Steenrod Algebra Seminar:
Construction of Steenrod Operations

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We will be discussing cohomology operations for cohomology with coefficients in $\mathbb{F}_p$, for $p$ a prime. A cohomology operation of degree $i$ is a natural transformation

$$H^n(X; \mathbb{F}_p) \to H^{n+i}(X; \mathbb{F}_p).$$

An obvious example is the identity transformation, of degree 0. Another important example is the $p$th power map $x \mapsto x^p$. Note that this is an additive operation since $(x+y)^p \equiv x^p + y^p \pmod{p}$. Yet another example is the Bockstein homomorphism

$$\beta : H^n(X; \mathbb{F}_p) \to H^{n+1}(X; \mathbb{F}_p).$$

This is just the connecting homomorphism in the long exact sequence in cohomology arising from the short exact sequence in coefficients

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

1. Basic properties

We will focus on the so-called Steenrod operations. They look a little different depending on whether $p$ is odd or 2, so we list the properties in the two cases separately.

**Theorem.** Let $p$ be odd. There are cohomology operations

$$P^i : H^n(X; \mathbb{F}_p) \to H^{n+2i(p-1)}(X; \mathbb{F}_p)$$

of degree $2i(p-1)$ satisfying the following properties.

1. $P^0$ is the identity
2. If $n = 2i$, then $P^i(x) = x^p$
3. (Instability) If $n < 2i + \varepsilon$, then $\beta^\varepsilon P^i(x) = 0$ for $\varepsilon \in \{0,1\}$.
4. (Stability) $P^i(\Sigma x) = \Sigma P^i(x)$
5. (Cartan formula) $P^k(xy) = \sum_{i+j=k} P^i(x)P^j(y)$ and $\beta(xy) = \beta(x)y + (-1)^{|x|}x\beta(y)$
6. (Adem relations) If \( \ell < pk \) then

\[
P^\ell P^k = \sum_{i=0}^{\lfloor \ell/p \rfloor} (-1)^{\ell+i} \binom{p-1}{k-i-1} \frac{\ell}{\ell - pi} P^{\ell+k-i} P^i,
\]

and if \( \ell < pk + 1 \) then

\[
P^\ell \beta P^k = \sum_{i=0}^{\lfloor \ell/p \rfloor} (-1)^{\ell+i} \binom{p-1}{k-i} \frac{\ell}{\ell - pi} \beta P^{\ell+k-i} P^i + \sum_{i=0}^{\lfloor (\ell-1)/p \rfloor} (-1)^{\ell+i-1} \binom{p-1}{k-i-1} \frac{\ell}{\ell - pi - 1} P^{\ell+k-i} \beta P^i,
\]

For the prime \( p = 2 \), we have operations

\[
\text{Sq}^i : H^n(X; \mathbb{F}_2) \longrightarrow H^{n+i}(X; \mathbb{F}_2)
\]
satisfying analogous conditions, with the following changes.

1. We add that \( \text{Sq}^1 = \beta \)
2. \( \text{Sq}^n(x) = x^2 \) (for any \( n \))
3. \( \text{Sq}^i(x) = 0 \) if \( i > n \)
4. The Cartan formula reads \( \text{Sq}^k(xy) = \sum_{i+j=k} \text{Sq}^i(x) \text{Sq}^j(y) \)
5. The Adem relations are that, for \( \ell < 2k \),

\[
\text{Sq}^\ell \text{Sq}^k = \sum_{i=0}^{\lfloor \ell/2 \rfloor} \binom{k-i-1}{\ell - 2i} \text{Sq}^{\ell+k-i} \text{Sq}^i.
\]

2. Outline

The homotopical construction of Steenrod operations involves two main steps:

(i) Construct the external reduced power operations

\[
P : H^{2n}(X; \mathbb{F}_p) \to H^{2np}(X \times B\Sigma_p; \mathbb{F}_p).
\]

(ii) Compute \( H^*(X \times B\Sigma_p; \mathbb{F}_p) \) in terms of \( H^*(X; \mathbb{F}_p) \).

We discuss step (ii) today, and we will handle (i), and show how (i) and (ii) give the operations, next time.
3. The cohomology of $B\Sigma_p$

Let $G$ be a finite group, and suppose given a subgroup $H \leq G$. Then, using suitable models for $BH$ and $BG$, there is a covering map $BH \to BG$ of degree $|H : G|$. In cohomology, any covering admits a "transfer map" $\tau : H^*(BH) \to H^*(BG)$ such that the composition

$$H^*(BG) \to H^*(BH) \xrightarrow{\tau} H^*(BG)$$

is multiplication by the degree of the covering. In particular, if $H$ is a $p$-Sylow subgroup of $G$ and we are working with $\mathbb{F}_p$ as our coefficients, the composition is an isomorphism, so that $H^*(BG)$ maps injectively into $H^*(BH)$. In fact, one can identify the image. Let $N_G(H)$ be the normalizer of $H$ in $G$. This acts on $H^*(BH)$ and also on the cohomology ring $H^*(BH)$. The inner conjugation action of $H$ on $BH$ is null-homotopic, so the quotient Weyl group $W_G(H) := N_G(H)/H$ acts on $H^*(BH)$, and it turns out that

$$H^*(BG) = H^*(BH)^{W_G(H)}.$$

Specializing now to $G = \Sigma_p$, the Sylow $p$-subgroup is $C_p$. We know that

$$H^*(BC_p; \mathbb{F}_p) \cong \mathbb{F}_p[u, v]/(u^2 = 0, \beta(u) = v), \quad |u| = 1, |v| = 2.$$

Recall that the units $\mathbb{F}_p^\times$ is a cyclic group $C_{p-1}$. Let $j \in \mathbb{F}_p$ be a generator for $\mathbb{F}_p^\times$. Then multiplication by $j$ in $\mathbb{F}_p$ corresponds to an element $\lambda \in \Sigma_p$ of order $p - 1$, and let $\sigma \in \Sigma_p$ correspond to addition by 1 in $\mathbb{F}_p$. Then $\sigma$ generates a Sylow subgroup $C_p$, and it turns out that $N_{\Sigma_p}(C_p) = \langle \sigma, \lambda \rangle$. It follows that $W_{\Sigma_p}(C_p) \cong C_{p-1}$. The generator $\lambda C_p$ of $W_{\Sigma_p}(C_p)$ acts on $\mathbb{F}_p$ as the element $\lambda$, and it follows that the induced action on the class $u$ is multiplication by $j$. Since $v = \beta(u)$, the same is true for $v$. More generally, the action on a class $u^i v^j$ is by $j^i \epsilon^j$. So a class in $H^*(BC_p)$ will be fixed by $W_{\Sigma_p}(C_p)$ if and only if it is a scalar multiple of either $v^{(p-1)m}$ or $uv^{(p-1)m-1}$ for some $m$. It follows that

$$H^*(B\Sigma_p; \mathbb{F}_p) \cong \mathbb{F}_p[w, z]/(w^2 = 0, \beta(w) = z), \quad |w| = 2(p - 1) - 1, |z| = 2(p - 1).$$