MULTI-VALUED FUNCTION THEOREM

**Theorem 0.1.** Let \( w = f(z) \) be analytic at \( z_0 \), and suppose that \( f(z) \) has a zero of order \( m \) at \( z_0 \). Then in some neighborhood of \( w = 0 \), each value \( w \) has \( m \) distinct pre-images in a neighborhood of \( z_0 \). Furthermore, we can write

\[
w = f(z) = g^m(z),
\]

where \( g(z) \) is univalent in a neighborhood of \( z_0 \).

**Proof.** Write

\[
w = f(z) = (z - z_0)^m h(z),
\]

where \( h(z) \) is analytic at \( z_0 \) and \( h(z_0) \neq 0 \). Because \( h(z) \neq 0 \) is some neighborhood of \( z_0 \), we can define an analytic \( m \)th root of \( h(z) \) near \( z_0 \). To that end, let

\[
H(z) := \{h(z)\}^{1/m}.
\]

Note that \( H(z_0) \neq 0 \), since \( h(z_0) \neq 0 \). Let

\[
g(z) := (z - z_0)H(z).
\]

Since

\[
g'(z) = H(z) + (z - z_0)H'(z),
\]

we have

\[
g'(z_0) = H(z_0) \neq 0.
\]

Hence, \( g(z) \) is univalent in a neighborhood of \( z_0 \). Moreover,

\[
w = (z - z_0)^m H^m(z) = g^m(z).
\]

Thus, the second part of our theorem has been proved.

Now consider a neighborhood of \( w = 0 \), say \( N = \{w : |w| < \delta\} \). We choose \( \delta \) sufficiently small so that \( N \) is in the range of \( f(z) \) and such that the corresponding values of \( H(z) \) are not equal to zero and such that \( g(z) \) is univalent. Now consider any point \( w_0 \in N \) such that \( w_0 \neq 0 \) and such that \( |w_0|^{1/m} < \delta \). Now

\[
w_0 = \{g(z)\}^m
\]

has \( m \) roots on \( |w| = |w_0|^{1/m} \). Each will be the image of some point \( z_j \), \( 1 \leq j \leq m \). These values will be distinct, because \( g(z) \) is univalent. This then concludes the proof of the first portion of the theorem. \( \Box \)