The subalgebras $A(n)$ & nilpotence in $A^*$.

Goal: Every elt in $A^{>0}$ is nilpotent.

- More generally, any finite collection $S_1, \ldots, S_n$ generates a finite subalgebra of $A^*$.

Will show $\{S_1^1, S_1^2, S_1^4, \ldots, S_1^{2^n}\}$

generates a finite subalgebra.

Recall Milnor basis for $A^*$. Start with Semi-Cartan basis at admissibles for $A^*$. Define classes $S_i \in A_{2i-1}^*$

dual to $S_1^{2i-1} S_1^{2i-2} \ldots S_1^2 S_1^1$.

Thm (Milnor) $A_* \cong F_2 [S_1, S_2, \ldots]$.

$\Delta(S_n) = \sum \limits_i S_{n-i} \otimes S_i$

Now define $S_r^1, r_2, \ldots, r_2 \in A^*$ of degree $r_1 + 3r_2 + 7r_3 + \ldots + (2^n-1)r_n$

dual to $S_1^r, S_2^r, \ldots, S_n^r$.

This is the Milnor basis for $A_*$. 
Be careful! $S_i$ dual to $S_q^{2^{-i}} \cdot S_q^{-1}$ and $S_q^{\frac{n}{2^i}}$ dual to $S_i$, but this does not imply that $S_q^{\frac{n}{2^i}} = S_q^{2^{-i}}$. E.g. What is $S_q^{\frac{n}{2^i}}$ in $Q^3$? Adm. basis is $\xi, S_q^3, S_q^2 S_q^1$.

Linear basis for $Q_3$ is $\xi, S_q^3, S_q^2$.

Write $<, > : Q_n \otimes A^n \to \mathbb{F}_2$ (evaluation).

By def. of $S_q^{\frac{n}{2^i}}$, have $< S_q^3, S_q^{\frac{n}{2^i}} > = 0$

$< S_q^2, S_q^{\frac{n}{2^i}} > = 1$.

Which lin. comb. of $S_q^3$ and $S_q^2 S_q^1$ satisfies this?

Calculates:

Recall $< x \cdot y, z >$

$= \langle x \otimes y, \Delta(z) \rangle$

$= \sum_i \langle x, z_i \rangle \langle y, z_i \rangle$

where $\Delta(z) = \sum_i z_i \otimes z_i$.

$S_n < S_q^3, S_q^3 > = < S_q^3 \otimes S_q^3, S_q^3 \otimes S_q^3 + S_q^1 \otimes S_q^2 + S_q^0 \otimes S_q^3 >$

$= < S_q^3, S_q^1 > \cdot < S_q^3, S_q^2 >$

$= \langle S_q^3 \otimes S_q^3, S_q^3 \otimes S_q^3 >$

$= < S_q^3, S_q^1 > \cdot < S_q^3, S_q^1 > = 1$. 

$\begin{array}{c|cc}
\xi & S_q^3 & S_q^2 \\
\hline
S_q^3 & 1 & 0 \\
S_q^2 S_q^1 & 1 & 1 \\
S_q^3 S_q^2 & 0 & 1 \\
\end{array}$
\[ \langle S_1^3, S_q^2 S_q' \rangle = \langle S_1 \circ S_1, (S_q^2 \circ 1 + S_q' \circ S_q + 1 \circ S_q^2) \circ (S_q \circ 1 + S_q' \circ 1 \circ S_q') \rangle \\
= \langle S_1, S_q' \rangle \langle S_1^2, S_q^2 S_q' + S_q^2 \rangle \\
= 1 \\

\underline{Conclusion:} \ S_q^{0,\infty} = S_q^3 + S_q^2 S_q'. \\

\underline{Remark} Easy to see that \ S_q^{n',0,0,0} = S_q^n.

Consider quotient \[ A_n \xrightarrow{1} \frac{A_n}{S_j^{2n+2-j}} \ \text{all} \ j \]

\[ \cong \mathbb{F}_2 [S_1, \ldots, S_{n+1}] \overline{\left( S_1^{2n+1}, S_2^{2n}, \ldots, S_{n+1}^{2n} \right)} \]

\underline{Note:} \ \lim_{\mathbb{F}_2} A_n = Z^{n+1} \cdot Z^n \cdots Z = Z^{n+1 + n + \cdots + 2} = Z^{\binom{n+2}{2}}.

\text{Top degree is} \ Z^{n+1} - 1 + (4-1)(Z^n - 1) + \cdots + (Z^{n+1} - 1) \\
= (n-1) Z^{n+2} + n + 5 \quad (\text{for } n \geq 1)

\text{Also, coproduct formula for } \Delta(S_k^{2n+2-k}) \text{ shows}

\text{this is a map of Hopf algebras.}
Dualizing gives an inclusion
\[ a(n)^* \hookrightarrow a^* \]
and a finite Hopf subalgebra.

Clear that \( S_q, S_q^2, \ldots, S_q^{2^n} \in a(n)^* \)
since these are dual to \( s_1, s_1^2, \ldots, s_1^{2^n} \in a(n) \).

\( \Rightarrow S_q, \ldots, S_q^{2^n} \) generate a subalgebra of \( a(n)^* \)
so this must be finite.

(In fact, they generate \( a(n)^* \). See Milnor, §8.)

\[ \text{e.g., } a(0) = \mathbb{F}_2 \left[ S_q \right] / (S_q)^2 \]

\( a(1) \) gen. by \( S_q, S_q^2 \).

Draw picture

In dim \( \leq 2 \), only one class.

In dim 3, 4 gen. basis
\[ S_q^2 S_q^1, S_q^3 (= S_q^1 S_q^2) \]

In dim 4, 5 gen. basis is
\[ S_q^4, S_q^3 S_q^1 \]

By Adem, \( S_q^2 S_q^2 = (1) S_q^4 + (0) S_q^3 S_q^1 = S_q^3 S_q^1 \)

Dim 5: \( S_q^2 S_q^2 S_q^1 = S_q^3 S_q^1 S_q^1 = 0, S_q^2 S_q^3 = (2) S_q^4 \)

Dim 6: \( S_q S_q^2 S_q^2 S_q = S_q^2 S_q^3 S_q^1 = S_q S_q^4 S_q^1 \)

\[ S_q^2 S_q^3 S_q^1 = S_q S_q^4 S_q^1 = S_q^5 S_q^1 \]
Already saw \( S_0^2 S_0^2 S_0^1 = 0 \), so no class in dim 7.

or dim 8.

So this picture represents \( A(1)^5 \), at total dimension 8

w/ dof class in dimension 6.

See website for picture of \( A(2) \).

Remark: In general, no formula for smallest \( K \) such

that \((S_0^n)^K = 0\). Known in some cases:

- \( (S_q^m)^{2m+2} = 0 \), \( (S_q^m)^{2m+1} \neq 0 \)

- \( (S_{q^{-1}})^{m+1} = 0 \), \( (S_{q^{-1}})^m \neq 0 \)

- \( S_q^{2m-2} \) not known