Computation of $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$

As the title suggests – our goal is to compute mod 2 cohomology of $K(\mathbb{Z}_2, n)$. The computation of the cohomology of the Eilenberg-MacLane spaces will be performed via induction on the connectivity of the space. The inductive step will be performed using Serre spectral sequence of the fiber sequence $K(\mathbb{Z}_2, n) \longrightarrow K(\mathbb{Z}_2, n+1)$. However, instead of tackling the problem directly, we will discuss some general properties of Serre spectral sequence and then apply those to our situation.

Suppose we are given a fibration $F \longrightarrow X \longrightarrow B$, and that it satisfies the usual conditions for existence of properly convergent Serre spectral sequence, i.e. the trivial action of $\pi_1(B, *)$ on $H^*(F)$. Set the coefficient group to be $G$. Let us look at the $E_n$-page of the spectral sequence for cohomology, where $n \geq 2$; there we have a differential $d^n : E_n^{0,n-1} \longrightarrow E_n^{n,0}$. This differential is often referred to as transgression. There is one transgression per dimension; therefore, we will write it as $\tau$, since most of the time the dimension is understood. Note that $E^{0,n-1}_n$ is a subgroup of $E^{0,n-1}_2 = H^0(B; H^{n-1}(F; G)) = H^{n-1}(F; G)$, since no non-trivial differentials enter $E^{0,n-1}_2$ spot of the spectral sequence. Similarly, $E^{0,0}_n$ is a quotient of $H^n(B; G)$. We can try to “manufacture” a map relating $H^{n-1}(F; G)$ to $H^n(B; G)$, using ordinary long exact sequence coboundary map. Here is what we will eventually arrive at:

$$
H^{n-1}(F; G) \xrightarrow{\delta} H^n(X, F; G) \xrightarrow{p^*} H^n(B, *; G) \xrightarrow{j^*} H^n(B; G)
$$

We get a map from $\delta^{-1}(im\, p^*)$ to $H^n(B; G)/j^*(ker\, p^*)$. Note also that $j^*$ is an isomorphism if $n > 0$. It is not unnatural to expect $\delta^{-1}(im\, p^*) = E^{0,n-1}_n \subset E^{0,n-1}_2 = H^{n-1}(F; G)$ and $H^n(B; G)/j^*(ker\, p^*) \cong E^{0,0}_n$ and the map $\mu$ be the same as $\tau$. In fact, the following statement is true.

**Proposition.** There exists a commutative diagram,

$$
\begin{array}{ccc}
\delta^{-1}(im\, p^*) & \xrightarrow{\mu} & H^n(B; G)/j^*(ker\, p^*) \\
\downarrow{\cong} & & \downarrow{\cong} \\
E^{0,n-1}_n & \xrightarrow{\tau} & E^{0,0}_n \\
\end{array}
$$

**Proof.** See [Hatcher, SSAT, Ch1, pp.21-22].

We set our coefficient group to be $\mathbb{Z}_2$ from now on. The action of Steenrod squares on $H^*(F; \mathbb{Z}_2)$ and $H^*(B; \mathbb{Z}_2)$ descend to an action on $E^{0,n-1}_n$ and $E^{0,0}_n$. The elements of $E^{0,n-1}_n$ in $H^*(F; \mathbb{Z}_2)$ are called transgressive. If $\alpha$ is transgressive then $\delta \alpha = p^* \beta$ for some $\beta \in H^n(B, *; \mathbb{Z}_2)$. Let $Sq$ be a Steenrod square. Then $\delta Sq \alpha = Sq \delta \alpha = Sq p^* \beta = p^* Sq \beta$, implying that $Sq \alpha$ is also transgressive. Let us emphasize that the coboundary operations and squares commute due to stability of squares. Similarly, an element of $j^*(ker\, p^*)$ is form, $j^*(\gamma)$, where $p^* \gamma = 0$. Then $Sq j^*(\gamma) = j^*(Sq \gamma)$ and $p^* Sq \gamma = Sq p^* \gamma = 0$. Thus, we have a well-defined action of Steenrod squares on $E^{0,0}_n$. Furthermore, we claim that squares commute with transgressions. Indeed, if $\delta \alpha = p^* \beta$, then $j^* \delta = \tau \alpha$. Then $Sq \alpha = p^* Sq \beta$, i.e. $\tau Sq \alpha = j^* Sq \beta = Sq j^* \beta = Sq \tau \alpha$. Thus, we can see that transgression work well with Steenrod squares.

In certain special situations transgressions can give enough information to compute $H^*(B; \mathbb{Z}_2)$ in terms of $H^*(\Omega B; \mathbb{Z}_2)$. The following theorem is due to Borel.

**Theorem.** Let $B$ be a simply-connected space and $H^*(\Omega B; \mathbb{Z}_2)$ have a simple system of transgressive generators, with finitely many of those generators in each dimension. Then $H^*(B; \mathbb{Z}_2)$ is a polynomial ring generated by any representatives of transgressions of Serre spectral sequence for the fiber sequence $\Omega B \longrightarrow * \longrightarrow B$, of this simple generators.

If $k$ is a field, a simple system of generators for a $k$-algebra $A$, is a subset $S$ of $A$, such that the elements of form $x_1 x_2 \ldots x_n$, where $x_i$’s are distinct elements of $S$, form a basis for $A$ as a $k$-module.

**Proof.** See [Mosher/Tangora, Ch9, pp.91-92] & [Hatcher, SSAT, Ch1, pp.54-58].
Note that a graded polynomial algebra $\mathbb{Z}_2[x_1, x_2, \ldots]$ has a simple system of generators, namely $\{x_i^2\}$. Thus, if we have that $H^*(\Omega B; \mathbb{Z}_2)$ is a polynomial algebra over a set of transgressive elements, then so is $H^*(B; \mathbb{Z}_2)$. This is a consequence of the fact that $x^2 = \text{Sq}^{2i-1} \ldots \text{Sq}^2 x$ ($n$ is the degree of $x$) and that the squares preserve transgressiveness. This is a good place to transition to a more concrete situation.

We begin with $K(\mathbb{Z}_2, 1)$. We know that $K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$. Using the standard cell structure of $\mathbb{R}P^\infty$ or the Gysin sequence of the universal line bundle $L \longrightarrow \mathbb{R}P^\infty$, we can determine that $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) = Z_2[\tau_1]$, where $\tau_1$ is the non-zero element of dimension 1 (and hence is the fundamental class). Furthermore, $\tau_1$ is transgressive, since the differentials before $\tau$ all map into a zero group. Thus, using the previous observation $H^*(K(\mathbb{Z}_2, 2); \mathbb{Z}_2)$ is polynomial algebra over $\mathbb{Z}_2$. In fact, we can continue the argument, and understand what the generators of this cohomology ring are using Borel’s theorem. The following theorem is due to Serre.

**Theorem.** $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\text{Sq}^i(\tau_n)]$, where $\tau_n$ is the generator of $H^n(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ and $I$ ranges over all admissible sequences of excess $e(I) < n$.

As a reminder a finite sequence $I = (i_1, i_2, \ldots)$ is admissible if $i_j \geq 2i_{j+1}$, $\text{Sq}^I x = \text{Sq}^i x \text{Sq}^{i_2} \ldots x$, and $e(I) = i_1 - \sum_{j \geq 2} i_j$. The theorem will follow directly from the following easy observations.

**Lemma.** (a) $\text{Sq}^I(\tau_n) = 0$ if $I$ is admissible and $e(I) > n$.

(b) The elements $\text{Sq}^I(\tau_n)$ with $I$ admissible and $e(I) = n$ are exactly the powers $(\text{Sq}^I(\tau_n))^2$ with $J$ admissible, $e(J) < n$, and $j > 0$.

**Proof.** (a) Note that $I_1 = e(I) + \sum_{j \geq 2} i_j > n + \sum_{j \geq 2} i_j = |\text{Sq}^I(\tau_n)|$, which implies that $\text{Sq}^I(\tau_n) = \text{Sq}^{i_1} \text{Sq}^{i_2} \ldots (\tau_n) = 0$.

(b) Let us pick $\text{Sq}^I(\tau_n)$, such that $e(I) = n$. Then $I_1 = n + \sum_{j \geq 2} i_j = |\text{Sq}^I(\tau_n)|$, where $\tilde{I} = (i_2, i_3, \ldots)$. Using the squaring property we get $\text{Sq}^I(\tau_n) = \text{Sq}^{i_1} \text{Sq}^I(\tau_n) = (\text{Sq}^I(\tau_n))^2$. The sequence $\tilde{I}$ is clearly admissible, and $e(\tilde{I}) = i_2 - \sum_{j \geq 3} i_j = e(I) + (2i_2 - i_1) < e(I) = n$. If $e(I) = n$, we repeat the process, otherwise we stop. The process will eventually stop, yielding an equality of form $\text{Sq}^I(\tau_n) = (\text{Sq}^I(\tau_n))^2$, where $e(J) < n$ and $j > 0$, which is what we want.

To prove the converse inclusion, we essentially reverse the argument. Suppose we are given an element $(\text{Sq}^I(\tau_n))^2$, where $J$ is admissible, $e(J) \leq n$ and $j > 0$. Let $\tilde{J}$ be the same as $J$ except with an extra $n + |J|$ on its left, where $|J|$ is the degree of $J$. Clearly, $e(\tilde{J}) = n$. Note that $n + |J| - 2i_1 = n - i_1 + \sum_{j \geq 2} i_j = n - e(J) > 0$, which implies that $\tilde{J}$ is admissible. We continue the process until we reach the equality $(\text{Sq}^I(\tau_n))^2 = \text{Sq}^I(\tau_n)$, where $I$ is admissible and $e(I) = n$.

**Proof of Serre’s Theorem.** We addressed the $n = 1$ case earlier. So suppose the statement of the theorem is true for $n$. Then $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ has a simple system of generators $\{(\text{Sq}^I(\tau_n))^2\}$, such that $e(I) < n$. $\tau_n$ is transgressive for the same reason that $\epsilon_1$ was. Therefore, the simple system of generators we have consists of transgressive elements. This set is the same as $\text{Sq}^I(\tau_n)$, where $e(I) \leq n$. Thus, $H^*(K(\mathbb{Z}_2, n + 1); \mathbb{Z}_2)$ is generated by representatives of elements $\tau(\text{Sq}^I(\tau_n)) = \text{Sq}^I(\tau(n)) = \text{Sq}^I(\tau(n))$, specifically, $\text{Sq}^I(\tau_{n+1})$, where $I$ is admissible and $e(I) < n + 1$. This completes the inductive step and the proof of the theorem.

Finally, let us mention what implications this theorem has on the structure of Steenrod algebra. We remind the reader that the Steenrod algebra, written as $A_2$, is the $\mathbb{Z}$-algebra of stable reduced cohomology operations. Here is a precise definition. First we consider general operations, which we define as natural transformations $H^k(\_; \mathbb{Z}_2) \longrightarrow H^{n+k}(\_; \mathbb{Z}_2)$ for various $k, n \in \mathbb{N}$. Let this set be denoted by $O_{n, k}$. Clearly, $O_{n, k}$ is a $\mathbb{Z}$-module. Assemble these modules into a bigger one $O_n = \bigoplus_{k=1}^{\infty} O_{n, k}$. Consider all the elements of form $\eta - \zeta \in O_n$, where $\eta \in O_{n, k}$ and $\zeta \in O_{n, k+r}$, such that for any cohomology class $\alpha$ of degree $k$, $\Sigma^r \eta(\alpha) = \zeta(\Sigma^r \alpha)$. These elements form a $\mathbb{Z}$-submodule. Let the quotient module of $O_n$ by this submodule be denoted by $S_n$. Define $A_2$ to be $\bigoplus_{n=0}^{\infty} S_n$. The algebra structure is given by composition.

Let $\eta$ be an operation in $O_{n, k}$. Using Yoneda type argument one can show that $\eta$ is completely determined by its value at the fundamental class $\eta(\epsilon_k) \in H^{n+k}(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$. By Serre’s theorem, $\eta(\epsilon_k)$ is a polynomial $p(\text{Sq}^l(\epsilon_k))$, where $I$ ranges over admissible sequences with excesses less than $n$, and $p$ does not contain any degree 0 terms. By naturality of Steenrod squares and cup product, we can conclude that $\eta(\alpha) = p(\text{Sq}^l(\alpha))$. Define another operation $\hat{\eta} \in O_{n, k+1}$ via the equation $\hat{\eta}(\alpha) = \hat{p}(\text{Sq}^l(\alpha))$, where $\hat{p}$ is the linear part of $p$. The elements $\eta$ and $\hat{\eta}$ represent the same class in $A_2$: $\Sigma(\eta) = \Sigma p(\text{Sq}^l(\alpha)) = \Sigma \hat{p}(\text{Sq}^l(\alpha)) = \hat{\eta}(\Sigma \alpha)$. The fact that only the linear part of the polynomial remains is a consequence of the fact that the cup product on suspended spaces is 0. Thus, any element of $A_2$ is represented by a linear polynomial on admissible
sequences of Steenrod operations. Now suppose \( \eta \in \mathcal{O}_{n,k} \) and \( \zeta \in \mathcal{O}_{n,k+r} \) represent the same element in \( A_2 \), and \( \eta(\alpha) = p(Sq^I(\alpha)) \) and \( \zeta(\beta) = q(Sq^I(\beta)) \), where \( p \) and \( q \) are linear. Then \( \Sigma^r p(Sq^I(\iota_k)) = \Sigma^r \eta(\iota_k) = \zeta(\Sigma^r \iota_k) = q(Sq^I(\Sigma^r \iota_k)) = \Sigma^r q(Sq^I(\iota_k)) \), which implies \( p = q \), since both of the polynomials are linear. Thus, any element of Steenrod algebra can be uniquely expressed as a \( \mathbb{Z}_2 \)-linear combination of admissible Steenrod squares. However, we know that using Adem relations one can convert any sequence of squares into an a sum of admissibles. Therefore, the following theorem holds.

**Theorem.** \( A_2 \) is the free associative algebra on symbols \( \{ Sq^n \}_{n=0}^{\infty} \) modulo the Adem relations. \( \blacksquare \)