ON THE TRANSFORMATION FORMULA FOR THE DEDEKIND ETA-FUNCTION

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Recall that the Dedekind eta-function is defined by

$$\eta(z) := e^{2\pi iz/24}\prod_{n=1}^{\infty}(1-e^{2\pi inz}), \qquad \text{Im } z>0.$$

The function $\eta(z)$ satisfies a certain transformation formula under any modular transformation $V(z) := (az+b)/(cz+d)$, where a, b, c, and d are integers such that $ad-bc=1$. For proofs, see, for example, T. M. Apostol's book [1, pp. 47–61] or Berndt's paper [2]. In particular, when $V(z)=-1/z$, the transformation formula for $\eta(z)$ can be written in the form

$$a^{1/4}e^{-a/12}\prod_{n=1}^{\infty}(1-e^{-2an}) = b^{1/4}e^{-b/12}\prod_{n=1}^{\infty}(1-e^{-2bn}), \tag{1}$$

where $\text{Re } a, \text{Re } b>0$ and $ab=\pi^2$. This is the formulation given by Ramanujan in Entry 27 of Chapter 16 in his second notebook [12], [5, p. 43]. Because the transformations $V(z)=z+1, -1/z$ generate the full modular group, in fact, the general transformation formula for $\eta(z)$ can be deduced from (1) and the trivial transformation formula for $V(z)=z+1$. For example, see M. I. Knopp's text [7, pp. 41–44, 49–60]. The purpose of this note is to present a new proof of (1) and to offer some connections with certain infinite series.

Proof of (1). We shall assume that $a, b>0$. The result for $\text{Re } a, \text{Re } b>0$ will then follow by analytic continuation.

Taking the logarithm of each side of (1) and using the Maclaurin series for $\log(1-z)$, we find that (1) takes the equivalent formulation

$$\sum_{n=1}^{\infty}\frac{1}{n(e^{2an}-1)}-\frac{1}{4}\log a+\frac{a}{12} = \sum_{n=1}^{\infty}\frac{1}{n(e^{2bn}-1)}-\frac{1}{4}\log b+\frac{b}{12}. \tag{2}$$

We now use the trivial equality

$$\frac{1}{e^z-1}=\frac{1}{2}\coth\frac{z}{2}-\frac{1}{2} \tag{3}$$

to recast (2) in the form

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\coth(an)-\coth(bn)\right)=\frac{1}{2}\log\frac{a}{b}-\frac{1}{6}(a-b). \tag{4}$$

By employing the partial fraction decomposition

$$\coth(\pi z)=\frac{1}{\pi z}+\frac{2z}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^2+z^2},$$

we can rewrite (4) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{m^2b+n^2a}-\frac{1}{m^2a+n^2b}\right)=\frac{1}{4}\log\frac{a}{b}, \tag{5}$$

where we have used the facts $\sum_{n=1}^{\infty} n^{-2}=\pi^2/6$ and $ab=\pi^2$. Now set $a=\pi e^{\gamma}$ and $b=\pi e^{-\gamma}$. Then, (5) is equivalent to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{m^2e^{-\gamma}+n^2e^{\gamma}}-\frac{1}{m^2e^{\gamma}+n^2e^{-\gamma}}\right)=\frac{\pi}{2}\gamma, \tag{6}$$

where γ is any real number.

We now prove (6). Let a_{mn} denote the summands in (6). Observe that $a_{mn}=-a_{nm}$. Hence,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{mn}=\lim_{n\to\infty}\sum_{\nu=1}^{n}\left(\sum_{\mu=1}^{n}+\sum_{\mu=n+1}^{\infty}a_{\mu\nu}\right)=\lim_{n\to\infty}\sum_{\nu=1}^{n}\sum_{\mu=n+1}^{\infty}a_{\mu\nu}. \tag{7}$$

Now the inequalities

$$\frac{1}{\mu^2e^{-\gamma}+\nu^2e^{\gamma}}<\int_{\mu-1}^{\mu}\frac{dx}{x^2e^{-\gamma}+\nu^2e^{\gamma}}<\frac{1}{(\mu-1)^2e^{-\gamma}+\nu^2e^{\gamma}}$$

give, on summing over $\mu, n+1\leq\mu<\infty$,

$$0<\int_{n}^{\infty}\frac{dx}{x^2e^{-\gamma}+\nu^2e^{\gamma}}-\sum_{\mu=n+1}^{\infty}\frac{1}{\mu^2e^{-\gamma}+\nu^2e^{\gamma}}<\frac{1}{n^2e^{-\gamma}+\nu^2e^{\gamma}}<\frac{e^{\gamma}}{n^2}.$$

Evaluating the integral above and then summing over $\nu, 1\leq\nu\leq n$, we deduce that

$$0<\sum_{\nu=1}^{n}\frac{1}{\nu}\tan^{-1}\left(\frac{\nu}{n}e^{\gamma}\right)-\sum_{\nu=1}^{n}\sum_{\mu=n+1}^{\infty}\frac{1}{\mu^2e^{-\gamma}+\nu^2e^{\gamma}}<\frac{e^{\gamma}}{n}. \tag{8}$$

Noting that the first sum above is simply a Riemann sum for $\frac{1}{x}\tan^{-1}x$ on $[0,e^{\gamma}]$ and letting n tend to ∞ in (8), we deduce that

$$\lim_{n\to\infty}\sum_{\nu=1}^{n}\sum_{\mu=n+1}^{\infty}\frac{1}{\mu^2e^{-\gamma}+\nu^2e^{\gamma}}=\int_{0}^{e^{\gamma}}\frac{\tan^{-1}x}{x}dx. \tag{9}$$

Replacing γ by $-\gamma$, we find that

$$\lim_{n\to\infty}\sum_{\nu=1}^{n}\sum_{\mu=n+1}^{\infty}\frac{1}{\mu^2e^{\gamma}+\nu^2e^{-\gamma}}=\int_{0}^{e^{-\gamma}}\frac{\tan^{-1}x}{x}dx. \tag{10}$$

Subtracting (10) from (9) and using (7), we find that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn} &= \lim_{n\to\infty}\sum_{\nu=1}^{n}\sum_{\mu=n+1}^{\infty}\left(\frac{1}{\mu^2e^{-\gamma}+\nu^2e^{\gamma}}-\frac{1}{\mu^2e^{\gamma}+\nu^2e^{-\gamma}}\right)\\ &=\int_{e^{-\gamma}}^{e^{\gamma}}\frac{\tan^{-1}x}{x}dx\\ &=\int_{-\gamma}^{\gamma}\tan^{-1}(e^t)dt\\ &=\frac{1}{2}\int_{-\gamma}^{\gamma}\left\{\tan^{-1}(e^t)+\tan^{-1}(e^{-t})\right\}dt\\ &=\frac{1}{2}\int_{-\gamma}^{\gamma}\frac{\pi}{2}dt\\ &=\frac{\pi}{2}\gamma,\end{aligned}$$

which completes the proof of (6).

Let $S(a,b)$ denote the double sum on the left side of (5). If we invert the order of summation in (5), we obtain the sum $S(b,a)$, which has the value $\frac{1}{4}\log\frac{b}{a}$. Thus, we do not obtain the same value when we change the order of summation in the conditionally convergent series $S(a,b)$. Moreover,

$$S(a,b)-S(b,a)=\frac{1}{2}\log\frac{a}{b}.$$

If we differentiate (5) with respect to a, we find that (since $ab=\pi^2$),

$$a\sum_{n=1}^{\infty}\operatorname{csch}^2(an)+b\sum_{n=1}^{\infty}\operatorname{csch}^2(bn)=\frac{1}{6}(a+b)-1. \tag{11}$$

To the best of our knowledge, in 1960, J. Lagrange [8] was the first person to give (11) in the literature. Another proof was given by Berndt [3, p. 164]. Setting $a=b=\pi$ in (11) yields

$$\sum_{n=1}^{\infty}\operatorname{csch}^2(n\pi)=\frac{1}{6}-\frac{1}{2\pi}, \tag{12}$$

which was evidently first proved by T. S. Nanjundiah [11] in 1951. Proofs of (12) have also been given by Berndt [3, p. 164], C.–B. Ling [9], K. Kiyek and H. Schmidt [6], B. Muckenhoupt [10], and R. E. Shafer [13].

Lastly, we remark that (4) has analogues. In particular, if $a,b>0$, $ab=\pi^2$, and N is any positive integer, then

$$\begin{aligned}a^{-N}\sum_{n=1}^{\infty}\frac{\coth(an)}{n^{2N+1}} &= (-b)^{-N}\sum_{n=1}^{\infty}\frac{\coth(bn)}{n^{2N+1}}\\ &-2^{2N+1}\sum_{k=0}^{N+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2N+2-2k}}{(2N+2-2k)!}a^{N+1-k}b^k,\end{aligned} \tag{13}$$

where $B_j, 0\le j<\infty$, denotes the jth Bernoulli number. By using (3), we can express (13) in terms of the Riemann zeta-function $\zeta(2N+1)$. There exist many proofs of (13) or its equivalent form in terms of ζ. See [3, pp. 153–155] or [4, pp. 276, 293] for many references.

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