#5 Steady state temp. in a cylinder

$$u_{xx}+u_{yy}+u_{zz}=0 \quad \text{Laplace d.e. in 3 dims.}$$

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+u_{zz}=0$$

Assume temp. indep. of θ

$$u_{rr}+\frac{1}{r}u_r+u_{zz}=0 \quad , \; u(r,z)$$

$u(r,0)=0,\; u(r,L)=u_0$ (top & bottom of cylinder)

$$u(a,z)=0$$

$$u(r,z) = R(r)Z(z)$$

$$R''Z+\frac{1}{r}R'Z+RZ''=0 \qquad \Big| \quad Z(0)=0,\; R(a)=0$$

$$\frac{R''}{R}+\frac{R'}{rR} = -\frac{Z''}{Z} = -\lambda^*$$

$$r^2R''+rR'+\lambda^* r^2R=0 \;, \quad Z''-\lambda^* Z=0$$

author does not explain

Assume $\lambda^*\geq 0$. $\quad \lambda^*=0$, $Z''=0$, $Z(z)=c_1z+c_2$

$$Z(0)=c_2=0$$

$$r^2R''+rR'=0 \quad s(s-1)+s=s^2=0 \quad R(r)=c_1+c_2\log r$$

temp finite at $r=0 \Rightarrow c_2=0$, $R(a)=0 \Rightarrow c_1=0$

$\therefore$ 0 is not e.v.

Let $\lambda^*=\lambda^2$ $\quad Z''-\lambda^2Z=0 \quad s^2-\lambda^2=0 \Rightarrow s=\pm\lambda$

$$Z(z)=c_1e^{\lambda z}+c_2e^{-\lambda z}$$

$$Z(0)=c_1+c_2=0 \Rightarrow c_2=-c_1$$

$$Z(z)=c_1(e^{\lambda z}-e^{-\lambda z})=2c_1\sinh\lambda z$$

$$r^2 R'' + r R' + \lambda^2 r^2 R = 0$$

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$$

temp. finite at origin $\Rightarrow c_2 = 0$

$$R(a) = c_1 J_0(\lambda a) = 0$$

$$\Rightarrow \lambda a = \gamma_n \ , \ \lambda_n = \gamma_n / a$$

$$u_n(r,z) = J_0(\lambda_n r) \sinh \lambda_n z$$

We seek a sol.

$$u(r,z) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) \sinh \lambda_n z$$

$\ni$:

$$u(r,L) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) \sinh \lambda_n L = f(r)$$

We know that

$$f(r) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r)$$

where

$$a_n = \frac{2}{a^2 J_1^2(\gamma_n)} \int_0^a r f(r) J_0(\lambda_n r)\, dr$$

$\therefore$

$$c_n \sinh \lambda_n L = a_n$$

$$u(r,z) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) \frac{\sinh(\lambda_n z)}{\sinh(\lambda_n L)}$$

$\lambda_n r = u$

Here $a_n = \dfrac{2u_0}{a^2 J_1^2(\gamma_n)} \displaystyle\int_0^a r J_0(\lambda_n r)\, dr = \dfrac{2u_0}{a^2 J_1^2(\gamma_n) \lambda_n^2} \displaystyle\int_0^{\lambda_n a} u J_0(u)\, du$

$$= \frac{2u_0}{J_1^2(\gamma_n)\gamma_n^2} \int_0^{\gamma_n} \frac{d}{du}\left(u J_1(u)\right) du = \frac{2u_0 \gamma_n J_1(\gamma_n)}{J_1^2(\gamma_n)\gamma_n^2} = \frac{2u_0}{\gamma_n J_1(\gamma_n)}$$