Problems

In each of Problems 1 through 12 determine the general solution of the given differential equation that is valid in any interval not including the singular point.

1. $x^2y'' + 4xy' + 2y = 0$
2. $(x + 1)^2y'' + 3(x + 1)y' + 0.75y = 0$
3. $x^2y'' - 3xy' + 4y = 0$
4. $x^2y'' + 3xy' + 5y = 0$
5. $x^2y'' - xy' + y = 0$
6. $(x - 1)^2y'' + 8(x - 1)y' + 12y = 0$
7. $x^2y'' + 6xy' - y = 0$
8. $2x^2y'' - 4xy' + 6y = 0$
9. $x^2y'' - 5xy' + 9y = 0$
10. $(x - 2)^2y'' + 5(x - 2)y' + 8y = 0$
11. $x^2y'' + 2xy' + 4y = 0$
12. $x^2y'' - 4xy' + 4y = 0$

In each of Problems 13 through 16 find the solution of the given initial value problem.

13. $2x^2y'' + xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 4$
14. $4x^2y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3$
15. $x^2y'' - 3xy' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 3$
16. $x^2y'' + 3xy' + 5y = 0, \quad y(1) = 1, \quad y'(1) = -1$

17. Find all values of α for which all solutions of $x^2y'' + \alpha xy' + (5/2)y = 0$ approach zero as $x \to 0$.
18. Find all values of β for which all solutions of $x^2y'' + \beta y = 0$ approach zero as $x \to 0$.
19. Find γ so that the solution of the initial value problem $x^2y'' - 2y = 0$, $y(1) = 1$, $y'(1) = \gamma$ is bounded as $x \to 0$.
20. Find all values of α for which all solutions of $x^2y'' + \alpha xy' + (5/2)y = 0$ approach zero as $x \to \infty$.
21. Consider the Euler equation $x^2y'' + \alpha xy' + \beta y = 0$. Find conditions on α and β so that
    (a) All solutions approach zero as $x \to 0$.
    (b) All solutions are bounded as $x \to 0$.
    (c) All solutions approach zero as $x \to \infty$.
    (d) All solutions are bounded as $x \to \infty$.
    (e) All solutions are bounded both as $x \to 0$ and as $x \to \infty$.
22. Using the method of reduction of order, show that if r_1 is a repeated root of $r(r - 1) + \alpha r + \beta = 0$, then x^{r_1} and $x^{r_1} \ln x$ are solutions of $x^2y'' + \alpha xy' + \beta y = 0$ for $x > 0$.
23. **Transformation to a Constant Coefficient Equation.** The Euler equation $x^2y'' + \alpha xy' + \beta y = 0$ can be reduced to an equation with constant coefficients by a change of the independent variable. Let $x = e^z$, or $z = \ln x$, and consider only the interval $x > 0$.
    (a) Show that

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}.$$

    (b) Show that the Euler equation becomes

$$\frac{d^2y}{dz^2} + (\alpha - 1)\frac{dy}{dz} + \beta y = 0.$$

Letting r_1 and r_2 denote the roots of $r^2 + (\alpha - 1)r + \beta = 0$, show that
    (c) If r_1 and r_2 are real and different, then

$$y = c_1e^{r_1z} + c_2e^{r_2z} = c_1x^{r_1} + c_2x^{r_2}.$$

    (d) If r_1 and r_2 are real and equal, then

$$y = (c_1 + c_2z)e^{r_1z} = (c_1 + c_2 \ln x)x^{r_1}.$$

5.6 **Series Sol**

(e) If r_1 and r_2 are complex conjugates, $r_1 = \lambda + i\mu$, then

$$y = e^{\lambda z}[c_1 \cos(\mu z) + c_2 \sin(\mu z)] = x^{\lambda}[c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)].$$

In each of Problems 24 through 29 use the method of Problem 23 to solve the given equation for $x > 0$.

24. $x^2y'' - 2y = 0$
25. $x^2y'' - 3xy' + 4y = \ln x$
26. $x^2y'' + 7xy' + 5y = x$
27. $x^2y'' - 2xy' + 2y = 3x^2 + 2 \ln x$
28. $x^2y'' + xy' + 4y = \sin(\ln x)$
29. $3x^2y'' + 12xy' + 9y = 0$
30. Show that if $L[y] = x^2y'' + \alpha xy' + \beta y$, then

$$L[(-x)^r] = (-x)^r F(r)$$

for all $x < 0$, where $F(r) = r(r-1) + \alpha r + \beta$. Hence conclude that if $r_1 \neq r_2$ are roots of $F(r) = 0$, then linearly independent solutions of $L[y] = 0$ for $x < 0$ are $(-x)^{r_1}$ and $(-x)^{r_2}$.

31. Suppose that x^{r_1} and x^{r_2} are solutions of an Euler equation, where $r_1 \neq r_2$, and r_1 is an integer. According to Eq. (24) the general solution in any interval not containing the origin is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$. Show that the general solution can also be written as $y = k_1x^{r_1} + k_2|x|^{r_2}$.
Hint: Show by a proper choice of constants that the expressions are identical for $x > 0$, and by a different choice of constants that they are identical for $x < 0$.

Complex Coefficients. If the constants α and β in the Euler equation $x^2y'' + \alpha xy' + \beta y = 0$ are complex numbers, it is still possible to obtain solutions of the form x^r. However, in general, the solutions are no longer real-valued. In each of Problems 32 through 34 determine the general solution of the given equation.

32. $x^2y'' + 2ixy' - iy = 0$
33. $x^2y'' + (1 - i)xy' + 2y = 0$
34. $x^2y'' + xy' - 2iy = 0$

5.6 Series Solutions near a Regular Singular Point, Part I

We now consider the question of solving the general second order linear equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the neighborhood of a regular singular point $x = x_0$. For convenience we assume that $x_0 = 0$. If $x_0 \neq 0$, the equation can be transformed into one for which the regular singular point is at the origin by letting $x - x_0$ equal t.

The fact that $x = 0$ is a regular singular point of Eq. (1) means that $xQ(x)/P(x) = xp(x)$ and $x^2R(x)/P(x) = x^2q(x)$ have finite limits as $x \to 0$, and are analytic at $x = 0$. Thus they have power series expansions of the form

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n, \tag{2}$$

which are convergent for some interval $|x| < \rho$, $\rho > 0$, about the origin. To make the quantities $xp(x)$ and $x^2q(x)$ appear in Eq. (1), it is convenient to divide Eq. (1) by $P(x)$ and then to multiply by x^2, obtaining

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0, \tag{3a}$$