Y: , 4, ~~[illegible]~~, 5, ~~[illegible]~~, 9, 10,

(1) $\frac{d}{dx}\{x^p J_p(x)\} = x^p J_{p-1}(x)$ (2) $\frac{d}{dx}\{x^{-p} J_p(x)\} = -x^{-p} J_{p+1}(x)$

(1) $x^p J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(p+k+1)} \frac{x^{2p+2k}}{2^{p+2k}}$

$$\frac{d}{dx}\{x^p J_p(x)\} = \sum_{k=0}^{\infty} \frac{(-1)^k\, 2(p+k)\, x^{2p+2k-1}}{k!\,\Gamma(p+k+1)\, 2^{p+2k}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(p+k)} \frac{x^{2p+2k-1}}{2^{p+2k-1}}$$

$$= x^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(p-1+k+1)} \left(\frac{x}{2}\right)^{p-1+2k}$$

read p. 208 section on properties of Bessel functions

2nd. Sol. when $p \in Z^+$ or 0

By thm.

$$y_2(x) = x^{-p} \sum_0^{\infty} b_n x^n + C J_p(x) \ln x$$

Let $p=0$. $x^2 y'' + x y' + x^2 y = 0$

$$y_2(x) = J_0(x) \ln x + \sum_{n=0}^{\infty} b_n x^n \quad \text{(WLOG take } C=1\text{)}$$

$$y_2'(x) = J_0'(x) \ln x + \frac{J_0(x)}{x} + \sum_{n=1}^{\infty} b_n n x^{n-1}$$

$$y_2''(x) = J_0''(x) \ln x + \frac{2 J_0'(x)}{x} - \frac{J_0(x)}{x^2} + \sum_{n=1}^{\infty} b_n n(n-1) x^{n-2}$$

$$L(u_2) = x^2 J_0''(x)\ln x + 2xJ_0'(x) - \cancel{J_0(x)} + \sum_{n=2}^{\infty} b_n n(n-1)x^n$$

$$+ xJ_0'(x)\ln x + \cancel{J_0}(x) + \sum_{n=1}^{\infty} b_n n x^n$$

$$+ x^2 J_0(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+2}$$

$$= 2xJ_0'(x) + \sum_{n=2}^{\infty} b_n n(n-1)x^n + \sum_{n=1}^{\infty} b_n n x^n + \sum_{n=2}^{\infty} b_{n-2}x^n$$

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{[n!]^2 2^{2n}}$$

$$J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}\, 2n}{[n!]^2 2^{2n}}$$

$$\therefore b_1 x + \sum_{n=2}^{\infty}[b_n n^2 + b_{n-2}]x^n = -2\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}\, 2n}{[n!]^2 2^{2n}}$$

Only even powers on right side

$$\therefore \quad b_1 = 0$$

$$b_n = \frac{-b_{n-2}}{n^2} + \text{Bessel series term}$$

Then $b_3 = 0$, $b_5 = 0$ etc.

Replace n by $2n$ on left side

$$b_{2n}(2n)^2 + b_{2n-2} = \frac{(-1)^{n+1}\, n}{[n!]^2 2^{2n-2}} \qquad n \geq 1$$

Take $b_0 = 0$.

$$4b_2 = 1, \quad b_2 = 1/2^2$$

$$16b_4 + b_2 = \frac{-2}{(2!)^2 2^2}$$

$$b_4 = \left[-\frac{1}{2^2} - \frac{2}{2^2(2!)^2}\right] \frac{1}{4^2}$$

$$= \frac{-1}{2^2 4^2}\left[1 + \frac{1}{2}\right]$$

$$36 b_6 + b_4 = \frac{3}{[3!]^2 2^4}$$

$$b_6 = \frac{1}{6^2}\left[\frac{3}{[3!]^2 2^4} + \frac{1}{2^2 4^2}\left(1+\frac{1}{2}\right)\right]$$

$$= \frac{1}{2^2 4^2 6^2}\left[\frac{1}{3} + 1 + \frac{1}{2}\right] = \frac{1}{2^2 4^2 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

In general, $b_{2n} = \dfrac{(-1)^{n+1}}{2^{2n}(n!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$

Sol.

$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{2^{2n}(n!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) = N_0($$

~~7.4C. 11, 13, 14, 15~~ 11, 13, 14

In 11 do not use method on p 210-214

EXTRA PROB

a. Show $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ ~~in the following way~~.
by using the fact that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

b. Show that $J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x$, $J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cdot \cos x$