MATH 448
EXAM #2
April 7, 2006

Name __________________________

If science is viewed as an industrial establishment, then mathematics is an associated power plant which feeds a certain kind of indispensable energy into the establishment. Salomon Bochner

SHOW ALL WORK. INDICATE ALL REASONING.

SCORES

1. _________
2. _________
3. _________
4. _________

Total _________
1. (20) Let

\[ f(z) = \frac{1}{z(z - 1)}. \]

a. Determine the Laurent expansion of \( f(z) \) in a neighborhood of \( z = 0 \).
b. Determine the Laurent expansion of \( f(z) \) in a neighborhood of \( z = 1 \).
c. Determine the Laurent expansion of \( f(z) \) in a neighborhood of \( z = \infty \).
d. In each case, determine where your Laurent series converges.
2. (10) Let $f(z)$ be analytic on a simply connected domain containing the closed unit disc centered at the origin. Suppose that $|f(z)| \leq 2$ for $|z| = 1$. Using one of the great theorems of complex analysis, prove that 

$$|f'''(0)| \leq 12.$$

3. (20) Let $f(z)$ be an entire function. Give a proof of Liouville’s theorem as follows:

a. Using one of the top five theorems in complex analysis, evaluate

$$\int_{|z|=R} \frac{f(z)dz}{(z-a)(z-b)},$$

(1)

where $a \neq b$ and $|a|, |b| < R$.

b. Suppose that $|f(z)| \leq M$, for all complex numbers $z$, and for some positive constant $M$. Determine an upper bound for the integral in (1).

c. Let $R \to \infty$ and use parts a. and b. to conclude your proof.
4. (50) Use the residue theorem to evaluate (giving all details)
\[
\int_{0}^{\infty} \frac{x^a}{x^2 + 1} \, dx, \quad -1 < a < 1.
\]
1. a. \[
\frac{1}{z(z-1)} = -\frac{1}{z(1-z)} = -\frac{1}{z} \sum_{n=0}^{\infty} z^{-n-1} = -\sum_{n=0}^{\infty} z^{-n}
\]
b. \[
\frac{1}{z(z-1)} = \frac{1}{(z-1)(z+1)} = \frac{1}{z-1} \sum_{n=0}^{\infty} (-1)^n (z-1)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^{n-1}
\]

c. Let \( w = 1/z \). So,
\[
\frac{1}{z(z-1)} \quad \frac{w}{(1-w)(1+w)} = \frac{w^2}{1-w} = \sum_{n=0}^{\infty} w^{n+2} = \sum_{n=0}^{\infty} z^{-n-2}
\]

d. a. nearest sing. to 0 is pole at \( z=1 \): conv. \( 0<|z|<1 \).
b. nearest sing. to 1 is pole at \( z=0 \): conv. \( 0<|z-1|<1 \)
c. nearest sing. to \( \infty \) is pole at \( z=1 \): conv. \( |z|>1 \).

2. By Cauchy's Integral formula for derivatives,
\[
|f^{(n)}(a)| = \left| \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{(z-a)^{n+1}} \, dz \right| \leq \frac{M}{\rho^n} \int_{|z|=\rho} \frac{1}{|z-a|} \, dz = \frac{2\pi \rho}{|z-a|} \leq \frac{2\pi \rho}{\rho} = 2\pi
\]

3. a. By the residue theorem,
\[
\int_{|z|=\rho} \frac{f(z)}{(z-a)(z-b)} \, dz = 2\pi i \left( \frac{f(a)}{a-b} + \frac{f(b)}{b-a} \right) = \frac{2\pi i}{a-b} (f(a) - f(b)).
\]
b. \[
\left| \int_{|z|=\rho} \frac{f(z)}{(z-a)(z-b)} \, dz \right| \leq \int_{|z|=\rho} \frac{1}{|z-a| |z-b|} \left| dz \right| \leq \frac{M \cdot 2\pi \rho}{(R-|a|)(R-|b|)}
\]

c. Let \( R \to \infty \). The integral tends to 0 by part b, and so must identically be equal to 0 because its value is independent of \( R \). Thus, by part a, \( f(a) - f(b) = 0 \) for all \( a \neq b \), i.e., \( f(a) = f(b) \), i.e., \( f \) is a constant.
4. \[ \int_{C_{1}}^{z} \, dz = 2\pi i \left( R_{z} + R_{-z} \right) \]
\[= 2\pi i \left( \frac{1}{2i} + \frac{(-i)^a}{2i} \right) \]
\[= \pi \left( e^{\pi i a/2} - e^{-\pi i a/2} \right) \]
\[= 2\pi i e^{\pi i a/2} \sin(\pi a/2) \]
\[\int_{l_{1}}^{l_{2}} \frac{z^{a}}{z^{2}+1} \, dz = \int_{l_{1}}^{l_{2}} \frac{R}{n^{2} e^{2\pi i} + 1} \, dz \rightarrow \int_{l_{1}}^{l_{2}} \frac{R}{n^{2} e^{2\pi i}} \, dz \]
\[\int_{l_{2}}^{l_{1}} \frac{z^{a}}{z^{2}+1} \, dz = \int_{l_{2}}^{l_{1}} \frac{(n e^{i(\pi-a)})^{a}}{n^{2} e^{2\pi i} - 1} \, dz \rightarrow \int_{l_{2}}^{l_{1}} \frac{(n e^{i\pi})^{a}}{n^{2} e^{2\pi i} - 1} \, dz \]
Thus, as \( n_{0} \rightarrow 0 \) and \( R \rightarrow \infty \)
\[\int_{C_{0}}^{C_{R}} \frac{z^{a}}{z^{2}+1} \, dz \rightarrow (1-e^{2\pi i a}) \int_{l_{1}}^{l_{2}} \frac{R}{n^{2}+1} \, dz. \]
\[\int_{C_{0}}^{C_{R}} \frac{z^{a}}{z^{2}+1} \, dz \leq \int_{C_{0}}^{C_{R}} \frac{|z|^{a}}{|z^{2}+1|} |dz| \leq \frac{R^{a}}{R^{2}-1} 2\pi R \rightarrow 0 \]
as \( R \rightarrow \infty \), because \( a < 1 \).
\[\int_{C_{0}}^{C_{R}} \frac{z^{a}}{z^{2}+1} \, dz \leq \int_{C_{0}}^{C_{R}} \frac{|z|^{a}}{|1-|z|^{2}} \, dz \leq \frac{n_{0}^{a}}{1-n_{0}^{2}} 2\pi n \rightarrow 0 \]
as \( n_{0} \rightarrow 0 \), because \( a > -1 \). Thus, by (4), as \( n_{0} \rightarrow 0 \) and \( R \rightarrow \infty \)
\[\int_{C_{0}}^{C_{R}} \frac{z^{a}}{z^{2}+1} \, dz \rightarrow 0+0 = -2\pi i e^{\pi i a/2} \sin(\pi a/2) \]
\[\int_{C_{0}}^{C_{R}} \frac{n_{0}^{a}}{n^{2}+1} \, dz = -\frac{2\pi i e^{-\pi i a/2} \sin(\pi a/2)}{e^{\pi i a/2} - 2i \sin(\pi a/2)} \]
\[= \frac{\pi \sin(\pi a/2)}{2 \cos(\pi a/2)}. \]