. R. Stembridge and J. Todd [78], and F. Calogero and A. M. Perelomov [21].

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