Last time, we began the discussion of the construction of Steenrod operations. Our first main focus today will be the construction of the “External reduced power operation”

\[ P : H^{2n}(X; \mathbb{F}_p) \rightarrow H^{2np}(X \times B\Sigma_p; \mathbb{F}_p). \]

1. Review of homotopy orbits

In order to define the external reduced power map, we will need to discuss the homotopy orbit construction.

Let \( G \) be a finite group. Recall that we write \( BG \) for a space \( K(G, 1) \). We write \( EG \) for a universal cover of \( BG \). Then \( EG \) is contractible, and \( G \) acts freely (through deck transformations) on \( EG \). More generally, if \( W \) is any \( G \)-space which is contractible and on which \( G \) acts freely, one tends to write \( EG \) for \( W \), and the orbit space \( W/G \) can be seen to be a \( K(G, 1) \), so that we may write \( BG = W/G \).

If \( Y \) is any \( G \)-space, we can think of \( G \) as acting ”diagonally” on \( EG \times Y \). That is, \( g \cdot (w, y) = (g \cdot w, g \cdot y) \). We write \( EG \times_G Y \) for the quotient by the \( G \)-action. This is sometimes called the Borel construction on \( Y \) or the homotopy orbit space (and written \( Y_{hG} \)). This has the feature that if \( Y \rightarrow Z \) is a \( G \)-equivariant map that is also a weak equivalence, then the induced map \( Y_{hG} \rightarrow Z_{hG} \) is also a weak equivalence.

Note: If we regard \( EG \) as a space with \( H \)-action, then it is a contractible space with a free \( H \)-action, so \( EG \) is a model for \( EH \) too. In particular, there is a natural quotient map \( EG/H \rightarrow EG/G \) that is a model for \( BG \rightarrow BG \). This is the model we had in mind in the last talk.

1.1. The external reduced power map

Assume given a factorization in the following diagram (all coefficients are assumed to be \( \mathbb{F}_p \))

\[
\begin{array}{c}
H^{2n}(X) \\
\Phi \\
\downarrow \\
H^{2np}(X \times \Sigma_p \times \mathbb{F}_p) \\
\Phi \\
\downarrow \\
H^{2pn}(X^p) \cong H^{2pn}(E \Sigma_p \times X^p)
\end{array}
\]
The diagonal arrow is the $p$th power map, and the vertical arrow is induced by the quotient map. The diagonal map $X \xrightarrow{\Delta} X^p$ is $\Sigma_p$ equivariant and so gives rise to a map

$$E\Sigma_p \times \Sigma_p X \cong B\Sigma_p \times X \longrightarrow E\Sigma_p \times \Sigma_p X^p.$$

Composing the induced map in cohomology with the map $\Phi$ above produces the external reduced power map

$$\text{H}^{2n}(X) \xrightarrow{\Phi} \text{H}^{2pn}(E\Sigma_p \times \Sigma_p X^p) \longrightarrow \text{H}^{2pn}(B\Sigma_p \times X).$$

It remains to define the map $\Phi$. Note that the Yoneda lemma implies that it suffices to do this in the case $X = K(F_p, 2n)$, in which case we are looking for a particular map

$$E\Sigma_p \times \Sigma_p K(F_p, 2n)^p \longrightarrow K(F_p, 2pm).$$

The existence of this map is a strengthening of the statement that the multiplication of (even-dimensional) classes is homotopy commutative. It now suffices to find a free $\Sigma_p$-space $W$ that is contractible and a $\Sigma_p$-equivariant map

$$W \times K(F_p, 2n)^p \longrightarrow K(F_p, 2pm)$$

where $\Sigma_p$ acts trivially on the right.

Define a new space $S$ ($S$ stands for Segal) by

$$S = \prod_{n} K(F_p, 2n).$$

It will be important for the following that we take a model $\tilde{F}_{p}(S^{2n})$ for $K(F_p, 2n)$ that is a topological abelian group (really topological $F_p$-vector space). Then $S$ becomes a graded topological $F_p$-vector space. We then define, for each $j \geq 0$, a $\Sigma_p$-space $M(j) \subseteq \text{Map}(S^j, S)$ as the space of multilinear graded maps $S^j \longrightarrow S$. Finally, we define $\mathcal{O}(j) \subseteq M(j)$ to be the component of the cup product.

Evidently this space splits up as a product of spaces $\mathcal{O}(j)[n_1, \ldots, n_j]$ parametrizing multilinear maps

$$K(F_p, 2n_1) \times \cdots \times K(F_p, 2n_j) \rightarrow K(F_p, 2(n_1 + \cdots + n_j))$$

inducing the cup product.

**Claim:** $\mathcal{O}(j)$ is contractible. It suffices to show that any $\mathcal{O}(j)[n_1, \ldots, n_j]$ is contractible, though for simplicity of notation we will only consider $\mathcal{O}[n, \ldots, n]$. Indeed, multilinear maps

$$K(F_p, 2n)^j \rightarrow K(F_p, 2jn)$$

correspond to linear maps

$$K(F_p, 2n)^{\otimes j} \rightarrow K(F_p, 2jn).$$

But note that

$$K(F_p, 2n)^{\otimes j} = \tilde{F}_{p}(S^{2n})^{\otimes j} \cong \tilde{F}_{p}((S^{2n})^{\wedge j}) \cong \tilde{F}_{p}S^{2jn}.$$
Thus
\[
\text{Map}_{\text{multilin}}(K(\mathbb{F}_p, 2n)^j, K(\mathbb{F}_p, 2jn)) \cong \text{Map}_{\text{TopVect}}(K(\mathbb{F}_p, 2n)^\otimes j, K(\mathbb{F}_p, 2jn)) \\
\cong \text{Map}_{\text{TopVect}}(K(\mathbb{F}_p, 2jn), K(\mathbb{F}_p, 2jn)) \\
\cong \text{Map}_*(S^{2jn}, K(\mathbb{F}_p, 2jn)) \cong \mathbb{F}_p
\]

Our space \(O[n, \ldots, n]\) is clearly a component of this mapping space and is therefore contractible.

**Lemma (Kozlowski).** The group \(\Sigma_j\) acts freely on \(O(j)\).

Thus the space \(O(p)\) is a model for \(E\Sigma_p\). Since \(O(p)\) is a subspace of \(\text{Map}_*(S^p, S)\), there is a natural \(\Sigma_p\)-equivariant map
\[
O(p) \times S^p \rightarrow S.
\]
By construction, it restricts to give an equivariant map
\[
O(p) \times K(\mathbb{F}_p, 2n)^p \rightarrow K(\mathbb{F}_p, 2pn).
\]
and we get the desired map.

2. **Putting it all together**

Combining the computation from last time with the Kunneth isomorphism gives the computation
\[
H^*(B\Sigma_p \times X; \mathbb{F}_p) \cong H^*(X; \mathbb{F}_p)p[w, z]/(w^2 = 0, \beta(w) = z), \quad |w| = 2(p-1)-1, |z| = 2(p-1).
\]
Then if \(P\) denotes the external reduced power operation
\[
H^{2n}(X) \xrightarrow{P} H^{2pn}(B\Sigma_p \times X),
\]
we can express the class \(P(x)\) as a polynomial in the classes \(w\) and \(z\). We define classes \(P^i(x)\) and \(B^i(x)\) as the coefficients:
\[
P(x) = P^n(x) + B^{n-1}(x)w + P^{n-1}(x)z + B^{n-2}(x)wz + P^{n-2}(x)z^2 + \ldots \\
\quad + B^1(x)wz^{n-2} + P^1(x)z^{n-1} + B^0(x)wz^{n-1} + P^0(x)z^n.
\]
Technically, the above only defines the reduced power on even dimensional classes. For \(y \in H^{2n+1}(X)\), we may suspend to get an even dimensional class \(\Sigma y \in H^{2(n+1)}(\Sigma X)\). Then \(P^i(\Sigma y)\) is a well-defined class in \(H^{2(n+1)+2i(p-1)}(\Sigma X)\), corresponding to a well-defined class in \(H^{2n+1+2i(p-1)}(X)\). We define \(P^i(y)\) to be this class.

Note that from the above construction it is fairly easy to see that \(P^n(x) = x^p\) if \(|x| = 2n\), but it is nontrivial to check that \(P^0(x) = x\). If one first shows that \(\beta P = 0\), it is then also easy to see, using that \(\beta(w) = z\), that the class \(B^i(x)\) is none other than \(\beta P^i(x)\); in particular, \(B^0(x) = \beta(x)\). Moreover, by definition, there are no operations \(P^i\) defined on \(x\) if \(i > n\).