How to Solve Boundary Value Problems

A boundary value problem consists of a partial differential equation together with boundary conditions which the desired unique solution must satisfy. There is but one procedure for solving boundary value problems. The details vary and primarily depend upon the particular partial differential equation. We shall, without loss of generality, assume that exactly one of the boundary conditions is homogeneous.

1. The first main goal is to find solutions which are products of functions of one variable. These solutions are to satisfy the pde and all homogeneous boundary conditions. Each homogeneous boundary condition for the function of two or more variables must be translated into a boundary condition for a function of one variable.

2. Separate variables. Each side will be equal to a constant, which we call $-\lambda$, but which you may want to rename, depending on the given problem.

3. We obtain boundary value problems for each of the variables. These problems will involve the constant λ. For most values of λ, there will not be any solutions to the problem, except the trivial solution identically equal to 0. Those values of λ which yield nontrivial solutions are called *eigenvalues*. The corresponding nontrivial solutions are called *eigenfunctions*. First solve the boundary value problem with the most boundary conditions attached to it. In many instances, it may not be clear which problem is to be solved first. Experience helps.

4. Having found the eigenvalues and eigenfunctions, we have obtained infinitely many product solutions. All solutions, and any linear combination of these solutions, will satisfy the pde and the homogeneous boundary conditions.

5. Up to this point, the inhomogeneous boundary condition has not been considered. We attempt to find that linear combination of the infinitely many product solutions which satisfies this last condition. Reflect a moment here, for possibly only a finite linear combination of the product solutions is needed. In general, form an infinite linear combination of product solutions with unknown coefficients c_n and invoke the inhomogeneous boundary condition.

6. Examine the sum arising from the inhomogeneous boundary condition. It will be either some kind of Fourier series (full Fourier series, cosine series, sine series, half-multiple cosine series, or half-multiple sine series), a series of Bessel series, or a series of Legendre polynomials.

7. Having identified the series above, write down what you know about such series, including formulas for the coefficients. You thus have two different series for the function of the inhomogeneous boundary condition. Equate coefficients to solve for the unknown coefficients c_n. Put these values of c_n into the infinite linear combination you formed in step 5.

8. If only a general function is given in the inhomogeneous boundary condition, you are finished. If a specific function is given, then you must calculate the coefficients according to the formula you wrote down in step 7.