1. \[ L(z_1) = \frac{z_1(dw_3 - \beta w_1) + (\beta w_1 z_3 - d z_1 w_3)}{z_1(\lambda - \beta) + (\beta z_3 - d z_1)} = \frac{-\beta z_1 w_1 + \beta w_1 z_3}{-\beta z_1 + \beta z_3} = \frac{z_1 w_1}{z_1 + z_3} \]

\[ L(z_2) = \frac{z_2(dw_3 - \beta w_1) + (\beta w_1 z_3 - d z_1 w_3)}{z_2(\lambda - \beta) + (\beta z_3 - d z_1)} = \frac{\lambda w_3(z_3 - z_1) + \beta w_1 z_3}{\lambda z_3 - \lambda z_1 + (\beta z_3 - d z_1)} \]

\[ = \frac{\lambda w_3 (z_3 - z_2) + \beta w_1 (z_3 - z_2)}{\lambda z_3 - \lambda z_2 + (\beta z_3 - d z_1)} = \frac{w_3 - \beta w_1}{z_3 - z_1} \]

\[ = \frac{\lambda w_3 - \lambda w_1}{\lambda z_3 - \lambda z_1} = \frac{\lambda w_3 - \lambda w_1}{\lambda z_3 - \lambda z_1} \]

\[ L(z_3) = \frac{z_3(dw_3 - \beta w_1) + (\beta w_1 z_3 - d z_1 w_3)}{z_3(\lambda - \beta) + (\beta z_3 - d z_1)} = \frac{\lambda w_3(z_3 - z_1)}{\lambda z_3 - \lambda z_1 + (\beta z_3 - d z_1)} \]

2. \[ S(z_1) = \frac{z_1 - z_1}{z_2 - z_1} = 0, \quad S(z_2) = \frac{z_2 - z_1}{z_2 - z_1} = 1, \quad S(z) = \frac{(z_2 - z_1)}{z_2 - z_1} = \]

4. In all cases, use the cross ratio \[
\frac{z - z_1}{z - z_2} = \frac{z_3 - z_2}{z_3 - z_1} = \frac{w - w_1}{w - w_2} = \frac{w_3 - w_2}{w_3 - w_1} \]

a. \[ \frac{z - 1}{z - 1} = \frac{1 - 1}{w - 1} \Rightarrow \frac{z - 1}{w - 1} = \frac{1 - 1}{w - 1} \]

\[ \Rightarrow \frac{z - 1}{w - 1} \]

b. \[ \frac{z - 1}{z - 1} = \frac{1 - 1}{z - 1} \Rightarrow \frac{z - 1}{z - 1} = \frac{1 - 1}{z - 1} \]

\[ \Rightarrow \frac{z - 1}{z - 1} \]

c. \[ \frac{z - 1}{z - 1} = \frac{w - 1}{w} \frac{1 + i}{1} \Rightarrow \frac{z - 1}{z - 1} = \frac{w - 1}{w} \frac{1 + i}{1} \]

\[ \Rightarrow \frac{z - 1}{z - 1} = \frac{w - 1}{w} \frac{1 + i}{1} \]

d. \[ \frac{z - 1}{z - 1} = \frac{w - 1}{w} \frac{1 + i}{1} \Rightarrow \frac{z - 1}{z - 1} = \frac{w - 1}{w} \frac{1 + i}{1} \]

\[ \Rightarrow \frac{z - 1}{z - 1} = \frac{w - 1}{w} \frac{1 + i}{1} \]

\[ \Rightarrow \frac{z - 1}{z - 1} = \frac{w - 1}{w} \frac{1 + i}{1} \]
\[ T(z) = \frac{az}{bz + d} \quad \text{as} \quad 0 \to 0 \]

\[ T(1) = \frac{a}{b + d} = 1 \quad \Rightarrow \quad a = b + d \quad \Rightarrow \quad T(z) = \frac{(b + d)z}{bz + d} \]

\[ T(c^\prime) = \infty \Rightarrow bc^\prime + d = 0 \quad \text{or} \quad d = -bc^\prime. \quad \text{Thus}, \]

\[ T(z) = \frac{(b - bc^\prime)z}{bz - bc^\prime} = \frac{(a - c^\prime)z}{z - c^\prime} \]

b. Because \((0, 0)\) lies on both the real and imaginary axes, \(T(0) = 0\) or 1.

Suppose \(T(0) = 0\). Then \(T(z) = \frac{az}{cz + d}\). Now \(\infty\) must map into a point on the given circle, and the point must be \(1\). Hence \(T(\infty) = 1\). Hence \(a = c\). WLOG we can then assume that \(T(z) = \frac{z}{z + b}\). Also \(T(x)\) is real. So \(b\) is real. Now \(T(0,1)\) must lie inside the circle by continuity. Since all of \([0,1]\) must be covered by \(T\), we must have \(T([0,1]) = [0,1]\). Thus \(T(x) \to 1\) as \(x \to 1^+\), and \(T(x) \to 1\) as \(x \to \infty\). This is impossible as \(T\) is one-to-one at \(z = 1\).

Hence, \(T(0) \neq 0\).

Thus, \(T(0) = 1\). Now \(T(\infty)\) is real and must lie on the given circle, so \(T(\infty) = 0\). Hence \(a = 0\) and \(T(z) = \frac{b}{cz + d}\). But \(T(0) = 1\). Hence, \(b = d\). WLOG we can assume that \(b = d = 1\). Thus, \(T(z) = \frac{z}{z + 1}\). Also, since \(T(x)\) is real, \(c\) must be real. Let us look at the image of an arbitrary point \(z = iy\) on the imaginary axis. Now

\[
\left| \frac{1}{cy + 1} - \frac{1}{2} \right| = \frac{1}{2} \quad \Rightarrow \quad \left| \frac{-cy + 1}{cy + 1} - \frac{1}{2} \right| = \frac{1}{2}
\]

or

\[
\left| \frac{1}{1 + cy^2} - \frac{1}{2} - \frac{cy}{1 + cy^2} \right| = \frac{1}{2}
\]
\[
\left( \frac{1}{1+c^2 y^2} - \frac{1}{2} \right)^2 + \frac{c^2 y^2}{(1+c^2 y^2)^2} = \frac{1}{4}
\]
\[
\frac{4}{(1+c^2 y^2)^2} - \frac{4}{1+c^2 y^2} + \frac{1}{4} + \frac{c^2 y^2}{(1+c^2 y^2)^2} = \frac{1}{4}
\]
\[
\frac{4 - (1+c^2 y^2) + c^2 y^2}{(1+c^2 y^2)^2} = 0 = 0
\]

Thus, for all \( C \neq 0 \) real, \( T(z) = \frac{1}{cz+i} \) maps the imaginary axis onto \( |W - \frac{1}{2}| = \frac{1}{2} \).

c. Since \( \Phi \) lies on both the real and imaginary axes, \( T(0) = \frac{1}{2} \) or \( T(0) = 2 \). Let \( T(z) = \frac{az+b}{cz+d} \).

Suppose \( T(0) = \frac{1}{2} \). Then \( b/d = \frac{1}{2} \) or \( d = 2b \). Suppose \( T(\infty) = 0 \). Then \( T(z) = \frac{az}{cz} \), because \( b = 0 \), and then we must also have \( d = 0 \), since \( d = 2b \). But \( T(z) \) cannot be a constant. Thus, \( T(\infty) \neq 0 \). Hence, \( T(\infty) = 2 \). But then, in both cases, \( T(z) = [2, \infty) \) or \( [\frac{1}{2}, 2] \), we will have \( T \) being not one-to-one. Thus, \( T(0) = \frac{1}{2} \) is not possible.

Suppose \( T(0) = 2 \). Hence \( b/d = 2 \) or \( b = 2d \). As above, if \( T(\infty) = 0 \), we find that \( T(z) \) is a constant, which is impossible. Hence, \( T(\infty) = \frac{1}{2} \). Hence \( a/c = 1/2 \), or \( c = 2a \). Hence, so far, we have \( T(z) = \frac{az + 2d}{2az + d} \). Suppose \( a \) is not real. Then if we let \( z \to \infty \), the imaginary part on the denominator is roughly twice as large as the imaginary part of the numerator.

Write \( T(z) = \frac{z + 2d/a}{2az + d} = \frac{z + 2d}{2az + d} \). Suppose that \( d \) is not real. Then let \( z = x \). We know, that \( T(x) = C \), say, is real. Thus, \( x + 2d = (2x + d)c \) or \( x(1-2c) = d(c-2) \). But \( x(1-2c) \) and \( (c-2) \) are real. Thus, \( x \) must be real. Let us examine the image of \( z = iy \) on the imaginary axis.
\[
\frac{4x^4 + 8x^2y^2 + 4y^4}{(x^2 + 4y^2)^2} - \frac{5(x^2 + y^2)}{x^2 + 4y^2} + \frac{25}{16} + \frac{9d^2y^2}{(x^2 + 4y^2)^2} = \frac{9}{16}
\]

\[
4x^4 + 8x^2y^2 + 4y^4 - 5(x^2 + y^2)(x^2 + 4y^2) + 9d^2y^2 = -1
\]

\[
4x^4 + 17x^2y^2 + 4y^4 - 5x^4 - 25d^2y^2 - 20y^4 = -1
\]

\[
-\frac{4x^4 - 8x^2y^2 - 16y^4}{(x^2 + 4y^2)^2} = -\frac{(x^2 + 4y^2)^2}{(x^2 + 4y^2)^2} = -1
\]

Thus, \( T(z) = \frac{z + 2d}{2z + d} \) has all the desired properties.

d. Let \( T(z) = \frac{az+b}{cz+d} \), since \((0,0)\) is on both the real axis and \( y = x \), \( T(0) = \pm 1 \).

Suppose \( T(0) = +1 \). Then \( b = d \), and

\[
T(z) = \frac{az+b}{cz+b}
\]

Now \( T([-1, \infty)) = [-1, 1] \).

In particular, \( T(\infty) = -1 \). Thus, \( a/c = -1 \), \( ac = -a \).

Hence, \( T(z) = \frac{az+b}{-az+b} \). When we substitute this value for \( w \) in the locus \( |w + i| = \sqrt{2} \) we find that the equality is not satisfied. Suppose that \( T(0) = -1 \), then \( b = -d \) and

\[
T(z) = \frac{az+b}{cz+b}
\]

Now \( T([-1, \infty)) = [-1, 1] \) and so...
Thus, \( a = c \), hence \( T(z) = \frac{az - b}{az + b} \). We check to see if \( |w + i| = \sqrt{a} \) is satisfied on the line \( y = x \). Thus, \( z = x + ix = x(a + i) \). Now
\[
\frac{|ax(1+i) - b + i|^2}{|ax(1+i) + b|^2} = \left| \frac{ax - b + axi}{ax + b + axi} + i \right|^2
\]
\[
= \left| \frac{(ax-b+axi)(ax+b-axi)}{(ax+b)^2 + a^2x^2} + i \right|^2
\]
\[
= \left| \frac{a^2x^2 - b^2 + a^2x^2 + 2abxi + i}{c} \right|^2
\]
\[
= \left| \frac{2a^2x^2 - b^2 + i(2abx + 2a^2x^2 + 2abx + b^2)}{c} \right|^2
\]
\[
= \left| \frac{2a^2x^2 - b^2 + i(2a^2x^2 + 4abx + b^2)}{c} \right|^2
\]
\[
= \frac{(2a^2x^2 - b^2)^2 + (2a^2x^2 + 4abx + b^2)^2}{c^2}
\]
\[
= \frac{4a^4x^4 - 4a^2b^2x^2 + b^4 + 4a^4x^4 + 16a^2b^2x^2 + b^4 + 16a^3b x^3 + 8ab^3 + 16b^2}{c^2}
\]
\[
= \frac{8a^4x^4 + 16a^2b^2x^2 + 2b^4 + 16a^3b x^3 + 8ab^3 x}{4a^4x^4 + 8a^2b^2x^2 + b^4 + 8a^3b x^3 + 4ab^3 x}
\]
\[
= 2
\]
\[
|w + i| = \sqrt{a}
\]
Hence, \( T(z) = \frac{az - b}{az + b} \) for any real \( a, b \neq 0 \), satisfies the mapping properties.
14. \[ T(1) = \frac{a+b}{c+d} = 1 \Rightarrow a+b = c+d \]
\[ T(-1) = \frac{-a+b}{-c+d} = 1 \Rightarrow -a+b = c-d \]

Add: \(2b = 2c\) or \(b = c\). Subtract: \(2a = 2d\) or \(a = d\)
\[ T(0) = \frac{a+0}{b+0} = \frac{a}{b} = \frac{a}{a} = 1 \]
\[ T(\delta) = \frac{a+\delta}{b+\delta} = \frac{a+b+\delta}{b+\delta} = 1 \]
\[ T(\delta) = 1 \]

Note \(T(0) = b/a, \) and if \(a = 0, \) then \(T(\delta) = 1/2.\)

12. \[ S^{-1} T 0 S(\Delta) = S^{-1} (T(p)) = S^{-1} (p) = 1 \]
\[ S^{-1} T 0 S(-\Delta) = S^{-1} T(q) = S^{-1} (q) = -1 \]

13. By Prob. 11, \( U(\Delta) = \frac{\Delta + \delta}{\delta \Delta + \delta}, \) \( U(\Delta) = \frac{\Delta - \delta^2}{(\delta \Delta + \delta)^2} \)

So \(U(\Delta) = \frac{\Delta - \delta^2}{(\delta \Delta + \delta)^2} = \frac{\Delta}{\delta + \delta} \)
\(U(-\Delta) = \frac{\Delta - \delta^2}{(-\delta \Delta + \delta)^2} = \frac{\Delta + \delta}{\delta - \delta} = \frac{\Delta}{U(\Delta)}\)

10a. We continue the discussion in the hint. Suppose \(U \in EC(1), \)
Then \(T(2,2,1)\) would map, by continuity, at least 1 point on \(C_2.\) But \(T\) is one-to-one and \(C_1\) is mapped onto \(C_2.\) Thus, at least 1 pt. on \(C_2\) is the image of 2 points, contradicting the fact that \(T\) is one-to-one.

b. We just need to check if \(z = -d/c \) lies on \(I(C_1)\) or \(EC(1), \) \(\delta d/c \in I(C_1), \) then the interior of \(C_1\) is mapped to the exterior of \(C_2. \) \(\delta d/c \in E(C_1), \) then \(T\) maps \(I(C_1)\) to \(I(C_2).\)

8a. From \#7c,
\[ T(x+iy) = \frac{x+iy+2x}{2x+2iy} = \frac{5(x+2y)(x+2y) - 2y^2}{(2x+2)^2 + 4y^2} \]

Now,
\[ \frac{5(x+2y)(x+2y) - 2y^2}{(2x+2)^2 + 4y^2} \]
\[ = 2x^2 + 5x^2 + 2x^2 + 2y^2 - 3iy^2 \]
\(\text{If } x > 0, \) the 1st quadrant maps into the fourth quadrant.