

RIEMANNIAN GEOMETRY

RICHARD L. BISHOP

Preface

These lecture notes are based on the course in Riemannian geometry at the University of Illinois over a period of many years. The material derives from the course at MIT developed by Professors Warren Ambrose and I M Singer and then reformulated in the book by Richard J. Crittenden and me, “Geometry of Manifolds”, Academic Press, 1964. That book was reprinted in 2000 in the AMS Chelsea series. These notes omit the parts on differentiable manifolds, Lie groups, and isometric imbeddings. The notes are not intended to be for individual self study; instead they depend heavily on an instructor’s guidance and the use of numerous problems with a wide range of difficulty.

The geometric structure in this treatment emphasizes the use of the bundles of bases and frames and avoids the arbitrary coordinate expressions as much as possible. However, I have added some material of historical interest, the Taylor expansion of the metric in normal coordinates which goes back to Riemann. The curvature tensor was probably discovered by its appearance in this expansion.

There are some differences in names which I believe are a substantial improvement over the fashionable ones. What is usually called a “distribution” is called a “tangent subbundle”, or “subbundle” for short. The name “solder form” never made much sense to me and is now labeled the descriptive term “universal cobasis”. The terms “first Bianchi identity” and “second Bianchi identity” are historically inaccurate and are replaced by “cyclic curvature identity” and “Bianchi identity” – Bianchi was too young to have anything to do with the first, and even labeling the second with his name is questionable since it appears in a book by him but was discovered by someone else (Ricci?).

The original proof of my Volume Theorem used Jacobi field comparisons and is not reproduced. Another informative approach is to use comparison theory for matrix Riccati equations and a discussion of how that works is included and used to prove the Rauch Comparison Theorem.

In July, 2013, I went through the whole file, correcting many typos, making minor additions, and, most importantly, redoing the index using the Latex option for that purpose.

Richard L. Bishop
University of Illinois at Urbana-Champaign
July, 2013.

CONTENTS

1. Riemannian metrics	3
1.1. Finsler Metrics	7
1.2. Length of Curves	7

1.3.	Distance	7
1.4.	Length of a curve in a metric space	8
2.	Minimizers	9
2.1.	Existence of Minimizers	9
2.2.	Products	10
3.	Connections	10
3.1.	Covariant Derivatives	10
3.2.	Pointwise in X Property	11
3.3.	Localization	11
3.4.	Basis Calculations	11
3.5.	Parallel Translation	11
3.6.	Existence of Connections	11
3.7.	An Affine Space	12
3.8.	The Induced Connection on a Curve	12
3.9.	Parallelizations	12
3.10.	Torsion	12
3.11.	Curvature	13
3.12.	The Bundle of Bases	14
3.13.	The Right Action of the General Linear Group	14
3.14.	The Universal Dual 1-Forms	14
3.15.	The Vertical Subbundle	15
3.16.	Connections	15
3.17.	Horizontal Lifts	15
3.18.	The Fundamental Vector Fields	15
3.19.	The Connection Forms	16
3.20.	The Basic Vector Fields	16
3.21.	The Parallelizability of BM	16
3.22.	Relation Between the Two Definitions of Connection	17
3.23.	The Dual Formulation	18
3.24.	Geodesics	18
3.25.	The Interpretation of Torsion and Curvature in Terms of Geodesics	19
3.26.	Development of Curves into the Tangent Space	21
3.27.	Reverse Developments and Completeness	21
3.28.	The Exponential Map of a Connection	22
3.29.	Normal Coordinates	22
3.30.	Parallel Translation and Covariant Derivatives of Other Tensors	22
4.	The Riemannian Connection	23
4.1.	Metric Connections	23
4.2.	Orthogonal Groups	23
4.3.	Existence of Metric Connections	23
4.4.	The Levi-Civita Connection	24
4.5.	Isometries	25
4.6.	Induced semi-Riemannian metrics	26
4.7.	Infinitesimal isometries–Killing fields	28
5.	Calculus of variations	30
5.1.	Variations of Curves–Smooth Rectangles	30
5.2.	Existence of Smooth Rectangles, given the Variation Field	31
5.3.	Length and Energy	31

6. Riemannian curvature	35
6.1. Curvature Symmetries	35
6.2. Covariant Differentials	36
6.3. Exterior Covariant Derivatives	36
6.4. Sectional Curvature	37
6.5. The Space of Pointwise Curvature Tensors	38
6.6. The Ricci Tensor	38
6.7. Ricci Curvature	39
6.8. Scalar Curvature	39
6.9. Decomposition of \mathcal{R}	40
6.10. Normal Coordinate Taylor Series	41
6.11. The Christoffel Symbols	41
7. Conjugate Points	44
7.1. Second Variation	45
7.2. Loops and Closed Geodesics	49
8. Completeness	51
8.1. Cut points	54
9. Curvature and topology	54
9.1. Hadamard manifolds	54
9.2. Comparison Theorems	56
9.3. Reduction to a Scalar Equation	58
9.4. Comparisons for Scalar Riccati Equations	58
9.5. Volume Comparisons	61
Index	62

1. RIEMANNIAN METRICS

Riemannian geometry considers manifolds with the additional structure of a Riemannian metric, a type $(0, 2)$ positive definite symmetric tensor field. To a first order approximation this means that a Riemannian manifold is a Euclidean space: we can measure lengths of vectors and angles between them. Immediately we can define the length of curves by the usual integral, and then the distance between points comes from the glb of lengths of curves. Thus, Riemannian manifolds become metric spaces in the topological sense.

Riemannian geometry is a subject of current mathematical research in itself. We will try to give some of the flavor of the questions being considered now and then in these notes. A Riemannian structure is also frequently used as a tool for the study of other properties of manifolds. For example, if we are given a second order linear elliptic partial differential operator, then the second-order coefficients of that operator form a Riemannian metric and it is natural to ask what the general properties of that metric tell us about the partial differential equations involving that operator. Conversely, on a Riemannian manifold there is singled out naturally such an operator (the Laplace-Beltrami operator), so that it makes sense, for example, to talk about solving the heat equation on a Riemannian manifold. The Riemannian metrics have nice properties not shared by just any topological metrics, so that in topological studies they are also used as a tool for the study of manifolds.

The generalization of Riemannian geometry to the case where the metric is not assumed to be positive definite, but merely nondegenerate, forms the basis for general relativity theory. We will not go very far in that direction, but some of the major theorems and concepts are identical in the generalization. We will be careful to point out which theorems we can prove in this more general setting. For a deeper study there is a fine book: O'Neill, *Semi-Riemannian geometry*, Academic Press, 1983. I recommend this book also for its concise summary of the theory of manifolds, tensors, and Riemannian geometry itself.

The first substantial question we take up is the existence of Riemannian metrics. It is interesting that we can immediately use Riemannian metrics as a tool to shed some light on the existence of semi-Riemannian metrics of nontrivial index.

Theorem 1.1 (Existence of Riemannian metrics). *On any smooth manifold there exists a Riemannian metric.*

The key idea of the proof is that locally we always have Riemannian metrics carried over from the standard one on Cartesian space by coordinate mappings, and we can glue them together smoothly with a partition of unity. In the gluing process the property of being positive definite is preserved due to the convexity of the set of positive definite symmetric matrices. What happens for indefinite metrics? The set of nonsingular symmetric matrices of a given index is *not* convex, so that the existence proof breaks down. In fact, there is a condition on the manifold, which can be reduced to topological invariants, in order that a semi-Riemannian metric of index ν exist: there must be a subbundle of the tangent bundle of rank ν . When $\nu = 1$ the structure is called a *Lorentz* structure; that is the case of interest in general relativity theory; the topological condition for a compact manifold to have a Lorentz structure is easily understood: the Euler characteristic must be 0.

The proof in the Lorentz case can be done by using the fact that for any simple curve there is a diffeomorphism which is the identity outside any given neighborhood of the curve and which moves one end of the curve to the other. In the compact case start with a vector field having discrete singularities. Then by choosing disjoint simple curves from these singularities to inside a fixed ball, we can obtain a diffeomorphism which moves all of them inside that ball. If the Euler characteristic is 0, then by the Hopf index theorem, the index of the vector field on the boundary of the ball is 0, so the vector field can be extended to a nonsingular vector field inside the ball.

Conversely, by the following Theorem 1.2, a Lorentz metric would give a nonsingular rank 1 subbundle. If that field is nonorientable, pass to the double covering for which the lift of it is orientable. Then there is a nonsingular vector field which is a global basis of the line field, so the Euler characteristic is 0.

In the noncompact case, take a countable exhaustion of the manifold by an increasing family of compact sets. Then the singularities of a vector field can be pushed outside each of the compact sets sequentially, leaving a nonsingular vector field on the whole in the limit. Thus, every noncompact (separable) manifold has a Lorentz structure.

Theorem 1.2 (Existence of semi-Riemannian metrics). *A smooth manifold has a semi-Riemannian metric of index ν if and only if there is a subbundle of the tangent bundle of rank ν .*

The idea of the proof is: the subbundle will be the directions in which the semi-Riemannian metrics will be negative definite. If we change the sign on the subbundle and leave it unchanged on the orthogonal complement, we will get a Riemannian metric. The construction goes both ways.

Although the idea for the proof of Theorem 1.2 is correct, there are some non-trivial technical difficulties to entertain us. One direction is relatively easy.

Proof of “if” part If M has a smooth tangent subbundle V of rank ν , then M has a semi-Riemannian metric of index ν .

Definition 1.3. A *frame* at a point p of a semi-Riemannian manifold is a basis of the tangent space M_p with respect to which the component matrix of the metric tensor is diagonal with -1 's followed by 1 's on the diagonal. A *local frame field* is a local basis of vector fields which is a frame at each point of its domain.

Lemma 1.4 (Technical Lemma 1). *Local frames exist in a neighborhood of every point.*

TL 1 is important for other purposes than the proof of the theorem at hand. For the proof of TL 1 one modifies the Gramm-Schmidt procedure, starting with a smooth local basis and shrinking the domain at each step if necessary to divide by the length for normalization.

Remark 1.5. If we write $g = g_{ij}\omega^i\omega^j$, then a local frame is exactly one for which the coframe of 1-forms (ϵ^i) satisfies

$$g = -(\epsilon^1)^2 - \dots - (\epsilon^\nu)^2 + \dots + (\epsilon^n)^2.$$

The Gramm-Schmidt procedure amounts to iterated completion of squares, viewing g as a homogeneous quadratic polynomial in the ω^i . The modifications needed to handle the negative signs are probably easier in this form.

Lemma 1.6 (Technical Lemma 2). *If V is a smooth tangent subbundle of rank ν and g is a Riemannian metric, then the $(1, 1)$ tensor field A and the semi-Riemannian metric h given as follows are smooth:*

$$(1) \quad A = \begin{cases} -1 & \text{on } V \\ 1 & \text{on } V^\perp. \end{cases}$$

$$h(v, w) = g(Av, w).$$

(Their expressions in terms of smooth local frames *adapted to* V for g are *constant*, hence smooth.)

Now the converse.

Proof of “only if” part If there is a semi-Riemannian metric h of index ν , then there is a tangent subbundle V of rank ν .

The outline of the proof goes as follows. Take a Riemannian metric g . Then h and g are related by a $(1, 1)$ tensor field A as above. We know that A is symmetric with respect to g -frames, and has ν negative eigenvalues (counting multiplicities) at each point. Thus, the subspace spanned by the eigenvectors of these negative eigenvalues at each point is ν -dimensional. The claim is that those subspaces form

a smooth subbundle, even though it may be impossible to choose the individual eigenvectors to form smooth vector fields.

Lemma 1.7 (Technical Lemma 3). *If $A : V \rightarrow V$ is a symmetric linear operator, smoothly dependent on coordinates x^1, \dots, x^n , and λ_0 is a simple eigenvalue at the origin, then λ_0 extends to a smooth simple eigenvalue function in a neighborhood of the origin having a smooth eigenvector field.*

Let $P(X) = \det(XI - A)$. Use the implicit function theorem to solve $P(X) = 0$, getting $X = \lambda(x^1, \dots, x^n)$. Then we can write $P(X) = (X - \lambda)Q(X)$, and any nonzero column of $Q(A)$ is an eigenvector, by the Cayley-Hamilton theorem.

Lemma 1.8 (Technical Lemma 4). *Suppose that $W : \mathbf{R}^n \rightarrow \bigwedge^\nu V$ is a smooth function with decomposable, nonzero values. Then locally there are smooth vector fields having wedge product equal to W .*

For $\omega \in \bigwedge^{\nu-1} V^*$, the interior product $i(\omega)W$ is always in the subspace carried by W . Choose ν of these ω 's which give linearly independent interior products with W at one point; then they do so locally.

Lemma 1.9 (Technical Lemma 5). *If $A : V \rightarrow V$ is a symmetric linear operator of index ν , smoothly dependent on coordinates x^i , then the extension of A to a derivation of the Grassmann algebra $\bigwedge^* V \rightarrow \bigwedge^* V$ has a unique minimum simple eigenvalue $\lambda_1 + \dots + \lambda_\nu$ on $\bigwedge^\nu V$. The (smooth!) eigenvectors $W : \mathbf{R}^n \rightarrow \bigwedge^\nu V$ are decomposable.*

Problem 1.10. Generalize the result of TL's 3, 4, 5: If there is a group of ν eigenvalues $\lambda_1, \dots, \lambda_\nu$ of $A : V \rightarrow V$ which always satisfy $a < \lambda_i < b$, then the subspace spanned by their eigenvectors is smooth.

(Consider $B = (aI - A)(bI - A)$. Can a, b be continuous functions of x^1, \dots, x^n too?)

Problem 1.11. Now order *all* of the eigenvalues of symmetric smooth $A : V \rightarrow V$, $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, defining uniquely n functions λ_i of x^1, \dots, x^n . Prove that the λ_i are continuous; on the subset where $\lambda_{i-1} < \lambda_i < \lambda_{i+1}$, λ_i is smooth and has locally smooth eigenvector fields.

Problem 1.12. Construct an example

$$A = \begin{pmatrix} f(x, y) & g(x, y) \\ g(x, y) & h(x, y) \end{pmatrix}$$

for which $\lambda_1(0, 0) = \lambda_2(0, 0) = 1$ and neither λ_1, λ_2 nor their eigenvector fields are smooth in a neighborhood of $(0, 0)$.

Remark 1.13. I have had to referee and reject two papers because the proofs were based on the assumption that eigenvector fields could be chosen smoothly. Take care!

Hard Problem. In Problem 1.12, can it be arranged so that there is no smooth eigenvector field on the set where $\lambda_1 < \lambda_2$? If that set is simply connected, then there is a smooth vector field; and in any case there is always a smooth subbundle of rank 1. However, in the non-simply-connected case, the subbundle may be disoriented in passing around some loop.

1.1. Finsler Metrics. Let $L : TM \rightarrow \mathbf{R}$ be a continuous function which is positive on nonzero vectors and is *positive homogeneous of degree 1*:

$$\text{for } \alpha \in \mathbf{R}, v \in TM, \text{ we have } L(\alpha v) = |\alpha|L(v).$$

Eventually we should also assume that the *unit balls*, that is, the subsets of the tangent spaces on which $L \leq 1$, are convex, but that property is not required to give the initial facts we want to look at here. In fact, it is usually assumed that L^2 is smooth and that its restriction to each tangent space has positive definite second derivative matrix (the *Hessian* of L^2) with respect to linear coordinates on that tangent space. This implies that the unit balls are strictly convex and smooth.

A function L satisfying these conditions is called a *Finsler metric* on M . If g is a Riemannian metric on M , then there is a corresponding Finsler metric, given by the norm with respect to g : $L(v) = \sqrt{g(v, v)}$. We use the letter L because it is treated like a Lagrangian in mechanics.

Finsler metrics were systematically studied by P. Finsler starting about 1918.

1.2. Length of Curves. For a piecewise C^1 curve $\gamma : [a, b] \rightarrow M$ we define the *length* of γ to be

$$|\gamma| = \int_a^b L(\gamma'(t)) dt.$$

The reason for assuming that L is positive homogeneous is that it makes the length of a curve independent of its parametrization. This follows easily from the change of variable formula for integrals.

Length is clearly additive with respect to chaining of curves end-to-end. Not every Finsler metric comes from a Riemannian metric. The condition for that to be true is the *parallelogram law*, well-known to functional analysts:

Theorem 1.14 (Characterization of Riemannian Finsler Metrics). *A Finsler metric L is the Finsler metric of a Riemannian metric g if and only if it satisfies the parallelogram law:*

$$L^2(v + w) + L^2(v - w) = 2L^2(v) + 2L^2(w),$$

for all tangent vectors v, w at all points of M . When this law is satisfied, the Riemannian metric can be recovered from L by the polarization identity:

$$2g(v, w) = L^2(v + w) - L^2(v) - L^2(w).$$

1.3. Distance. When we have a notion of lengths of curves satisfying the additivity with respect to chaining curves end-to-end, as we do when we have a Finsler metric, then we can define the *intrinsic metric* (the word metric is used here as it is in topology, a distance function) by specifying the *distance from p to q* to be

$$d(p, q) = \inf\{|\gamma| : \gamma \text{ is a curve from } p \text{ to } q\}.$$

Generally this function is only a semi-metric, in that we could have $d(p, q) = 0$ but not $p = q$. The symmetry and the triangle inequality are rather easy consequences of the definition, but the nondegeneracy in the case of Finsler lengths of curves is nontrivial.

Theorem 1.15 (Topological Equivalence Theorem). *If d is the intrinsic metric coming from a Finsler metric, then d is a topological metric on M and the topology given by d is the same as the manifold topology.*

If M is not connected, then the definition gives $d(p, q) = \infty$ when p and q are not in the same connected components. We simply allow such a value for d ; it is a reasonable extension of the notion of a topological metric.

Lemma 1.16 (Nondegeneracy Lemma). *If $\varphi : U \rightarrow \mathbf{R}^n$ is a coordinate map, where U is a compact subset of M , then there are positive constant A, B such that for every curve γ in U ,*

$$|\gamma| \geq A|\varphi \circ \gamma|, \quad |\varphi \circ \gamma| \geq B|\gamma|.$$

Here $|\varphi \circ \gamma|$ is the standard Euclidean length of $\varphi \circ \gamma$.

The key step in the proof is to observe that the union EU of the unit (with respect to the Euclidean coordinate metric) spheres at points of U forms a compact subset of the tangent bundle TM . Since L is positive on nonzero vectors and continuous on TM , on EU we have a positive minimum A and a finite maximum $1/B$ for L on EU .

The result breaks down on infinite-dimensional manifolds modeled on a Banach space, because there the set of unit vectors EU will not be compact. So in that case the inequalities $L(v) \geq A\|v\|$, and $\|v\| \geq BL(v)$ must be taken as a local hypothesis on L .

The nondegeneracy of d uses the lemma in an obvious way, although there is a subtlety that could be overlooked: the nondegeneracy of the intrinsic metric on \mathbf{R}^n defined by the standard Euclidean metric must be proved independently.

Aside from the Hausdorff separation axiom, the topology of a manifold is determined as a local property by the coordinate maps on compact subsets U . Since the nondegeneracy lemma tells us that there is nesting of d -balls and coordinate-balls, the topologies must coincide.

Problem 1.17. Prove that the intrinsic metric on \mathbf{R}^n defined by the standard Euclidean metric is nondegenerate, and, in fact, coincides with the usual distance formula.

Hint: For a curve γ from p to q , split the tangent vector γ' into components parallel to \vec{pq} and perpendicular to \vec{pq} . The integral of the parallel component is always at least the usual distance.

1.4. Length of a curve in a metric space. If we have a topological metric space M with distance function d and a curve $\gamma : [a, b] \rightarrow M$, then the *length* of γ is

$$|\gamma| = \sup \sum d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all partitions of the interval $[a, b]$. Due to the triangle inequality, the insertion of another point into the partition does not decrease the sum, so that in particular the length of a curve from p to q is at least $d(p, q)$. If the length is finite, we call the curve *rectifiable*. It is obvious from the definition that the length of a curve is independent of its parametrization and satisfies the additivity property with respect to chaining curves. Following the definition of the length of a curve, we can then define the *intrinsic metric generated by d* , as we defined the intrinsic metric for a Finsler space. Clearly the intrinsic metric is at least as great as the metric d we started with.

2. MINIMIZERS

A curve whose length equals the intrinsic distance between its endpoints is called a *minimizer* or a *shortest path*. In Riemannian manifolds they are often called *geodesics*, but we will avoid that term for a while because there is another definition for geodesics. One of our major tasks will be to establish the relation between minimizers and geodesics. They will turn out to be not quite the same: geodesics turn out to be only *locally* minimizing, and furthermore, there are technical reasons why we should require that geodesics have a special parametrization, a constant-speed parametrization.

2.1. Existence of Minimizers. We can establish the existence of minimizers for a Finsler metric within a connected compact set by using convergence techniques developed by nineteenth century mathematicians to show the existence of solutions of ordinary differential equations by the Euler method. The main tool is the Arzelà-Ascoli Theorem. Cesare Arzelà (1847-1912) was a professor at Bologna, who established the case where the domain is a closed interval, and Giulio Ascoli (1843-1896) was a professor at Milan, who formulated the definition of uniform equicontinuity.

Definition 2.1. A collection of maps $\mathcal{F} = \{f : M \rightarrow N\}$ from a metric space M to a metric space N is *uniformly equicontinuous* if for every $\epsilon > 0$, there is $\delta > 0$ such that for every $f \in \mathcal{F}$ and every $x, y \in M$ such that $d(x, y) < \delta$ we have $d(f(x), f(y)) < \epsilon$.

The word “uniform” refers to the quantification over all x, y , just as in “uniform continuity”, while “equicontinuous” refers to the quantification over all members of the family. The definition is only significant for infinite families of functions, since a finite family of uniformly continuous functions is always also equicontinuous. Moreover, the application is usually to the case where M is compact, so that uniform continuity, but not uniform equicontinuity, is automatic. If M and N are subsets of Euclidean spaces and the members of the family have a uniform bound on their gradient vector lengths, then the family is uniformly equicontinuous. Even if those gradients exist only piecewise, this still works, which explains why the theorem below can be used to get existence of solutions by the Euler method.

Theorem 2.2 (Arzelà-Ascoli Theorem). *Let K, K' be compact metric space and assume that K has a countable dense subset. Let \mathcal{F} be a collection of continuous functions $K \rightarrow K'$. Then the following properties are equivalent:*

- (a) \mathcal{F} is uniformly equicontinuous.
- (b) Every sequence in \mathcal{F} has a subsequence which is uniformly convergent on K .

There is a proof for the case where K and K' are subsets of Euclidean spaces given in R. G. Bartle, “The Elements of Real Analysis”, 2nd Edition, Wiley, page 189. No essential changes are needed for this abstraction to metric spaces.

Theorem 2.3 (Local Existence of minimizers in Finsler Spaces). *If M is a Finsler manifold, each $p \in M$ has nested neighborhoods $U \subset V$ such that every pair of points in U can be joined by a minimizer which is contained in V .*

To start the proof one takes V to be a compact coordinate ball about p , and U a smaller coordinate ball so that, using the curve-length estimates from the

Nondegeneracy Lemma, any curve which starts and ends in U cannot go outside V unless it has greater length than the longest coordinate straight line in U . Now for two points q and r in U we define a family of curves \mathcal{F} parametrized on the unit interval $[0, 1] = I$ (so that K of the theorem will be I with the usual distance on the line). We require that the length of each curve to be no more than the Finsler length of the coordinate straight line from q to r . We parametrize each curve so that it has constant “speed”, which together with the uniform bound on length, gives the uniform equicontinuity of \mathcal{F} . We take K' to be the outer compact ball V .

By the definition of intrinsic distance $d(q, r)$ there exists a sequence of curves from q to r for which the lengths converge to the infimum of lengths. If the coordinate straight line is already minimizing, then we have our desired minimizer. Otherwise the lengths will be eventually less than that of the straight line, and from there on the sequence will be in \mathcal{F} . By the AA Theorem there must be a uniformly convergent subsequence. It is fairly easy to prove that the limit is a minimizer.

Remark 2.4. The minimizers do not have to be unique, even locally. For an example consider the l_1 or l_∞ norm on R^2 to define the Finsler metric (the “taxicab metric”). Generally the local uniqueness of minimizers can only be obtained by assuming that the unit tangent balls of the Finsler metric are strictly convex. We won’t do that part of the theory in such great generality, but we will obtain the local uniqueness in Riemannian manifolds by using differentiability and the calculus of variations.

The result on existence of minimizers locally can be abstracted a little more. Instead of using coordinate balls we can just assume that the space is locally compact. Thus, in a locally compact space with intrinsic metric there are always minimizers locally. The proof is essentially the same.

2.2. Products. If we have two metric spaces M and N , then the Cartesian product has a metric $d_{M \times N}$, whose square is $d_M^2 + d_N^2$. We use the same idea to get a Finsler metric or a Riemannian metric on the product when we are given those structures on the factors. When we pass to the intrinsic distance function of a Finsler product there is no surprise, the result is the product distance function.

Problem 2.5. Prove that the projection into each factor of a minimizer is a minimizer. Conversely, if we handle the parametrizations correctly, then the product of two minimizers is again a minimizer.

3. CONNECTIONS

We define an additional structure to a manifold structure, called a connection. It can be given either in terms of covariant derivative operators D_X or in terms of a horizontal tangent subbundle on the bundle of bases. Both ways are important, so we will give both and establish the way of going back and forth. Eventually our goal is to show that on a semi-Riemannian manifold (including the Riemannian case) there is a canonical connection called the Levi-Civita connection.

3.1. Covariant Derivatives. For a connection in terms of covariant derivatives we give axioms for the operation $(X, Y) \rightarrow D_X Y$, associating a vector field $D_X Y$ to a pair of vector fields (X, Y) .

1. If X and Y are C^k , then $D_X Y$ is C^{k-1} .

2. $D_X Y$ is \mathcal{F} -linear in X , for \mathcal{F} = real-valued functions on M . That is, $D_{fX} Y = fD_X Y$ and $D_{X+X'} Y = D_X Y + D_{X'} Y$.

3. D_X is a derivation with respect to multiplication by elements of \mathcal{F} . That is, $D_X(fY) = (Xf)Y + fD_X Y$ and $D_X(Y + Y') = D_X Y + D_X Y'$.

We say that $D_X Y$ is the *covariant derivative of Y with respect to X* . It should be thought of as an extension of the defining operation $f \rightarrow Xf$ of vector fields so as to operate on vector fields Y as well as functions f .

3.2. Pointwise in X Property. Axiom 2 conveys the information that for a vector $x \in M_p$, we can define $D_x Y \in M_p$. We extend x to a vector field X and prove, using 2, that $(D_X Y)(p)$ is the same for all such extensions X .

3.3. Localization. If Y and Y' coincide on an open set, then $D_X Y$ and $D_X Y'$ coincide on that open set.

3.4. Basis Calculations. If $(X_1, \dots, X_n) = B$ is a local basis of vector fields, we define n^2 1-forms φ_j^i locally by

$$D_X X_j = \sum_{i=1}^n \varphi_j^i(X) X_i.$$

These φ_j^i are called the *connection 1-forms with respect to the local basis B* . If we let $\omega^1, \dots, \omega^n$ be the dual basis to B of 1-forms, and arrange them in a column ω , and let $\varphi = (\varphi_j^i)$, then we can specify the connection locally in terms of φ by

$$D_X Y = B(X + \varphi(X))\omega(Y).$$

3.5. Parallel Translation. If $D_x Y = 0$, we say that Y is *infinitesimally parallel* in the direction x . If $\gamma : [a, b] \rightarrow M$ is a curve and $D_{\gamma'(t)} Y = 0$ for all $t \in [a, b]$, we say that Y is *parallel along γ* and that $Y(\gamma(b))$ is the *parallel translate* of $Y(\gamma(a))$ along γ .

Theorem 3.1 (Existence, Uniqueness, and Linearity of Parallel Translation). *For a given curve $\gamma : [a, b] \rightarrow M$ and a vector $y \in M_{\gamma(a)}$, there is a unique parallel translate of y along γ to $\gamma(b)$. This operation of parallel translation along $\gamma, M_{\gamma(a)} \rightarrow M_{\gamma(b)}$, is a linear transformation.*

If γ lies in a local basis neighborhood of B , then the coefficients $\omega(Y \circ \gamma)$ of Y along γ , when Y is parallel, satisfy a system of linear homogeneous differential equations:

$$\frac{d\omega(Y \circ \gamma)}{dt} = -\varphi(\gamma'(t))\omega(Y \circ \gamma).$$

Globally we chain together the local parallel translations to span all of γ in finitely many steps, using the fact that $\gamma([a, b])$ is compact.

3.6. Existence of Connections. It is trivial to check that for a local basis B we can set $\varphi = 0$ and obtain a connection in the local basis neighborhood. Then if $\{U_\alpha\}$ covers M , D^α is a connection on U_α , and $\{f_\alpha\}$ is a subordinate partition of unity (we do not even have to require that $0 \leq f_\alpha \leq 1$, only that the sum $\sum f_\alpha = 1$ be locally finite), then the definition

$$D_X Y = \sum_{\alpha} f_{\alpha} D_X^{\alpha} Y$$

defines a connection globally on M .

3.7. An Affine Space. If D and E are connections, then for any $f \in \mathcal{F}$ we have that $fD + (1 - f)E$ is again a connection. As f runs through constants this gives a straight line in the collection of all connections on M . Moreover, $S_X Y = D_X Y - E_X Y$ is \mathcal{F} -linear in Y , and so defines a (1,2) tensor field S such that $E = D + S$. Conversely, for any (1,2) tensor field S , $D + S$ is a connection. We interpret this to say:

The set of all connections on M is an affine space for which the associated vector space can be identified with the space of all (1,2) tensor fields on M .

3.8. The Induced Connection on a Curve. If γ is a curve in a manifold M which has a connection D , then we can define what we call the induced connection on the pullback of the tangent bundle to the curve. This is a means of differentiating vector fields Y along the curve with respect to the velocity of the curve. Thus, for each parameter value t , $Y(t) \in M_{\gamma(t)}$, and $D_{\gamma'} Y$ will be another field, like Y , along γ . There are various ways of formulating the definition, and there is even a general theory of pulling back connections along maps (see Bishop & Goldberg, pp.220–224), but there is a simple method for curves as follows. Take a tangent space basis at some point of γ and parallel translate this basis along γ to get a parallel basis field (E_1, \dots, E_n) for vector fields along γ . Then we can express Y uniquely in terms of this basis field, $Y = \sum f^i E_i$, where the coefficient functions f^i are smooth functions of t . We define $D_{\gamma'} Y = \sum f^{i'} E_i$. From the viewpoint of the theory of pulling back connections it would be more appropriate to write $D_{d/dt} Y$ instead of $D_{\gamma'}$. In fact, the Leibnitz rule for this connection on γ is

$$D_{\gamma'} fY = \frac{df}{dt} Y + f D_{\gamma'} Y,$$

and we also have the strange result that even if $\gamma'(t) = 0$ it is possible to have $(D_{\gamma'} Y)(t) \neq 0$. Even a field on a constant curve (which is just a curve in the tangent space of the constant value of the curve) can have nonzero covariant derivative along the curve.

3.9. Parallelizations. A manifold M is called *parallelizable* if the tangent bundle TM is trivial as a vector bundle: $TM \cong M \times \mathbf{R}^n$. For a given trivialization (\cong) the vector fields X_1, \dots, X_n which correspond to the standard unit vectors in \mathbf{R}^n are called the corresponding *parallelization* of M . Conversely, a global basis of vector fields gives a trivialization of TM . If we set $\varphi = 0$, we get the *connection of the parallelization*. For this connection parallel translation depends only on the ends of the curve. The parallel fields are $\sum a^i X_i$, a^i constants.

An even-dimensional sphere is *not* parallelizable.

Lie groups are parallelized by a basis of the left-invariant vector fields.

We shall see that for any manifold M its bundle of bases BM and various frame bundles FM are parallelizable.

3.10. Torsion. If D is a connection, then

$$T(X, Y) = D_X Y - D_Y X - [X, Y]$$

defines an \mathcal{F} -linear, skew-symmetric function of pairs of vector fields, called the *torsion* of D . Hence T is a (1,2) tensor field. For a local basis B with dual ω and

connection form φ , the torsion $T(X, Y)$ has B -components

$$\Omega(X, Y) = d\omega(X, Y) + \varphi \wedge \omega(X, Y).$$

The \mathbf{R}^n -valued 2-form Ω is called the *local torsion 2-form* and its defining equation

$$\Omega = d\omega + \varphi \wedge \omega$$

is called the *first structural equation*.

A connection with $T = 0$ is said to be *symmetric*.

For any connection D , the connection $E = D - \frac{1}{2}T$ is always symmetric.

3.11. Curvature. For vector fields X, Y we define the *curvature operator* $R_{XY} : Z \rightarrow R_{XY}Z$, mapping a third vector field Z to a fourth one $R_{XY}Z$ by

$$R_{XY} = D_{[X, Y]} - D_X D_Y + D_Y D_X.$$

(Some authors define this to be $-R_{XY}$.) As a function of three vector fields $R_{XY}Z$ is \mathcal{F} -linear and so defines a $(1, 3)$ tensor field. However, the interpretation as a 2-form whose values are linear operators on the tangent space is the meaningful viewpoint. For a local basis we have *local curvature 2-forms*, with values which are $n \times n$ matrices:

$$\Phi = d\varphi + \varphi \wedge \varphi,$$

which is the *second structural equation*. The sign has been switched, so that the matrix of R_{XY} with respect to the basis B is $-\Phi(X, Y)$. The wedge product is a combination of matrix multiplication and wedge product of the matrix entries, just as it was in the first structural equation, so that $(\varphi \wedge \varphi)(X, Y) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X)$.

Problem 3.2. Calculate the torsion tensor for the connection of a parallelization, relating it to the brackets of the basis fields.

Problem 3.3. Calculate the law of change of a connection: that is, if we have a local basis B and its connection form φ and another local basis $B' = BF$ and its connection form φ' , find the expression for φ' on the overlapping part of the local basis neighborhoods in terms of φ and F . In the case of coordinate local bases the matrix of functions F is a Jacobian matrix.

Problem 3.4. Check that the axioms for a connection are satisfied for the connection specified by the partition of unity and local connections, in the proof of existence of connections.

Problem 3.5. Verify that the definition of $T(X, Y)$ leads to the local expression for Ω given by the first structural equation; that is, T and Ω are assumed to be related by $T = B\Omega$.

3.12. The Bundle of Bases. We let

$$BM = \{(p, x_1, \dots, x_n) : p \in M, (x_1, \dots, x_n) \text{ is a basis of } M_p\}.$$

This is called the *bundle of bases of M* , and we make it into a manifold of dimension $n + n^2$ as follows. Locally it will be a product manifold of a neighborhood U of a local basis (X_1, \dots, X_n) with the general linear group $Gl(n, \mathbf{R})$ consisting of all $n \times n$ nonsingular matrices. Since the condition of nonsingularity is given by requiring the continuous function *determinant* to be nonzero, $Gl(n, \mathbf{R})$ can be viewed as an open set in \mathbf{R}^{n^2} , so that it gets a manifold structure from the single coordinate map. Then if $p \in U$ and $g = (g_i^j) \in Gl(n, \mathbf{R})$, we make the element (p, g) of the product correspond to the basis $(p, \sum_j X_j(p)g_1^j, \dots, \sum_j X_j(p)g_n^j)$. It is routine to prove that if M has a C^k structure, then the structure defined on BM by using C^{k-1} local bases is a C^{k-1} manifold structure on BM .

The *projection map* $\pi : BM \rightarrow M$ given by $(p, x_1, \dots, x_n) \rightarrow p$ is given locally by the product projection, so is a smooth map. The *fiber over p* is $\pi^{-1}(p)$, a submanifold diffeomorphic to $Gl(n, \mathbf{R})$.

Each local basis $B = (X_1, \dots, X_n)$ can be thought of as a smooth map $B : U \rightarrow BM$, called a *cross-section of BM over U* . The composition with π , $\pi \circ B$ is the identity map on U .

3.13. The Right Action of the General Linear Group. Each matrix $g \in Gl(n, \mathbf{R})$ can be used as a change of basis matrix on every basis of every tangent space. This simultaneous change of all bases in the same way is a map $R_g : BM \rightarrow BM$, given by $(p, x_1, \dots, x_n) \rightarrow (p, \sum_j x_j g_1^j, \dots, \sum_j x_j g_n^j)$. For $b \in BM$ we will also write $R_g b = bg$. It is called the *right action of g on BM* . For two elements $g, h \in Gl(n, \mathbf{R})$ we clearly have $(bg)h = b(gh)$, that is, $R_h \circ R_g = R_{gh}$.

3.14. The Universal Dual 1-Forms. There is a column of 1-forms on BM , existing purely due to the nature of BM itself, an embodiment of the idea of a dual basis of the basis of a vector space. These 1-forms are called the *universal dual 1-forms*, are denoted $\omega = (\omega^i)$, and are defined by the equation:

$$\pi_*(v) = \sum_i \omega^i(v)x_i,$$

where v is a tangent vector to BM at the point $(p, x_1, \dots, x_n) = b$. We can also use multiplication of the row of basis vectors by the columns of values of the 1-forms to write the definition as $\pi_*(v) = b\omega(v)$. Thus, the projection of a tangent vector v to BM is referred to the basis at which v lives, and the coefficients are the values of these canonical 1-forms on v . (The canonical 1-forms are usually called the *solder forms of BM* .)

It is a simple consequence of the definition of ω that if B is a local basis on M , then the pullback $B^*\omega$ is the local dual basis of 1-forms. This justifies the name for ω .

We clearly have that $\pi \circ R_g = \pi$, so that

$$\pi_* \circ R_{g_*}(v) = bg\omega(R_{g_*}(v)) = b\omega(v),$$

where v is a tangent at $b \in BM$. Rubbing out the “ b ” and “ (v) ” on both sides of the equation leaves us with an equation for the action of R_g on ω :

$$g \cdot R_g^* \omega = \omega, \quad \text{or} \quad R_g^* \omega = g^{-1} \omega.$$

3.15. The Vertical Subbundle. The tangent vectors to the fibers of BM , that is, the tangent vectors in the kernel of π_* , form a subbundle of TBM of rank n^2 . This is called the *vertical* subbundle of TBM , and is denoted by V .

3.16. Connections. A *connection on BM* is a specification of a complementary subbundle H to V which is smooth and invariant under all right action maps R_g . The idea is that moving in the direction of H on BM represents a motion of a basis along a curve in M which will be defined to be parallel translation of that basis along the curve. We can then parallel translate any tangent vector along the curve in M by requiring that its coefficients with respect to the parallel basis field be constant. The invariance of H under the right action is needed to make parallel translation of vectors be independent of the choice of initial basis.

3.17. Horizontal Lifts. Since H is complementary to the kernel of π_* at each point of BM , the restriction of π_* to $H(b)$ is a vector space isomorphism to $M_{\pi b}$. Hence we can apply the inverse to vectors and vector fields on M to obtain *horizontal lifts* of those vectors and vector fields. We usually lift single vectors to single horizontal vectors, but for a smooth vector field X on M we take all the horizontal lifts of all the values of X , thus obtaining a *smooth* vector field \bar{X} . These vector field lifts are compatible with π and all R_g , so that if Y is another vector field on M , then $[\bar{X}, \bar{Y}]$ is right invariant and can be projected to $[X, Y]$. However, we have not assumed that H is involutive, so that $[\bar{X}, \bar{Y}]$ is not generally the horizontal lift of $[X, Y]$.

The construction of the pull-back bundle and its horizontal subbundle is the bundle of bases version of the induced connection on the curve γ . More generally, the induced connection on a map has a bundle of bases version defined in just the same way.

We can also get horizontal lifts of smooth curves in M . This is equivalent to getting the parallel translates of bases along the curve. If the curve is the integral curve of a vector field X , then a horizontal lift of the curve is just an integral curve of \bar{X} . However, curves can have points where the velocity is 0, making it impossible to realize the curve even locally as the integral curve of a smooth vector field. Thus, we need to generalize the bundle construction a little to obtain horizontal lifts of arbitrary smooth curves $\gamma : [a, b] \rightarrow M$. We define the *pull-back bundle* γ^*BM to be the collection of all bases of all tangent spaces at points $\gamma(t)$, and give it a manifold with boundary structure, diffeomorphic to the product $[a, b] \times Gl(n, \mathbf{R})$, just as we did for BM , along with a smooth map into BM . The horizontal subbundle H can also be pulled back to a horizontal subbundle of rank 1. Then we have a vector field d/dt on $[a, b]$ whose horizontal lift has integral curves representing the desired horizontal lift of γ . This structure on γ^*BM corresponds to the connection on γ given above.

When we relate all of this to the other version of connections in terms of covariant derivative operators, we see that the differential equations problem for getting parallel translations has turned into the familiar problem of getting integral curves of a vector field on a different space.

3.18. The Fundamental Vector Fields. Corresponding to the left-invariant vector fields on $Gl(n, \mathbf{R})$ we have some canonically defined vertical vector fields on BM ; each fiber is a copy of $Gl(n, \mathbf{R})$ and these canonical fields are carried over as copies of the left invariant fields. Generally on a Lie group the left-invariant vector fields

are identified with the tangent space at the identity: in one direction we simply evaluate the vector field at the identity; in the other direction, if we are given a vector at the identity as the velocity $\gamma'(0)$ of a curve, then we can get the value at any other point g of the group as the velocity of the curve $g \cdot \gamma$ at time 0. Note that whereas g multiplies the curve on the left, which is what makes the vector field *left invariant*, what we see nearby g is the result of multiplying g on the *right* by γ . It is this process of multiplying on the right by a curve through the identity that we can imitate in the case of BM , since we have the right action of $Gl(n, \mathbf{R})$ on BM . It is natural to view the tangent space of $Gl(n, \mathbf{R})$ at the identity matrix I as being the set of all $n \times n$ matrices, which we denote by $gl(n, \mathbf{R})$. If we let E_j^i be the matrix with 1 in the ij position and 0's elsewhere, we get a standard basis of the Lie algebra. For a curve with velocity E_j^i we can take simply $\gamma(t) = I + tE_j^i$. The *fundamental vector fields* on BM are the vector fields E_j^i defined by

$$E_j^i(b) = \gamma'(0), \text{ where } \gamma(t) = b \cdot (I + tE_j^i).$$

We sometimes also call the constant linear combinations of these basis fields *fundamental*.

3.19. The Connection Forms. If we are given a connection H on BM , we define a matrix of 1-forms $\varphi = (\varphi_j^i)$ to be the forms which are dual to the vector fields E_j^i on the vertical subbundle V of TBM and are 0 on the connection subbundle H . This means that if $\gamma(t) = b \cdot (I + ta)$, where a is an $n \times n$ matrix, then $\varphi(\gamma'(0)) = a$.

Clearly the connection forms completely determine H as the subbundle they annihilate. Thus, in order to give a connection it is adequate to specify the connection form φ . In order to say what matrices of 1-forms on BM determine a connection, besides the property that the restriction to the vertical V gives forms dual to the fundamental fields E_j^i , we have to spell out the condition that H is right invariant in terms of φ . The name for this condition is *equivariance*, and things have been arranged so that it is expressed easily in terms of the differential action of R_g and matrix operations:

$$R_g^* \varphi = g^{-1} \varphi g \quad \text{for all } g \in Gl(n, \mathbf{R}).$$

3.20. The Basic Vector Fields. The universal dual cobasis is nonzero on any nonvertical vector, so that if we restrict it to the horizontal subspace of a connection it gives an isomorphism: $\omega : H(b) \rightarrow \mathbf{R}^n$. If we invert this map and vary b , we get the *basic vector fields* of the connection. In particular, using the standard basis of \mathbf{R}^n , we get the basic vector fields E_i ; they are the horizontal vector fields such that $\omega^i(E_j) = \delta_j^i$.

3.21. The Parallelizability of BM. Since a connection always exists, we now know that BM is parallelizable; specifically, $\{E_i^j, E_i\}$ is a parallelization.

Theorem 3.6 (Existence of Connections (again)). *There exists a connection on BM .*

Locally we have that BM is defined to be a product manifold. We can define a connection locally by taking the horizontal subspace to be the summand of the tangent bundle given by the product structure, complementary to the tangent spaces of $Gl(n, \mathbf{R})$. In turn this will give us some local connection forms which satisfy the equivariance condition. Then we combine these local connection forms by using a

partition of unity for the covering of M by the projection of their domains. (This is no different than the previous proof of existence of a connection.)

3.22. Relation Between the Two Definitions of Connection. If we are given a connection on BM and a local basis $B : U \rightarrow BM$, then we can pull back the connection form on BM by B to get a matrix of 1-forms on U . This pullback $B^*\varphi$ will then be the connection form of a connection on U in the sense of covariant derivatives. It requires some routine checking to see that these local connection forms all fit together, as B varies, to make a global connection on M in the sense of covariant derivatives.

Conversely, if we are given a connection on M in the sense of covariant derivatives, then we can define φ on the image of a local basis B by identifying it with the local connection form under the diffeomorphism. Then the extension to the rest of BM above the domain of B is forced on us by the equivariance and the fact that φ is already specified on the vertical spaces. The geometric meaning of the relation between the local connection form and the form on BM is clear: on the image of B the form φ measures the failure of B to be horizontal; by the differential equation for parallel translation the local form measures the failure to be parallel. But “parallel on M ” and “horizontal on BM ” are synonymous. A form on BM is *horizontal* if it gives 0 whenever any vertical vector is taken as one of its arguments. This means that it can be expressed in terms of the ω^i with real-valued functions as coefficients. A form η on BM with values in \mathbf{R}^n is *equivariant* if $R_g^*\eta = g^{-1}\eta$. A form ψ on BM with values in $gl(n, \mathbf{R})$ is *equivariant* if $R_g^*\psi = g^{-1}\psi g$. The significance of these definitions is that the horizontal equivariant forms on BM correspond to tensorial forms on M : η corresponds to a tangent-vector valued form on M , ψ corresponds to a form on M whose values are linear transformations of the tangent space. The rules for making these correspondences are rather obvious: evaluate η or ψ on lifts to a basis b of the vectors on M and use the result as coefficients with respect to that basis for the result we desire on M . The horizontal condition makes this independent of the choice of lifts; the equivariance makes it independent of the choice of b .

For example, the form ω corresponds to the 1-form on M with tangent-vector values which assigns a vector x to itself.

Theorem 3.7 (The Structural Equations). *If φ is a connection form on BM , then there is a horizontal equivariant \mathbf{R}^n -valued 2-form Ω and a horizontal equivariant $gl(n, \mathbf{R})$ -valued 2-form Φ such that*

$$d\omega = -\varphi \wedge \omega + \Omega,$$

and

$$d\varphi = -\varphi \wedge \varphi + \Phi.$$

The structural equations have already been proved in the form of pullbacks of the terms of the equations by a local basis. This shows how the forms Ω and Φ are given on the image of a local basis. The fact that these local forms yield the tensors T and R which live independently of the local basis can be interpreted as establishing the equivariance properties of Ω and Φ since they are horizontal. It is also not difficult to prove the structural equations directly from the specified equivariance of φ . If we restrict the structural equations to vertical vectors, or one vertical and one horizontal vector, we get information that has nothing to do with

connections. The first one tells how $Gl(n, \mathbf{R})$ operates on \mathbf{R}^n . The second one is more interesting: it is the equations of Maurer-Cartan for $Gl(n, \mathbf{R})$, which are essentially its Lie algebra structure in its dual packaging.

3.23. The Dual Formulation. The dual to taking exterior derivatives of 1-forms is essentially the operation of bracketing vector fields. If we bracket two fundamental fields or a fundamental and a basic field, we obtain nothing new, only a repeat of the Lie algebra structure and its action on Euclidean space. The brackets that actually convey information about the connection are the brackets of basic vector fields. By using the exterior derivative formula $d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y])$, we see that the first structural equation tells what the horizontal part of a basic bracket is and the second tells us what the vertical part is. Most of the terms are 0:

$$\begin{aligned} d\omega(E_i, E_j) &= E_i\omega(E_j) - E_j\omega(E_i) - \omega([E_i, E_j]) \\ &= -\omega([E_i, E_j]) \\ &= \Omega(E_i, E_j). \end{aligned}$$

Similarly,

$$-\varphi([E_i, E_j]) = \Phi(E_i, E_j).$$

We can immediately get some important geometrical information about a connection. The condition for the subbundle to be integrable is that the brackets of its vector fields again be within the subbundle. The basic fields are a local basis for H , so the condition for H to be integrable is just that curvature be 0. This means that locally there are horizontal submanifolds, which are local bases with a very special property: whenever we parallel translate around a small loop the result is the identity transformation; or, parallel translation is locally independent of path. The fact that setting curvature to 0 gives this local independence of path is not very obvious from the covariant derivative viewpoint of connections. If we go one step further and impose both curvature and torsion equal 0, then the result is also easy to interpret from basic manifold theory applied to the fields on BM . Indeed, when a set of independent vector fields has all brackets vanishing, there are coordinates so that these fields are coordinate vector fields. When these are the E_i of a connection on BM the coordinates correspond to coordinates on the leaves of H , which get transferred down to coordinates on M such that the coordinate vector fields are parallel along every curve. The geometry is exactly the same as the usual geometry of Euclidean space, at least locally.

3.24. Geodesics. A *geodesic* of a connection is a curve for which the velocity field is parallel. Hence, a geodesic is also called an *autoparallel curve*. Notice that the parametrization of the curve is significant, since a reparametrization of a curve can stretch or shrink the velocity by different factors at different points, which clearly destroys parallelism. (There is a trivial noncase: the constant curves are formally geodesics. But then a reparametrization does nothing.)

If we are given a point p and a vector x at p , we can take a basis $b = (x_1, \dots, x_n)$ so that $x = x_1$. Then the integral curve of E_1 starting at b is a horizontal curve, so represents a parallel field of bases along its projection γ to M . Moreover, the velocity field of γ is the projection of E_1 at the points of the integral curve; but the projection of $E_1(b)$ always gives the first entry of b . We conclude that γ has parallel velocity field. The steps of this argument can be reversed, so that the geodesics of

M are exactly the projections of integral curves of E_1 . Any other basic field could be used instead of E_1 .

Geodesics do not have to go on forever, since the field E_i may not be complete. If E_i is complete, so that all geodesics are extendible to all of \mathbf{R} as geodesics, then we say that M is *geodesically complete*.

Theorem 3.8 (Local Existence and Uniqueness of Geodesics). *For every $p \in M$ and $x \in M_p$ there is a geodesic γ such that $\gamma'(0) = x$. Two such geodesics coincide in a neighborhood about $0 \in \mathbf{R}$. There is a maximal such geodesic, defined on an open interval, so that every other is a restriction of this maximal one.*

This theorem is an immediate consequence of the same sort of statements about vector fields.

3.25. The Interpretation of Torsion and Curvature in Terms of Geodesics.

Recall the geometric interpretation of brackets: if we move successively along the integral curves of $X, Y, -X, -Y$ by equal parameter amounts t , we get an endpoint curve which returns to the origin up to order t^2 , but gives the bracket in its second order term. We apply this to the vector fields E_1, E_2 on BM . The meaning of the construction of the “small parallelogram” on BM is that we follow some geodesics below on M , carrying along a second vector by parallel translation to tell us what geodesic we should continue on when we have reached the prescribed parameter distance t . If we were to do this in Euclidean space, the parallelogram would always close up, but here the amount it fails to close up is of order t^2 and is measured by the horizontal part of the tangent to the endpoint curve in BM above. But we have seen that the horizontal part of that bracket is given by $-\Omega(E_1, E_2)$ relative to the chosen basis. When we eliminate the dependence on the basis we conclude the following:

Theorem 3.9 (The Gap of a Geodesic Parallelogram). *A geodesic parallelogram generated by vectors x, y with parameter side-lengths t has an endgap equal to $-T(x, y)t^2$ up to terms of order t^3 .*

The other part of the gap of the bracket parallelogram on BM is the vertical part. What that represents geometrically is that failure of parallel translation around the geodesic parallelogram below to bring us back to the identity. That failure is what curvature measures, up to terms cubic in t . We can't quite make sense of this as it stands, because the parallel translation in question is not quite around a loop; however, if we close off the gap left due to torsion in any non-roundabout way, then the discrepancies among the various ways of closing up, as parallel translation is affected, are of higher order in t . That is the interpretation we place on the following theorem.

Theorem 3.10 (The Holonomy of a Geodesic Parallelogram). *Parallel translation around a geodesic parallelogram generated by vectors x, y with parameter side-lengths t has second order approximation $I + R_{xy} \cdot t^2$.*

It seems to me that a conventional choice of sign of the curvature operators to make the “+” in the above theorem turn into a “-” is in poor taste. The only other guides for which sign should be chosen seem to be merely historic. The word “holonomy” is used in connection theory to describe the failure of parallel translation to be trivial around loops. By chaining one loop after another we get

the product of their holonomy transformations, so that it makes sense to talk about a holonomy group as a measure of how much the connection structure fails to be like Euclidean geometry.

Problem 3.11. Decomposition of a Connection into Geodesics and Torsion. Prove that if two connections have the same geodesics and torsion they are the same. Furthermore, the geodesics and torsion can be specified independently.

In regard to the meaning of the last statement, we intend that the torsion can be any tangent-vector valued 2-form; the specification of what families of curves on a manifold can be the geodesics of some connection has been spelled out in an article by W. Ambrose, R.S. Palais, and I.M. Singer, *Sprays*, Anais da Academia Brasileira de Ciencias, vol. 32, 1960. But for the problem you are required only to show that the geodesics of a connection to be specified can be taken to be the same as some given connection.

Problem 3.12. Holonomy of a Loop. Prove the following more general and precise version of the Theorem on Holonomy of Geodesic Parallelograms. Let $h : [0, 1] \times [0, \tau] \rightarrow M$ be a smooth homotopy of the constant loop $p = h(0, v)$ to a loop $\gamma(v) = h(1, v)$ with fixed ends, so that $h(u, 0) = h(u, \tau) = p$. Fix a basis $b = (p, x_1, \dots, x_n)$ and let $g(u) \in Gl(n, \mathbf{R})$ be the matrix which gives parallel translation $bg(u)$ of b around the loop $v \rightarrow h(u, v)$. Let $X = \frac{\partial h}{\partial u}, Y = \frac{\partial h}{\partial v}$ and let $\bar{h} : [0, 1] \times [0, \tau] \rightarrow BM$ be the lift of h given by lifting each loop $v \rightarrow h(u, v)$ horizontally with initial point b . In particular, $\bar{h}(u, \tau) = bg(u)$. Prove that

$$\int_0^1 g(u)^{-1} g'(u) du = \int_0^1 \int_0^\tau \bar{h}(u, v)^{-1} \circ R_{XY} \circ \bar{h}(u, v) dv du.$$

The meaning of the integrand on the right is as follows: We interpret a basis $b' = (q, y_1, \dots, y_n)$ to be the linear isomorphism $\mathbf{R}^n \rightarrow M_q$ given by $(a^1, \dots, a^n) \rightarrow \sum_i a^i y_i$. Thus, for each u, v , we have a linear map

$$\bar{h}(u, v)^{-1} \circ R_{X(u,v)Y(u,v)} \circ \bar{h}(u, v) : \mathbf{R}^n \rightarrow M_{h(u,v)} \rightarrow M_{h(u,v)} \rightarrow \mathbf{R}^n.$$

As a matrix this can be integrated entry-by-entry. Hint: Pullback the second structural equation via \bar{h} and apply Stokes' theorem on the rectangle.

Problem 3.13. The General Curvature Zero Case. Suppose that M is connected and that we have a connection H on BM for which $\Phi = 0$, that is, H is completely integrable. Let \tilde{M} be a leaf of H , that is, a maximal connected integrable submanifold. Show that the restriction of π to \tilde{M} is a covering map. Moreover, if we choose a base point $b \in \tilde{M}$, then we can get a homomorphism of the fundamental group $\pi_1(M) \rightarrow Gl(n, \mathbf{R})$ as follows: for a loop based at $\pi(b)$ we lift the loop into \tilde{M} , necessarily horizontally, getting a curve in \tilde{M} from b to bg . Then g depends only on the homotopy class of the loop.

A connection with curvature zero is called *flat* and the homomorphism of Problem 3.13 is called the *holonomy map* of that flat connection.

3.26. Development of Curves into the Tangent Space. Let $\gamma : [0, 1] \rightarrow M$ and let D be a connection on M . We let $\bar{\gamma} : [0, 1] \rightarrow BM$ be a parallel basis field along γ starting at $b = \bar{\gamma}(0)$ and express the velocity of γ in terms of this parallel basis, getting a curve of velocity components $\beta(t) = \bar{\gamma}(t)^{-1}\gamma'(t) \in \mathbf{R}^n$. Then we let $\sigma(t) = b \int_0^t \beta(u) du$. Thus, the velocity field of σ in the space $M_{\gamma(0)}$ bears the same relation to Euclidean parallel translates of b as does the velocity field of γ to the parallel translates of b given by the connection. We call σ the *development of γ into $M_{\gamma(0)}$* .

Problem 3.14. Show that the development is independent of the choice of initial basis b .

Problem 3.15. Show that the lift $\bar{\gamma}$ of γ in the definition of the development is the integral curve of the time-dependent vector field $\sum \beta^i(t)E_i$ on BM starting at b .

3.27. Reverse Developments and Completeness. Starting with a curve σ in M_p such that $\sigma(0) = 0$, we choose a basis b and let $\beta = b^{-1}\sigma'$. By Problem 3.15, we can then get a curve γ in M whose development is σ , at least locally. We call γ the *reverse development* of σ . Since the vector field $\sum \beta^i(t)E_i$ need not be complete, it is not generally true that every curve in M_p can be reversely developed over its entire domain.

We say that (M, D) is *development-complete* at p if every smooth curve σ in M_p such that $\sigma(0) = 0$ has a reverse development over its entire domain.

Problem 3.16. Suppose that M is connected. Show that the condition that (M, D) be development-complete at p is independent of the choice of p .

Problem 3.17. Show that the development of a geodesic is a ray with linear parametrization, and hence, if M is development complete, then M is geodesically complete.

Problem 3.18. For the connection of a parallelization (X_1, \dots, X_n) show that the geodesics are integral curves of constant linear combinations $\sum a^i X_i$.

Problem 3.19. For the connection of a parallelization $B = (X_1, \dots, X_n)$ show that the reverse development of σ for which $\beta = b^{-1}\sigma'$ is an integral curve of $B\beta$.

Problem 3.20. Show that if $f > 0$ grows fast enough along a curve γ in \mathbf{R}^2 , then the development of γ with respect to the parallelization $(f \frac{\partial}{\partial x}, f \frac{\partial}{\partial y})$ is bounded. Hence there is no reverse development having unbounded continuation.

Problem 3.21. Show that the geodesics of the parallelization of Problem 3.20 are straight lines except for parametrization, and on regions where $f = 1$ even the parametrization is standard. Then by taking γ to be an unbounded curve with a neighborhood U such that any straight line meets U at most in a bounded set, and f a function which is 1 outside U and grows rapidly along γ , it is possible to get a connection (of the parallelization) which is geodesically complete but not development-complete.

Remark 3.22. Parallel translation along a curve γ is independent of parametrization.

Remark 3.23. The only reparametrizations of a nonconstant geodesic which are again geodesics are the affine reparametrizations: $\sigma(s) = \gamma(as + b)$.

A curve which can be reparametrized to become a geodesic is called a *pregeodesic*.

Problem 3.24. Show that a regular curve γ is a pregeodesic if and only if $D_{\gamma'}\gamma' = f\gamma'$ for some real-valued function f of the parameter.

3.28. The Exponential Map of a Connection. For $p \in M$, $x \in M_p$ let $\gamma_{p,x}$ be the geodesic starting at p with initial velocity $\gamma'_{p,x}(0) = x$. The *exponential map* at p is defined by

$$\exp_p : U \rightarrow M \quad \text{by} \quad \exp_p x = \gamma_{p,x}(1),$$

where U is the subset of $x \in M_p$ for which $\gamma_{p,x}(1)$ is defined.

Proposition 3.25. *The domain U of \exp_p is an open star-shaped subset of M_p . Exponential maps are smooth.*

It is called the exponential map because it generalizes the matrix exponential map, and also the exponential map of a Lie group. These are obtained when the connection is taken to be the connection of the parallelization by a basis of the Lie algebra (the left-invariant vector fields or the right invariant vector fields; both give the same geodesics through the identity, namely, the one-parameter subgroups). More specifically, the multiplicative group of the complex numbers is a two-dimensional real Lie group for which the ordinary complex exponential map coincides with the one given by the invariant (under multiplication) connection. The geodesics are concentric circles, open rays, and loxodromes (exponential spirals).

3.29. Normal Coordinates. Since M_p is a vector space, the tangent space $(M_p)_0$ is canonically identified with M_p itself. Using this identification, it is easily seen that the tangent map of \exp_p at 0 may be considered to be the identity map. In particular, by the inverse function theorem, there is a neighborhood V of p on which the inverse of \exp_p is a diffeomorphism. Referring M_p to a basis b gives us an isomorphism b^{-1} to \mathbf{R}^n , and the composition gives a *normal coordinate map* at p :

$$b^{-1} \circ \exp_p^{-1} : V \rightarrow \mathbf{R}^n.$$

For normal coordinates it is clear that the coordinate rays starting at the origin of \mathbf{R}^n correspond to the geodesic rays starting at p .

Problem 3.26. Suppose that torsion is 0 and that (x^i) are normal coordinates at p . For any $x \in M_p$ show that $D_x \frac{\partial}{\partial x^i} = 0$, and hence the operation of covariant differentiation of vector fields with respect to vectors at p reduces to operating on the components of the vector fields by the vectors at p .

3.30. Parallel Translation and Covariant Derivatives of Other Tensors.

We have so far only been concerned with parallel translation and covariant derivatives of vectors and vector fields. For tensors of other types we simply reduce to the vector field case: a tensor field is *parallel along a curve* γ if the components of the tensor field are constant with respect to a choice of parallel basis field along γ . We calculate $D_x A$, where A is a tensor field and $x \in TM$ by taking a curve γ with $\gamma(0) = x$, referring A to a parallel basis along γ , and differentiating components at $t = 0$. These definitions are independent of the choice of basis.

4. THE RIEMANNIAN CONNECTION

4.1. Metric Connections. We now return to the study of semi-Riemannian metrics, and in particular, Riemannian metrics. If g is such a metric, then we say that a connection is a *metric connection* if parallel translation along any curve preserves inner products with respect to g . It is easy to see that there are several equivalent ways of expressing that same condition:

Equivalent to a connection being metric are:

- (1) The parallel translation of a frame is always a frame.
- (2) The tensor field g is parallel along every curve.
- (3) $D_x g = 0$ for all tangent vectors x .
- (4) $x(g(Y, Z)) = g(D_x Y, Z) + g(Y, D_x Z)$ for all tangent vectors x and all vector fields Y and Z .
- (5) Let FM be the *frame bundle* of g , consisting of all frames at all points of M . It is easily shown that FM is a submanifold of BM of dimension $n(n+1)/2$. The condition equivalent to a connection H being metric is that at points of FM the horizontal subspaces are contained in TFM .

4.2. Orthogonal Groups. The frame bundle is invariant under the action of the orthogonal group with the corresponding index ν . This is the group of linear transformations which leaves invariant the standard bilinear form on \mathbf{R}^n of that index:

$$g_\nu(x, y) = \sum_{i=1}^{n-\nu} x^i y^i - \sum_{i=n-\nu+1}^n x^i y^i.$$

Thus, $A \in O(n, \nu)$ if and only if $g_\nu(Ax, Ay) = g_\nu(x, y)$ for all $x, y \in \mathbf{R}^n$. In case $\nu = 0$ this reduces to the familiar condition for orthogonality: $A^T \cdot A = I$. For other indices the matrix transpose should be replaced by the adjoint with respect to g_ν , which we will denote by A' . The corresponding Lie algebra consists of the matrices which are skew-adjoint:

$$so(n, \nu) = \{A : A' = -A\}.$$

No matter what ν is, the dimension of $O(n, \nu)$ is $n(n-1)/2$, and since FM is locally a product of open sets (the domains of local frames!) in M times $O(n, \nu)$, FM has dimension $n(n+1)/2$.

4.3. Existence of Metric Connections. In general, the definition and existence of a connection on a principal bundle (this means that the fiber is a Lie group acted on the right by a model fiber) can be carried out by imitating the case of BM . However, for FM we can obtain a connection by restricting a connection on BM and then retaining only the skew-adjoint part. Thus, if φ is a connection form on BM , then for $x \in TFM$ we let

$$\varphi_a(x) = \frac{1}{2}(\varphi(x) - \varphi(x)').$$

For the Riemannian case this means that we decompose the matrix $\varphi(x)$ into its symmetric and skew-symmetric parts and discard the symmetric part. Since the decomposition into these parts is invariant under the action of the orthogonal group

by conjugation (similarity transform), it follows that the form φ_a on FM satisfies the equivariance condition:

$$R_g^* \varphi_a = g^{-1} \varphi_a g,$$

for all $g \in O(n, \nu)$. On the vertical subspaces of TFM the values of φ were already skew-adjoint, so that φ_a is an isomorphism of each vertical space onto $so(n, \nu)$. The number of independent entries of φ_a is $n(n-1)/2$. Consequently, the annihilated subspaces are a complement to the vertical ones, and by the equivariance condition, they are invariant under R_{g_*} . That is, we have a connection on FM .

What this means in terms of covariant derivatives on M is that we can start with any connection D ; then for any frame field (E_i) with respect to the metric g we have $D_x E_i = \sum_j \varphi_i^j(x) E_j$, defining the local connection forms φ_i^j . Then the local connection forms of a metric connection are obtained by taking the skew-adjoint part of φ .

4.4. The Levi-Civita Connection.

Theorem 4.1 (The Fundamental Theorem of Riemannian Geometry). *For a semi-Riemannian metric g there exists a unique metric connection with torsion 0.*

In fact, the metric connections are parametrized bijectively by their torsions. To see this it is necessary to make the correspondence between the torsion form Ω and the form $\tau = \varphi - \varphi_0$, where φ is an arbitrary metric connection form and φ_0 will be the one with torsion 0. According to the first structural equation (which pulls back to FM unchanged in appearance) we would have

$$d\omega = -\varphi \wedge \omega + \Omega = -\varphi_0 \wedge \omega,$$

or

$$\Omega = \tau \wedge \omega.$$

It is clear that τ determines Ω , so the problem reduces to showing that Ω determines τ . The condition that the values of the connection forms, and hence, of τ , are skew-adjoint with respect to the standard bilinear form g_ν must be used.

The trick used to determine τ in terms of Ω is to alternately apply the skew-adjointness: $g_\nu(\tau(x)\omega(y), \omega(z)) = -g_\nu(\omega(y), \tau(x)\omega(z))$ and first-structural equation relation: $\Omega(x, y) = \tau(x)\omega(y) - \tau(y)\omega(x)$ three times:

$$\begin{aligned} g_\nu(\tau(x)\omega(y), \omega(z)) &= -g_\nu(\omega(y), \tau(x)\omega(z)) \\ &= -g_\nu(\omega(y), \Omega(x, z)) - g_\nu(\omega(y), \tau(z)\omega(x)) \\ &= -g_\nu(\omega(y), \Omega(x, z)) + g_\nu(\tau(z)\omega(y), \omega(x)) \\ &= -g_\nu(\omega(y), \Omega(x, z)) + g_\nu(\Omega(z, y), \omega(x)) + g_\nu(\tau(y)\omega(z), \omega(x)) \\ &= -g_\nu(\omega(y), \Omega(x, z)) + g_\nu(\Omega(z, y), \omega(x)) - g_\nu(\omega(z), \tau(y)\omega(x)) \\ &= -g_\nu(\omega(y), \Omega(x, z)) + g_\nu(\Omega(z, y), \omega(x)) - g_\nu(\omega(z), \Omega(y, x)) - g_\nu(\omega(z), \tau(x)\omega(y)). \end{aligned}$$

By symmetry of g_ν , the expression we began with and last term are the same, so

$$2g_\nu(\tau(x)\omega(y), \omega(z)) = -g_\nu(\omega(y), \Omega(x, z)) + g_\nu(\Omega(z, y), \omega(x)) - g_\nu(\omega(z), \Omega(y, x)).$$

This establishes the unique determination of τ by Ω since g_ν is nondegenerate and the tangent vectors y, z to FM can be chosen freely to give all possible values for $\omega(y), \omega(z)$. We can use this formula in both directions: we can start with φ and determine τ and hence φ_0 ; or we can start with φ_0 and a given torsion Ω and determine φ .

There are several variants on the trick used to determine the Levi-Civita connection. Levi-Civita used it to determine the Christoffel symbols for a local coordinate basis. There is a basis-free version of it known as the *Koszul formula* for covariant derivatives:

$$2g(D_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) + g([Z, X], Y) + g(X, [Z, Y]).$$

If we restrict attention to X, Y, Z chosen from a local frame, then the first three terms of the Koszul formula vanish; for coordinate basis fields the last three vanish and we recapture Levi-Civita's formula.

Frequently the most efficient way to calculate the Levi-Civita connection is to use a local coframe (ω^i) and the first structural equation with Ω assumed to be zero and φ forced to be skew-adjoint. The fundamental theorem tells us that the information contained in the first structural equation is enough to determine φ . Often the equations can be manipulated so as to apply Cartan's Lemma on differential forms:

$$\text{If } \sum \theta^i \omega^i = 0$$

and the ω^i are linearly independent 1-forms, then the 1-forms θ^i must be expressible in terms of the ω^i with a symmetric matrix of coefficients.

4.5. Isometries. An *isometry* between metric spaces is a mapping which preserves the distance function and is 1-1 onto. In particular, it is a homeomorphism between the underlying topological spaces. If it is merely 1-1, then it is called an *isometric imbedding*. We use the same words for the mappings which preserve a semi-Riemannian metric:

An *isometry* $F : M \rightarrow N$ from a semi-Riemannian manifold M with metric tensor g onto a semi-Riemannian manifold N with metric tensor h is a diffeomorphism such that for all tangents $v, w \in M_p$, for all $p \in M$, we have $h(F_* v, F_* w) = g(v, w)$.

An *isometric imbedding* is a map satisfying the same condition relating the tangent map and the metrics, but requiring only that it be a differentiable imbedding; this does not mean that the topology has to be the one induced by the map, only that the map be 1-1 and regular. Finally, for an *isometric immersion* we drop the requirement that it be 1-1.

Since covariant tensor fields (those of type $(0, s)$) can be pulled back by tangent maps in the same way that differential forms are, we can also write the condition for isometric immersions as: $F^* h = g$.

We have seen that there are auxiliary structures uniquely determined by a semi-Riemannian metric or a Riemannian metric. Thus, the Levi-Civita connection is uniquely determined by the metric tensor g , and in the Riemannian case, lengths of curves and Riemannian distance are determined by the metric tensor g . Moreover, the curvature tensor is uniquely determined by the Levi-Civita connection. These additional structures are naturally carried from one manifold to another by a diffeomorphism. Thus, it is obvious that these auxiliary structures are preserved by isometries.

In particular, geodesics of the Levi-Civita connection are carried to geodesics by an isometry; this includes their distinguished parametrizations. Immediately we

have that isometries commute with exponential maps:

$$F \circ \exp_p = \exp_{Fp} \circ F_{*p}.$$

Theorem 4.2. *On a connected semi-Riemannian manifold an isometry is determined by its value and its tangent map at one point. The group of isometries is imbedded in FM .*

Corollary 4.3. *The group of isometries of a semi-Riemannian manifold into itself is a Lie group. The dimension of the group of isometries is at most $n(n+1)/2$, where n is the dimension of the original manifold.*

4.6. Induced semi-Riemannian metrics. If we have a differentiable immersion $F : M \rightarrow N$ and N has a semi-Riemannian metric h , then we get an induced symmetric tensor field of type $(0,2)$ on M , F^*h . In the Riemannian case F^*h is always positive definite, hence a Riemannian metric on M ; in the semi-Riemannian case it could even happen that F^*h is degenerate, or even if we assume that doesn't happen, on different connected components of M , F^*h could have different indices. When F^*h is indeed a semi-Riemannian metric, we say that it is the metric on M induced by F . Of course, then F becomes an isometric immersion (or imbedding or isometry, depending on what other set-theoretic properties it has).

Example 4.4 (Euclidean and semi-Euclidean spaces). When we consider \mathbf{R}^n with its standard coordinate vector fields $X_i = \frac{\partial}{\partial x^i}$, we have first of all a parallelization, and hence the connection of that parallelization. It serves to give the usual identification of each tangent space of \mathbf{R}^n with \mathbf{R}^n itself. In turn, the standard inner product g_ν of index ν can be considered as defined on each tangent space, so that we have a semi-Riemannian structure of index ν , called the semi-Euclidean space of index ν . We denote this by \mathbf{R}_ν^n . When $\nu = 0$ it is Euclidean space. When $\nu = 1$ it is called *Minkowski* space (although there are other things, Finsler manifolds, called Minkowski space).

The dual 1-forms of the parallelization X_i are the coordinate differentials $\omega^i = dx^i$. When we put them into a column ω and take exterior derivative we get $d\omega = 0$. Setting $\varphi = 0$ clearly gives us the connection forms of the parallelization; but parallel translation also clearly preserves g_ν , and torsion is obviously 0. By the fundamental theorem it follows that this same connection is also the Levi-Civita connection for all of these semi-Riemannian metrics.

It is obvious that all of the translations $T_a : x \rightarrow x + a$ are isometries of g_ν . They form a subgroup of dimension n of the isometry group. Almost as obvious (compute the tangent map!), for any orthogonal transformation $A \in O(n, \nu)$ the inner product g_ν , viewed as a semi-Riemannian metric, is preserved by A . This gives another subgroup of isometries, of dimension $n(n-1)/2$. Together the products of these two kinds of isometries form the full isometry group of \mathbf{R}_ν^n :

The *semi-Euclidean motion group*: $\{T_a \circ A : x \rightarrow Ax + a : a \in \mathbf{R}^n, A \in O(n, \nu)\}$. The motion group is transitive on \mathbf{R}^n , and at each point, the tangent maps of isometries which fix that point are transitive on the frames at that point. Thus, the induced group on the bundle of frames is transitive; in fact, if we fix a base point of $F\mathbf{R}_\nu^n$, then the motion group becomes identified with $F\mathbf{R}_\nu^n$ with the base point as the identity of the group.

Example 4.5 (Round spheres). The points at distance R from the origin in $E^n = R_0^n$ form the $n - 1$ -dimensional sphere S^{n-1} of radius R . The induced Riemannian metric has $O(n)$ as a group of isometries, and it is easy to check that it is transitive on points and transitive on frames at (some conveniently chosen) point. Thus, we can identify FS^{n-1} with $O(n)$; the bundle projection can be taken to be the map which takes an orthogonal matrix to R (first column). Since the isometry group is transitive on frames, and isometries preserve the curvature tensor of the Levi-Civita connection, the curvature tensor has the same components with respect to every frame. Along with symmetries shared by every Riemannian curvature, the invariance under change of frame is enough to determine the curvature tensor up to a scalar multiple. Certainly this is enough excuse to say that S^{n-1} has “constant curvature”. However, we shall amplify the meaning of constancy of curvature when we discuss sectional curvature.

Problem 4.6. Assume the symmetries of the general Riemannian curvature tensor with respect to a frame:

$$R^i_{jhk} = -R^j_{ihk} = -R^i_{jkh} = R^h_{kij}$$

and

$$R^i_{jhk} + R^i_{hjk} + R^i_{kjh} = 0.$$

For a curvature tensor which has the same components with respect to every frame, discover what these components must be, up to a scalar multiple.

In the case $R = 1$, the unit sphere, the structural equations of the Levi-Civita connection on the bundle of frames $O(n)$ are just the Maurer-Cartan equations of $O(n)$. One has to separate the left-invariant 1-forms on the group into those which correspond to the universal coframe and those which correspond to the connection forms. This gives a way of calculating the curvature components for problem 23, except for the multiple.

Example 4.7 (Other quadric hypersurfaces). Analogously to what we did with round spheres, we consider quadric surfaces $\{x : g_\nu(x, x) = R\}$. For nonzero R this always gives a submanifold for which the induced semi-Riemannian metric is nondegenerate and of constant index. That new index is $\nu - 1$ if $R < 0$ and ν if $R > 0$. In any case this gives us examples of metrics having a maximal group of isometries and “constant curvature” in the sense that the components of the curvature tensor are the same relative to any frame.

Especially important is the case $\nu = 1, R < 0$, for then we get a Riemannian manifold “dual” to the round sphere, with constant curvature (which we will call negative curvature). There are two connected components; retaining only the upper component, we get the *quadric surface model of hyperbolic $n - 1$ -space*.

There is an elementary argument to show that the geodesics of any of the quadric hypersurfaces are the intersections of the hypersurface with planes through the origin of \mathbf{R}^n . We use the fact that the isometries are so plentiful and they are induced by linear transformations of the surrounding space, which take planes into planes. Moreover, for any such plane, the isometries which leave it invariant are transitive on the intersection with the quadric surface. Thus, the parallel field generated by a tangent to that intersection must be carried into another parallel field tangent

to that intersection, which can only be itself or its negative. In particular, the geodesics of a round sphere are the great circles.

Problem 4.8. Describe examples of submanifolds of a semi-Riemannian manifold such that the metric induces tensor fields on the submanifolds which are degenerate, or are nondegenerate but not of constant index.

Remark 4.9 (Geodesics of quadric hypersurface). The idea of using the plethora of isometries to show that the geodesics of a quadric hypersurface $g_\nu(x, x) = R$ are the intersections with planes through the origin is valid, but the argument given in the last paragraph does not work. Here is a correct argument. We show that there is an isometry leaving the intersection of the plane and hypersurface pointwise fixed, but taking every tangent vector perpendicular to the intersection into its negative. We must assume that the tangent vectors v to the intersection are not null vectors (for which $g_\nu(v, v) = 0$). The desired isometry is described in terms of a frame for the vector space \mathbf{R}^n with the bilinear form g_ν . The first frame vector is a normal to the hypersurface $\text{grad } g_\nu$ (as a quadratic form g_ν can be considered to be a real-valued function on \mathbf{R}^n , and the gradient is taken relative to the inner product g_ν) at some point of the intersection. The second frame member is to be tangent to the intersection at that same point. Then the frame is filled out in any way. The isometry then takes this frame into the frame having the same first two members (so that it fixes the plane) and the remaining members of the frame replaced by their negatives.

The existence of such an isometry forces the intersection curve to be a geodesic. The parallel field along that intersection generated by the tangent to the geodesic at the base point, i.e., the second frame member, must be carried to a parallel field along the (fixed) intersection curve by the isometry. Since one vector, at the base point, of the parallel field is fixed, the whole field is fixed. But the only vectors fixed by the isometry are tangent to the intersection. Hence the (unit) tangent field along the intersection is parallel, making that curve a geodesic.

In case the induced metric on the quadric hypersurface has nonzero index, there will be null vectors, and the proof given will not apply to planes through the origin tangent to those null vectors. But the result is still true, since the limit of geodesics is still a geodesic, and the excluded planes can be obtained as limits of the others.

4.7. Infinitesimal isometries—Killing fields. A one-parameter subgroup in the isometry group of a semi-Riemannian manifold is, in particular, a one-parameter group of diffeomorphisms of the manifold. Hence, it is the flow of a complete vector field. More generally there will be vector fields whose local flows will be isometries of the open sets on which they are defined. These are called *infinitesimal isometries* or *Killing fields*. We give an equation which describes Killing fields, derived by using the idea of a Lie derivative. When we differentiate tensors carried along by a flow with respect to the flow parameter, we get the Lie derivative of the tensor field with respect to the vector field generating the flow. When the flow consists of local isometries and the tensor field is the metric tensor g itself, the tensors being differentiated are constant, so the derivative is 0. Hence, we have:

Proposition 4.10. *If J is a Killing field, then $L_J g = 0$. Consequently, a Killing field is characterized by the fact that the linear map on each tangent space given by $v \rightarrow D_v J$ is skew-adjoint with respect to g .*

We derive the consequence of $L_J g = 0$, the *equation of a Killing field*, namely, $g(D_x J, y) = -g(x, D_y J)$, as follows:

$$\begin{aligned} 0 &= (L_J g)(X, Y) = Jg(X, Y) - g(L_J X, Y) - g(X, L_J Y) \\ &= g(D_J X, Y) + g(X, D_Y) - g([J, X], Y) - g(X, [J, Y]) \\ &= g(D_J X, Y) + g(X, D_J Y) - g(D_J X - D_X J, Y) - g(X, D_J Y - D_Y J) \\ &= g(D_X J, Y) + g(X, D_Y J). \end{aligned}$$

In terms of a local coordinate basis, applied with x, y chosen from all pairs of coordinate vector fields, the equation of a Killing field becomes a system of linear first order partial differential equations for the coefficients of J . Since the group of isometries has dimension at most $n(n+1)/2$, we know *a priori* that there are at most that many linearly independent solutions for J . In the case of Euclidean space we already know all the isometries, and hence can calculate the Killing fields from that knowledge too, but it is an interesting exercise to calculate them as solutions of a system of PDE.

The Killing fields are named after Wilhelm Killing (1847-1923), who discovered the above equations.

Problem 4.11. For \mathbf{E}^n , calculate the Killing fields by solving the system of PDE's.

There is a simple, but useful, relation between geodesics and Killing fields, generalizing a theorem of Clairaut on surfaces of revolution.

Lemma 4.12 (Conservation Lemma). *Let γ be a geodesic and J a Killing field. Then $g(\gamma', J) = c$, a constant along γ . If γ has unit speed, then c is a lower bound on the length $|J|$ of J along γ , and hence, γ lies in the region where $|J| \geq c$. If γ is perpendicular to J at one point, then it is perpendicular to J along its extent.*

Corollary 4.13 (Clairaut's Theorem). *If γ is a geodesic on a surface of revolution, r is the distance from the axis, and φ is the angle γ makes with the parallels, then $r \cos \varphi = r_0$ is constant along γ . Hence, γ can never pass inside the "barrier" $r = r_0$.*

Of course, the proposition can be interpreted as saying that the level hypersurfaces of $|J|$ form barriers to geodesics even in the general case. For the surface of revolution we take $J = \frac{\partial}{\partial \theta}$, where θ is the angular cylindrical coordinate about the axis of revolution in space; J is a Killing field of Euclidean space and its restriction to any surface of revolution is tangent to that surface.

Clairaut's theorem is a very powerful tool for analyzing the qualitative behavior of geodesics on a surface of revolution. We develop this theme in the following problems, in which we suppose that the profile curve is expressed parametrically in terms of its arclength u by giving r and z as functions of u : $r = f(u)$, $z = h(u)$.

Problem 4.14. If u_0 is a critical point of f , then the parallel corresponding to this value of u is a geodesic. If u_0 is not a critical point of f , then on any geodesic tangent to the corresponding parallel r has a nondegenerate local minimum at the point of tangency.

Problem 4.15. The meridians $\theta = \text{constant}$ are geodesics. On the other geodesics, θ is strictly monotonic, and u is strictly monotonic on arcs which contain no barrier point.

Example 4.16. On the usual donut torus r is not monotonic between barriers for most geodesics.

Problem 4.17. Classify the geodesics according to whether u is periodic or not along the geodesic, and, if not, according to how it behaves relative to its barrier parallels.

Problem 4.18. Suppose that we have a geodesic which traverses the gap between its barriers, neither of which are geodesics. Show that the angular change $\Delta\theta$ in θ between successive collisions with barriers is a continuous function of the geodesic for nearby geodesics (obtained, say, by varying the angle φ at a point through which we assume they all pass). Hence, either $\Delta\theta$ is constant or there are nearby geodesics which fill up the barrier strip densely and others which are periodic (closed) with arbitrarily long periods. The case of $\Delta\theta$ constant can actually occur. How?

5. CALCULUS OF VARIATIONS

An important technique for discovering the extremal properties of functionals (usually looking for minimums of length, energy, area, etc.) is the calculus of variations. A putative extremum is presumed to be within a family, whereupon the derivative of the functional with respect to the parameter(s) of the family must be zero. Using the arbitrariness of the choice of family, we then obtain the *Euler equations* for extremals of the functional. These are usually differential equations which the mapping representing the extremal must satisfy. It is also called the *first variation condition*; “first” refers to “first derivative with respect to the parameter”. Once the condition for the first variation to be zero is satisfied, the analogue of the second derivative test for a minimum is often employed; hence we must calculate the “second variation”, similar to the Hessian of a function at a critical point. Thus, the second variation is a quadratic form on the (infinite-dimensional) tangent space to the space of all mappings in question. The condition for a nondegenerate minimum is then that the second variation be positive definite. More subtle results are obtained by determining the *index* of the second variation, that is, the maximal dimension of a subspace on which the second variation is negative definite. If the Euler equations are elliptic, then the index will usually be finite. We won’t get into this subtler analysis very much; it is known as “Morse theory”, named after Marston Morse, who developed it for the length functional on the space of paths with remarkable success.

5.1. Variations of Curves—Smooth Rectangles. . For a given curve $\gamma : [a, b] \rightarrow M$ a *variation* of γ is a one-parameter family of curves such that γ is the curve obtained by taking the parameter value 0. Formally this is a mapping $Q : [a, b] \times [0, \epsilon] \rightarrow M$, smooth as a function of two variables, such that $\gamma(s) = Q(s, 0)$. We call Q a *smooth rectangle* with base curve γ . The curves we get by fixing the second variable t and varying s are called the *longitudinal* curves of Q ; the curves obtained by fixing the first variable s are called the *transverse* curves of Q . The velocity fields of the longitudinal curves are united in the *longitudinal vector field*, which is formally a vector field on the map Q [see Bishop and Goldberg, §5.7], and can be denoted either $\frac{\partial Q}{\partial s}$ or $Q_*(\frac{\partial}{\partial s})$. Similarly, we have the *transverse vector field* $\frac{\partial Q}{\partial t}$ or $Q_*(\frac{\partial}{\partial t})$. The *variation vector field* is the vector field on γ obtained by restricting

$\frac{\partial Q}{\partial t}$ to the points $(s, 0)$. The first variation of length or energy depends only on the variation vector field; and when the base curve is critical with respect to these functionals, and we restrict to variations which satisfy reasonable end conditions, then the second variation also depends only on the variation vector field. (This is analogous to the situation where a smooth function has a critical point, whereupon the second derivative becomes tensorial.)

5.2. Existence of Smooth Rectangles, given the Variation Field. . If we are given a vector field V on a curve γ , then we can obtain a rectangle having V as its variation field by using the exponential map of some connection:

$$Q(s, t) = \exp_{\gamma(s)}(tV(s)).$$

5.3. Length and Energy. If we have a Riemannian metric, we have already defined the length of a curve γ as the integral $|\dot{\gamma}|$ of its speed. The *energy* of γ is

$$E(\gamma) = \int_a^b g(\dot{\gamma}, \dot{\gamma}) ds.$$

By Schwartz inequality applied to the functions $f = 1$ and $\sqrt{g(\dot{\gamma}, \dot{\gamma})}$ on the interval $[a, b]$ we obtain

$$|\gamma|^2 \leq (b - a)E(\gamma).$$

The condition for equality is that $g(\dot{\gamma}, \dot{\gamma})$ be proportional to $f = 1$ in the sense of integration, i.e., at all but a set of measure zero. Thus, we are allowed to apply this condition on piecewise smooth curves, so the conclusion is that we have equality for piecewise smooth curves if and only if the speed is constant. Hence, if we reparametrize curves so that they have constant speed, which doesn't change their lengths, then the energy functional becomes practically the same as the length functional for the purposes of calculus of variations. There is a technical advantage gained from using energy instead of length because the formulas for derivatives are simplified—similar to what happens in calculus when you choose to differentiate the square root of a function implicitly.

Proposition 5.1. *If M is a Riemannian manifold, then a curve has minimum energy if and only if the curve has minimum length and is parametrized by constant speed. (The comparison is among smooth curves connecting the same two points, parametrized on the same interval $[a, b]$.)*

In semi-Riemannian manifolds the concept of length does not have a meaning, so that the significant functional for the calculus of variations of curves is energy. However, in a Lorentz manifold of index $n - 1$ the curves γ for which $g(\dot{\gamma}, \dot{\gamma}) > 0$, which are called time-like curves, have a special significance: they represent paths of “events” which an object could experience as time passes. For these curves the usual expression for length is called the “elapsed time”, measuring the amount that a clock would change as it moved with the object. In a Lorentzian vector space the direction of the Cauchy-Schwartz and triangle inequalities, restricted to time-like vectors, is reversed. Thus, in Lorentz geometry we find that the time-like geodesics are the curves which locally *maximize* energy among nearby time-like curves. If one person moves on a geodesic while a second person starts out at the same time and place, accelerates away, and then steers back to join the first person, the first

person will age more than the second! A discussion of the inequalities is given in O'Neill, *Semi-Riemannian Geometry*, p 144.

Theorem 5.2. [*First Variation of Energy and Arclength*] Let Q be a smooth rectangle in a semi-Riemannian manifold, with base curve γ and variation field V . Denote the longitudinal curves by $Q_t : s \rightarrow Q(s, t)$. Then the first variation of energy is given by

$$\frac{dE(Q_t)}{dt}(0) = 2 \left[g(V(b), \gamma'(b)) - g(V(a), \gamma'(a)) - \int_a^b g(V(s), D_{\gamma'(s)} \gamma') ds \right].$$

If $g(\gamma', \gamma') = C^2$ is positive, then the first variation of arclength also makes sense and is given by

$$\frac{dL(Q_t)}{dt}(0) = \frac{dE(Q_t)}{dt}(0)/2C.$$

Remark 5.3. Covariant derivatives of vector fields along curves are defined in terms of parallel frames along curves. This amounts to lifting the curves to the bundle of frames, pulling back the universal coframe ω and connection form φ , then using the usual formula ($\omega(D_X Y) = (X + \varphi(X))\omega(Y)$). For an alternative approach which does not use bundles, but instead develops the idea of connections on maps and their pullbacks, see Bishop and Goldberg, Chapter 5. In particular, the fact that the torsion 0 exchange rule $D_{\frac{\partial}{\partial t}} \frac{\partial Q}{\partial s} = D_{\frac{\partial}{\partial s}} \frac{\partial Q}{\partial t}$ has meaningful terms and is true follows from the invariance of exterior calculus under pullbacks.

We define a curve to be *energy-critical* or *length-critical* within spaces of curves by the requirement that for all variations Q of the curve in the space the first variation of energy or length is 0. By elementary calculus it then follows that energy-minimal and length-minimal curves in the space are also critical. The space of curves used can be the smooth curves connecting two fixed points and parametrized on a fixed interval $[a, b]$. More generally, we can let the ends of the curves vary on submanifolds. The generality of the domain interval $[a, b]$ is convenient because it allows us to apply results to subintervals immediately: if a curve is critical or minimal for certain endpoint conditions on the interval $[a, b]$, then its restriction to $[a', b'] \subset [a, b]$ is critical or minimal for fixed endpoint variations over $[a', b']$. Thus, after analyzing the fixed endpoint case we can easily discover what additional condition is needed for the variable endpoint case.

Theorem 5.4. *If γ is energy-critical (length-critical) for smooth curves from p to q , then γ is a geodesic (pregeodesic).*

There is a subtle mistake which should be avoided at this point in the development: we cannot immediately assert from the theorem that minimizing curves are geodesics. There are two gaps in the argument: minimizers may not exist, or if they do they may not be smooth. For the case at hand, dealing with the length of curves, we have taken care of the existence problem using the Arzelà-Ascoli Theorem. However, the example of the taxicab metric shows that we are still required to establish smoothness. Moreover, the failure to take care of these gaps in other contexts has generated some famous mistakes: recall the 19th century dispute over the Dirichlet principle; and in our era, the Yamabe “theorem” turned into the Yamabe *problem* precisely because the convergence to a critical map failed, yielding a nonsmooth object which was a generalized function, not a smooth map.

For the length functional on curves we handle the difficulty by imbedding the geodesic in a field of geodesics and applying Gauss's Lemma. For the special case of Euclidean space Problem 1.17 and its hint shows how this works by using as the field of geodesics a field of parallel lines.

Lemma 5.5 (Gauss's Lemma). *Let Q be a variation of a geodesic γ such that all the longitudinal curves Q_t are geodesics with the same speed and having variation field V . Then $g(\gamma', V)$ is constant. In particular, if V is orthogonal to γ at one point, it is orthogonal everywhere.*

Remark 5.6. We have already seen a special case of Gauss's Lemma, namely, the proposition on Killing fields which we specialized to get Clairaut's Theorem. Given a Killing field J and a geodesic segment γ , we generate a variation Q by applying the flow of J to γ ; this Q satisfies the hypotheses of Gauss's Lemma. The fact that the variations of geodesics generated by the flow of a Killing field always yields geodesics for the longitudinal curves shows that a Killing field, restricted to any geodesic, is always a Jacobi field, defined as follows.

Definition 5.7 (Jacobi fields). If γ is a geodesic, a *Jacobi field* along γ is a vector field J on γ such that on each subinterval of γ , J is the variation field of a variation for which the longitudinal curves are geodesics.

The longitudinal curves in this definition do not all have to have the same speed, so that the Jacobi fields dealt with by Gauss's Lemma are a little special. It is rather trivial to analyze the Jacobi fields which come from variations which don't move the base curve out of its trace, instead simply sliding and stretching the geodesic along itself. We have already stated how a geodesic can be reparametrized to remain a geodesic, and that's all there is to it. Thus, a Jacobi field which is tangent to the geodesic it sits on has the form $J(s) = (as + b)\gamma'(s)$ for some constants a and b . We also have a clear idea of how freely we can vary geodesics in general: we can move the initial point to any neighboring point and the initial velocity to any neighboring velocity. Thus, the geodesic variations correspond to a neighborhood of the initial velocity in the tangent bundle, and the Jacobi fields, in turn, correspond to the tangents to the tangent bundle at that point.

Proposition 5.8. *The Jacobi fields along a given geodesic form a space of vector fields of dimension $2n$.*

Problem 5.9. (Rather trivial) Show that a vector field J on M is a Killing field if and only if along every geodesic J has constant inner product with the geodesic's velocity.

In fact, we shall soon calculate that the Jacobi fields satisfy a homogeneous linear second-order differential equation, so that the space of them is actually a vector space of dimension $2n$. The tangential ones form a subspace of dimension 2, and Gauss's Lemma says, in effect, that the ones orthogonal to the base at every point form a subspace of dimension $2n - 2$, complementary to the tangential ones.

Proposition 5.10. *In a normal coordinate neighborhood of p the images under \exp_p of the hypersurfaces $g(v, v) = c$ form a family of hypersurfaces orthogonal to the radial geodesics from p . The tangent map \exp_{p*} takes vectors orthogonal to the rays from the origin of M_p to vectors orthogonal to the corresponding geodesic from p .*

Note that both Gauss's Lemma and the above Proposition hold in the semi-Riemannian case. In the Riemannian case the local minimizing property of geodesics now follows by using as a field of geodesics the radial geodesics of the same speed starting from the initial point and applying the property given about that field in the Proposition. The local maximizing property of time-like geodesics in a Lorentz manifold of index $n - 1$ is done in just the same way, using the fact that removing the orthogonal component of the tangent to a nearby curve then *increases* rather than decreases the elapsed time.

Theorem 5.11. *Let M be a Riemannian manifold, $p \in M$, and R the radius of a normal coordinate ball B at p . Then for any $q \in B$ the radial geodesic segment from p to q is the shortest smooth curve from p to q .*

The "radius" referred to in the theorem is the distance measured in terms of the Euclidean formula with the normal coordinates. The value of that radius function at q is the length of the radial geodesic. Hence we have

Corollary 5.12. *If the ball of radius R is normal, then the distance function $q \rightarrow d(p, q)$ is smooth on $B - \{p\}$, and its square is smooth at p as well.*

Theorem 5.13 (The Jacobi equation). *Let J be a Jacobi field along a geodesic γ . We denote covariant derivatives with respect to γ' by a prime. Then*

$$J'' + R_{\gamma'J}\gamma' = 0.$$

The proof proceeds in a straightforward fashion by applying the first and second structural equations to a smooth rectangle generating the Jacobi field.

Example 5.14. In Euclidean space or in \mathbf{R}_n^n the connection is flat, $R = 0$. Thus, the Jacobi equation is $J'' = 0$. But covariant differentiation is just differentiation of components with respect to the standard Cartesian coordinates. Hence the Jacobi fields are linear fields, $J(s) = sA + B$, where s is a linear parameter on a straight line.

Example 5.15. In a sphere of radius 1 the curvature tensor is given for a unit speed geodesic γ by

$$R_{\gamma'J}\gamma' = \begin{cases} J, & \text{if } J \perp \gamma'; \\ 0, & \text{if } J \text{ is tangent to } \gamma. \end{cases}$$

Hence the Jacobi differential equation splits into uncoupled second order differential equations for the components of J with respect to a parallel frame along γ : $J_i'' + J_i = 0$ for the components orthogonal to γ , and $J_n'' = 0$ for the component tangent to γ .

Example 5.16. For hyperbolic geometry the curvature is opposite in sign from that on the sphere: $J_i'' - J_i = 0$, if $i < n$, $J_n'' = 0$.

Example 5.17. On a surface the curvature operator is expressed in terms of one component, the Gaussian curvature K . If J is orthogonal to unit vector γ' we have $R_{\gamma'J}\gamma' = KJ$. Thus, the Jacobi equation for a Jacobi field orthogonal to a geodesic is $J'' + KJ = 0$, which is regarded as a scalar differential equation for the one component of J .

6. RIEMANNIAN CURVATURE

6.1. Curvature Symmetries. *The matrix of R_{XY} is given by the value $-\Phi(X, Y)$ of a 2-form, so that it is skew-symmetric in X, Y : (skew-symmetry in arguments)*

$$(2) \quad R_{XY} = -R_{YX}.$$

The matrices $-\Phi$ have values in $so(n, \nu)$, so on M_p the operators are skew-adjoint with respect to g :

$$(3) \quad g(R_{XY}Z, W) = -g(Z, R_{XY}W).$$

If we take the exterior derivative of the first structural equation and substitute, using both structural equations, for $d\omega$ and $d\varphi$, we get

$$\begin{aligned} 0 &= d^2\omega = -d\varphi \wedge \omega + \varphi \wedge d\omega \\ &= \varphi \wedge \varphi \wedge \omega - \Phi \wedge \omega - \varphi \wedge \varphi \wedge \omega = -\Phi \wedge \omega. \end{aligned}$$

Evaluating this on X, Y, Z , using the shuffle permutation rule for exterior products, gives

$$(4) \quad R_{XY}Z + R_{YZ}X + R_{ZX}Y = 0.$$

We call this the *cyclic curvature symmetry*. Many references call it the “first Bianchi identity” (or even worse, “Bianchi’s first identity”), but that name is solely due to its formal resemblance to the Bianchi identity (see below). It was known to Christoffel and Lipschitz in 1871 (when Bianchi was 13) and probably to Riemann in 1854.

A fourth identity, *bivector symmetry* or *symmetry in pairs*, is a consequence of (2), (3), (4):

$$(5) \quad g(R_{XY}Z, W) = g(R_{ZW}X, Y).$$

The name “bivector symmetry” comes from a standard identification of $\bigwedge^2 M_p$ with $so(M_p, g)$, the skew-adjoint endomorphisms of M_p . We extend g to bivectors by the usual determinant method:

$$g(X \wedge Y, Z \wedge W) = g(X, Z)g(Y, W) - g(X, W)g(Y, Z).$$

Then if $A : M_p \rightarrow M_p$ is skew-adjoint, we can interpret A as a $(1, 1)$ tensor, then raise the covariant index to get a skew-symmetric $(2, 0)$ tensor, also labeled A ; this amounts to

$$g(A, X \wedge Y) = g(AX, Y).$$

Thus, $R : X \wedge Y \rightarrow R_{XY}$ is interpreted as a linear map $\bigwedge^2 M_p \rightarrow so(M_p, g) \approx \bigwedge^2 M_p$, for which (5) yields

$$(6) \quad g(R_{XY}, Z \wedge W) = g(X \wedge Y, R_{ZW}).$$

In this way we have an interpretation of R as a self-adjoint linear map of $\bigwedge^2 M_p$.

Problem 6.1. Let $S : M_p \rightarrow M_p$ be a symmetric (with respect to g) linear transformation. Extend S to act on the Grassmann algebra over M_p as a homomorphism, which we denote on $\bigwedge^2 M_p$ by $S \wedge S$. Show that $R = S \wedge S$, turned into a $(1, 1)$ tensor-valued 2-form by equation (6) satisfies all the identities of a curvature tensor.

6.2. Covariant Differentials. If T is a tensor field of type (r, s) , then we define a tensor field DT of type $(r, s + 1)$ by letting $DT(\dots, X) = D_X T(\dots)$. For example, the Riemannian Hessian of a function f is Ddf , given explicitly by $Ddf(X, Y) = XYf - (D_X Y)f$. At a critical point of f this coincides with the natural Hessian, and it is always symmetric.

6.3. Exterior Covariant Derivatives. If a tensor field T is skew-symmetric in some of its vector arguments, say the last t of the s arguments, then after forming DT we can skew-symmetrize on the last $t + 1$ vector arguments, obtaining a tensor field which we denote dT . This is called the *exterior covariant derivative* of T viewed as an $(r, s - t)$ tensor valued t -form. The second exterior derivative satisfies an identity $d^2T = R \wedge \cdot T$, which requires an explanation, so although it is not 0, it does not depend on derivatives of components of T , only the pointwise value of T and the curvature of the space.

Problem 6.2. (a) Explain how the explicit expression for Ddf comes from a product rule for the “product” $(df, X) \rightarrow df(X)$. (b) Find the corresponding explicit expression for $(DR(X, Y, Z))W$, taking as conventional: X, Y are the 2-form arguments of R which appear in R_{XY} , W is the vector on which R_{XY} operates, and Z is the additional argument for D .

Remark 6.3. In the semi-Riemannian case, keeping track of the signs $g(e_i, e_i) = \epsilon_i$ is a source of considerable irritation. In what follows we do not generally sum on a repeated index when it is attached only to ϵ and one other letter; sometimes we use the sum convention in other settings, sometimes we stick in sum signs. Usually it should be clear from context whether a sum is intended.

Theorem 6.4 (The Bianchi Identity). *The exterior covariant derivative of the curvature 2-form is 0. The formulation $dR = 0$ can be expanded to give the usual expressions as follows. The covariant differential DR is already skew-symmetric in the 2-form arguments it inherited from R , so that in order to skew-symmetrize on those two and the additional one we only need to throw in the cyclic permutation of the three. Thus, we get the equation*

$$DR(X, Y, Z) + DR(Y, Z, X) + DR(Z, X, Y) = 0,$$

for all vector fields X, Y, Z . In terms of components with respect to a basis (not necessarily a frame), we take the first two indices of R to be the indices of its matrix as a linear operator, so that one is up the other down, then the next two are its indices as a 2-form, hence subscripts. Taking the covariant differential adds another subscript, which is customarily separated from the others by a semicolon “;” or a solidus “/”. Thus, in index notation the Bianchi identity is written

$$R^i_{j|hk|l} + R^i_{j|kl|h} + R^i_{j|lh|k} = 0.$$

The usual application of the Bianchi identity is to prove Schur’s theorem that a semi-Riemannian manifold with pointwise constant curvature and dimension at least 3 has constant curvature. We state it precisely and then combine the proof with a proof of the Bianchi identity.

Theorem 6.5 (Schur’s Theorem). *Let M be a connected semi-Riemannian manifold of dimension > 2 such that there is a function $K : M \rightarrow \mathbf{R}$ such that the local expression for the curvature forms is $\Phi_j^i = \epsilon_j K \omega^i \wedge \omega^j$. Then K is constant.*

The local coframe expression given is what we call “pointwise constant” curvature.

Proof. The hypothesis about local coframe expression passes over to a claim that the global expression for the curvature forms on FM looks the same, $\Phi_j^i = \epsilon_j K \omega^i \wedge \omega^j$, where now K is the pullback to FM of the former K . Thus, K is a real-valued function which is constant on fibers, so that to show it is constant we only need to show that its derivatives in the horizontal directions are 0, i.e., $E_k K = 0$.

In general we have

$$\begin{aligned} d\Phi_j^i &= d(d\varphi_j^i + \varphi_k^i \wedge \varphi_j^k) \\ &= d\varphi_k^i \wedge \varphi_j^k - \varphi_k^i \wedge d\varphi_j^k. \end{aligned}$$

Now the general procedure for calculating the exterior covariant derivative of a tensor-valued form is to pass to the corresponding equivariant form on FM , take exterior derivative, and then take the horizontal part. We note that every term of $d\Phi_j^i$ has vertical factors, so the horizontal part is 0. This proves the Bianchi identity.

Now we continue the calculation supposing that the curvature forms have the pointwise constant curvature expression.

$$\begin{aligned} d(K\omega^i \wedge \omega^j) &= dK \wedge \omega^i \wedge \omega^j + K(-\varphi_k^i \wedge \omega^k \wedge \omega^j + \omega^i \wedge \varphi_k^j \wedge \omega^k) \\ &= (E_k K)\omega^k \wedge \omega^i \wedge \omega^j + K(-\varphi_k^i \wedge \omega^k \wedge \omega^j + \omega^i \wedge \varphi_k^j \wedge \omega^k). \end{aligned}$$

Now we see that in order for this expression to have no horizontal component the first term must vanish for all i and j . If $n > 2$, we can choose i and j different from any given k , so that we must have $E_k K = 0$ for all k , as required. \square

6.4. Sectional Curvature. The curvature tensor is the major invariant of Riemannian geometry; it entirely determines the local geometry in a sense spelled out precisely by the Cartan Local Isometry Theorem. However, it is too complicated to use directly in the formulation of local hypotheses which have significant geometric consequences. Thus, it is important to repackage the information it conveys in a more tractable form, the sectional curvature function. In two-dimensional spaces this reduces to a function on points, since there is only one 2-plane section at each point, namely, the tangent plane; the sectional curvature is then just the Gaussian curvature, $K = R_{1212}$, which is a component of R with respect to a frame. In higher dimensions the sectional curvature is still a real-valued function, but the domain consists of all 2-dimensional subspaces of all the tangent spaces, which we call *sections*.

Let σ be a section and let (v, w) be a frame for σ . Then the sectional curvature of σ is

$$(7) \quad K(\sigma) = g(R_{vw}v, w).$$

In the semi-Riemannian case this must be modified slightly: it is not defined if the section σ is degenerate, so there is no frame for it; but even in the case where the metric is indefinite on σ we change the sign, so that the result will conform to the more general formula for $K(\sigma)$ in terms of an arbitrary basis (x, y) of σ . Then we can write $v = ax + by$, $w = cx + dy$, and we get, using the symmetries of R

$$(8) \quad K(\sigma) = (ad - bc)^2 g(R_{xy}x, y).$$

A straightforward calculation, starting with the inverse expressions for x and y in terms of v and w , gives

$$(9) \quad g(x, x)g(y, y) - g(x, y)^2 = \pm(ad - bc)^{-2}.$$

The sign is positive if g is definite (positive or negative) on σ , and negative if g is indefinite. In the Riemannian case the geometric meaning of the expression (9) is that it is the square of the area of the parallelogram with edges x, y . If we take another frame for (x, y) , then the calculation shows that the result for $K(\sigma)$ is the same whether we use (7) or (8), showing that we have really defined K . Following the prescription given for the sign, the general formula given for sectional curvature in terms of an arbitrary basis is thus

$$K(\sigma) = \frac{g(R_{xy}x, y)}{g(x, x)g(y, y) - g(x, y)^2}.$$

For sections in the direction of pairs of frame vectors, σ_{ij} spanned by e_i, e_j , we introduce the signs $\epsilon_i = g(e_i, e_i)$ and get expressions for sectional curvature in terms of curvature components:

$$K_{ij} = K(\sigma_{ij}) = g(R_{e_i e_j} e_i, e_j) \epsilon_i \epsilon_j = \epsilon_i \epsilon_j R_{ijij} = \epsilon_i R_{ijj}^j = R_{ij}^{ij}.$$

Remark 6.6. If the curvature at $p \in M$ is constant in the sense of having the same components with respect to every frame, then as a map $R : \bigwedge^2 M_p \rightarrow \bigwedge^2 M_p$ it must be a constant multiple $K \cdot I$ of the identity map. (*A priori* this claim is true for possibly different multiples depending on the signature of the section, but one checks that it works in general by making some Galilean boost change of frames.) Hence the sectional curvature is that same constant K for all sections.

6.5. The Space of Pointwise Curvature Tensors. Let \mathcal{R} denote the subspace of $(1, 3)$ tensors over a semi-Euclidean vector space V of dimension n satisfying the curvature symmetries (2), (3), (4), and hence (5). Let $W = \bigwedge^2 V$. By lowering the first index we identify \mathcal{R} with the symmetric tensors of degree 2 over W satisfying (4). The dimension of W is $N = n(n-1)/2$, so that $\dim(\mathcal{S}^2(W)) = N(N+1)/2$; here \mathcal{S} denotes the space of symmetric tensors. In terms of a basis we get an independent linear restriction from the cyclic symmetry (4) for each choice of 4 distinct indices $i < j < h < k$. Hence,

$$\dim \mathcal{R} = N(N+1)/2 - \binom{n}{4} = n^2(n^2-1)/12.$$

6.6. The Ricci Tensor. For $R \in \mathcal{R}$, for all $v, w \in V$, we consider the linear map $A_{vw} : V \rightarrow V, x \rightarrow R_{vx}w$. The trace gives us a bilinear form $tr A_{vw} = Ric(v, w)$ on $V \times V$ called the Ricci tensor of R . In terms of components R_{ijk}^j with respect to a basis the components of Ric are the contraction $Ric_{ih} = \sum_j R_{ijh}^j$. With respect to a frame e_i we write $R_{ijhk} = g(R_{e_i e_j} e_h, e_k)$, which makes the superscript index correspond to the *fourth* index of the covariant form. To take care of the indefinite case we use the signs ϵ_i , whereupon $R_{ijhk} = \epsilon_k R_{ijh}^k$, hence $Ric_{ih} = \sum_j \epsilon_j R_{ijh}^j = \sum_j \epsilon_j R_{hji} = Ric_{hi}$. That is, Ric is a symmetric bilinear form.

6.7. Ricci Curvature. The Ricci tensor is determined by the corresponding quadratic form $v \rightarrow Ric(v, v)$. For a unit vector v , $Ric(v, v)g(v, v)$ is called the *Ricci curvature* of v . We can then take v to be a frame member $v = e_i$, so that

$$Ric(v, v)g(v, v) = \epsilon_i \sum_j \epsilon_j R_{ijij} = \sum_{j \neq i} K_{ij},$$

the sum of sectional curvatures of frame sections containing v . When the normal space to v is definite with respect to g we could average the sectional curvatures of planes containing v over the whole normal unit sphere, obtaining a standard multiple (depending only on dimension) of $Ric(v, v)$.

The geometric content of the Ricci curvature is that it measures the acceleration of volume contraction with respect to the flow along certain fields of geodesics. We formulate this precisely as follows.

Definition 6.7. The *divergence* of a vector field V is a concept which depends only on an unsigned volume element, that is, a *density* μ . If Ω is an n -form such that locally $\mu = |\Omega|$, then $div V$ is defined by taking the Lie derivative of Ω :

$$L_V \Omega = (div V)\Omega.$$

For any given unit vector v we define a canonical unit vector field extension V , by forcing V to satisfy

- (1) The integral curves of V are unit speed geodesics.
- (2) If v^\perp is the hyperplane in M_p orthogonal to v , then V is orthogonal to the hypersurface $\exp_p v^\perp$.

Theorem 6.8 (Theorem on Volume and Ricci Curvature). *The divergence of the canonical extension V of v is 0 at p ; its derivative in the direction of v is*

$$v \operatorname{div} V = -Ric(v, v).$$

Problem 6.9. [The Divergence Theorem.] Suppose that M is compact and orientable. Prove that for any smooth vector field V

$$\int_M (div V)\Omega = 0.$$

6.8. Scalar Curvature. If we contract the curvature a second time, we get the *scalar curvature*

$$S = \sum_{i,j} R_{ij}^{ij} = \sum_{i \neq j} K(\sigma_{ij}) = \operatorname{tr} Ric.$$

It is twice the sum of all the frame sectional curvatures, independent of the choice of frame. In Riemannian geometry it gives the discrepancy of the measure of a sphere or ball from the Euclidean value:

$$\begin{aligned} \mu_{n-1}(S(p, r)) &= r^{n-1} \Omega_{n-1}(1 - c_n S(p)r^2 + \dots), \\ \mu_n(B(p, r)) &= \int_0^r \mu_{n-1}(S(p, t)) dt = \frac{1}{n} r^n \Omega_{n-1} \left(1 - \frac{n}{n+2} c_n S(p)r^2 + \dots\right), \end{aligned}$$

where Ω_{n-1} is the $n-1$ -dimensional measure of the Euclidean unit sphere in E^n and c_n is another constant depending only on n . The argument to prove these approximations is based on the fact that S is the only linear scalar invariant of R , as well as expressions for the second-order terms of the metric in normal coordinates

which we develop later. (Cartan, *Leçons sur la Géométrie des Espaces de Riemann*, discusses this, pp 255-256.)

6.9. Decomposition of \mathcal{R} . Tensor spaces over an inner product space are naturally inner product spaces, with the induced action of $O(n, \nu)$ leaving the inner product invariant. That is, if e_i is a frame of V , then we get a frame for the tensor space by forming all the products of the e_i . When the tensor space involves the Grassmann algebra over V , then it is customary to use the determinant inner product, so that for $e_i \wedge e_j = e_i \otimes e_j - e_j \otimes e_i$ (in conformity with the shuffle-permutation definition of \wedge), we have $g(e_i \wedge e_j, e_i \wedge e_j) = g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)^2$, not twice that.

The operations of raising and lowering indices and contractions are equivariant under the action of $O(n, \nu)$. Thus, the kernel of the Ricci contraction $C_{Ric} : \mathcal{R} \rightarrow \mathcal{S}^2$, $R \rightarrow Ric$, is a subspace $ker C_{Ric} = \mathcal{W} \subset \mathcal{R}$ invariant under $O(n, \nu)$. The orthogonal projection of \mathcal{R} onto \mathcal{W} gives us the *Weyl conformal curvature tensor*. It is easy to show that C_{Ric} is *onto*, so that we have an orthogonal direct sum $\mathcal{R} = \mathcal{W} \oplus \mathcal{S}^2$.

The second contraction to get scalar curvature, $tr : \mathcal{S}^2 \rightarrow \mathbf{R}$ splits the Ricci curvature summand further into a constant-curvature tensor and a trace-free Ricci tensor in $\mathcal{S}_0^2 = ker tr$. Thus, we always have an orthogonal direct sum decomposition

$$\mathcal{R} = \mathcal{W} \oplus \mathbf{R} \oplus \mathcal{S}_0^2.$$

This decomposition is irreducible under $SO(n, \nu)$ except when $n = 4$. Then there is a Hodge $*$ -operator $* : \bigwedge^2 V \rightarrow \bigwedge^2 V$ satisfying $* \circ * = I$, which has a natural isometric extension to \mathcal{R} , and consequently gives an a further splitting of \mathcal{R} into the $+1$ and -1 eigenspaces of $*$, invariant under the orientation-preserving maps $SO(n, \nu)$. Curvature tensors which are eigenvalues of $*$ are called self-dual and anti-self-dual; they have become very important in recent years because the Yang-Mills extremals are just the connections with self-dual or anti-self-dual curvature tensor, and there were surprising relations with the multitude of differentiable structures on 4-manifolds.

Problem 6.10. (a) Calculate the dimensions of the summands in the splitting of \mathcal{R} . (b) For self-adjoint linear transformations $A, B : M_p \rightarrow M_p$ we have seen in Problem 6.1 that $A \wedge A \in \mathcal{R}$, hence $A \wedge B + B \wedge A = (A + B) \wedge (A + B) - A \wedge A - B \wedge B \in \mathcal{R}$. Taking $B = I$, the identity transformation, show that

$$C_{Ric}(A \wedge I + I \wedge A) = (n - 2)A + (tr A)I.$$

(c) Hence,

$$C_{Ric}\left(\frac{1}{n-2}(A \wedge I + I \wedge A) - \frac{1}{n-1}(tr A)I \wedge I\right) = A,$$

and

$$W^\perp(A) = \frac{1}{n-2}(A \wedge I + I \wedge A) - \frac{1}{n-1}(tr A)I \wedge I$$

gives a monomorphism $W^\perp : \mathcal{S}^2 \rightarrow \mathcal{R}$ equivariant under $O(n, \nu)$.

From the problem we conclude that the Weyl conformal curvature tensor is given by

$$W(R) = R - W^\perp(Ric).$$

6.10. Normal Coordinate Taylor Series. It was Riemann who invented Riemannian geometry and defined normal coordinates, calculated the second-order expressions for the metric coefficients. Maybe that's how he discovered the Riemann tensor (i.e., the curvature tensor); but anyway he knew that those second-order coefficients are given in terms of curvature components.

Theorem 6.11 (Riemann's Normal Coordinate Expansion of Metric Coefficients). *The second-order Taylor expansion of the metric in terms of normal coordinates x^i is given in terms of components of R as follows:*

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij} - \frac{1}{3} \sum_{h,k} R_{ihjk} x^h x^k + \dots$$

An interesting feature is how sparse these quadratic parts are, as well as the fact that there are no linear terms. The components of R are supposed to be evaluated at the origin, with respect to the frame which defines the normal coordinates. There is probably a version for semi-Riemannian geometry, but as it stands the formula given is only for the Riemannian case. We can now continue by obtaining the Christoffel symbols of the normal coordinate vector field basis, stopping at the linear terms. It is quite easy to use the Koszul formula for this, since the inverse of the matrix g_{ij} which is needed is just the identity matrix, up to second-order terms.

6.11. The Christoffel Symbols. The Christoffel symbols for the coordinate vector field local basis have as the linear terms of their Taylor expansions

$$\Gamma_{jk}^i = \frac{1}{3} (R_{jikh} + R_{kijh}) x^h + \dots$$

On a normal coordinate neighborhood we define a canonical frame field (E_i) as follows. At the origin p the frame coincides with the coordinate basis $\partial_i = \frac{\partial}{\partial x^i}$. Then we obtain E_i at other points by parallel translation along the radial geodesics from p . Thus, $D_{x^i \partial_i} E_j = 0$. These canonical frames are vital in the proof of the Cartan Local Isometry Theorem.

Problem 6.12. Show that the second order Taylor expansion of the canonical frame is given by

$$E_i = \sum_h (\delta_i^h + \frac{1}{6} \sum_{j,k} R_{ijhk} x^j x^k + \dots) \partial_h,$$

the coframe by

$$\omega^i = \sum_h (\delta_h^i - \frac{1}{6} \sum_{j,k} R_{ijhk} x^j x^k + \dots) dx^h.$$

Moreover, the connection forms of this frame field are

$$\varphi_j^i = \frac{1}{4} \sum_{h,k} R_{ijhk} (x^h dx^k - x^k dx^h) + \dots$$

Problem 6.13. Explain why for each h, k , along every radial geodesic $x^h \partial_k - x^k \partial_h$ is a field.

We can prove Riemann's theorem by doing the 2-dimensional case first. Let $ds^2 = E dx^2 + 2F dx dy + G dy^2$ be the metric in terms of Riemannian normal coordinates x, y . Then the result to be proved is that the second order Taylor expansions are

$$E = 1 - \frac{1}{3}Ky^2 + \dots,$$

$$F = \frac{1}{3}Kxy + \dots,$$

$$G = 1 - \frac{1}{3}Kx^2 + \dots,$$

where K is the Gaussian curvature at the origin p . Once we have the 2-dimensional case, we can prove the general case by a multitude of polarizations; that is, we apply the 2-dimensional case to surfaces obtained by exponentiating tangent planes spanned by (∂_i, ∂_j) , $(\partial_i, \partial_j + \partial_k)$, and $(\partial_i + \partial_h, \partial_j + \partial_k)$. To make this approach work we need to know that the Gaussian curvature of these surfaces is the sectional curvature of their tangent planes, and that the normal coordinates of these surfaces are the restrictions to the surfaces of the appropriate normal coordinates of the space.

In the Riemannian case, the fact that a geodesic of the ambient space which lies in a submanifold is a geodesic of the induced metric on that submanifold follows immediately from the characterization of geodesics as locally length-minimizing curves. However, we also would like to have the results in the semi-Riemannian case, so a more computational proof that geodesics are inherited is in order. In fact, we need to know how the Levi-Civita connections are related.

Theorem 6.14 (Theorem on the Connection of an Isometric Imbedding). *Let $N \subset M$ be a submanifold such that the induced metric from M on N is nondegenerate. (This is automatic in the Riemannian case.) Then the Levi-Civita connection of N is given by projecting the covariant derivatives for the Levi-Civita connection of M orthogonally onto the tangent planes of N . That is, if Δ is the connection of N , D is the connection of M , and $\Pi : M_p \rightarrow N_p$ is the orthogonal projection for each $p \in N$, then for vector fields X, Y on N*

$$\Delta_X Y = \Pi D_X Y.$$

The theorem can be proved by a straightforward verification of the axioms and characteristic properties for Δ (connection axioms, torsion 0, metric). It can also be done by using *adapted frame fields*: these are frames of M at points of N such that the first $q = \dim N$ of the fields are a frame for N . Locally we can always extend adapted frame fields to a frame field of M in a neighborhood. Then the local coframe ω^i , when restricted to N , satisfies

$$\omega^1, \dots, \omega^q$$

is a coframe of N , and

$$\omega^{q+1} = 0, \dots, \omega^n = 0$$

on N . Now consider the connection forms φ_j^i of ω^i . The first $q \times q$ block is skew-adjoint with respect to the semi-Euclidean metric induced on the tangent spaces of

N by $(\omega^1, \dots, \omega^q)$. The restriction of the first structural equation of M to N then gives

$$\omega^i = - \sum_{j=1}^q \varphi_j^i \wedge \omega^j,$$

which shows that the $q \times q$ block is that connection form of N and that torsion is 0. Then we have that orthogonal projection of $Z \in M_p$ to N_p is the vector with components $\omega^i(Z)$, $i = 1, \dots, q$, and hence

$$\begin{aligned} \omega^i(\Delta_X Y) &= X\omega^i(Y) + \sum_{j=1}^q \varphi_j^i(X)\omega^j(Y) \\ &= \omega^i(D_X Y), \quad i = 1, \dots, q. \end{aligned}$$

This verifies the formula $\Delta_X Y = \Pi D_X Y$.

Corollary 6.15. *A geodesic of N is a curve γ in N such that the M -acceleration $D_{\gamma'}\gamma'$ is always orthogonal to N .*

This suggests a numerical algorithm for approximate geodesics on a surface N in 3-space. Given a velocity tangent to N , move a small distance on N in the direction of that velocity. Then rotate the velocity in the normal plane to N at the new point so that the velocity will be tangent to N again. Then repeat the procedure. It is not hard to show that as the “small distance” goes to 0, the sequence of points generated converges to a geodesic.

Corollary 6.16 (Riemann’s Theorem–Pointwise Realizability of Curvature Tensors). *If we are given components R_{ijhk} satisfying the symmetries (2), (3), (4), then there exists a metric on a neighborhood of $O \in \mathbf{R}^n$ such that these R_{ijhk} are the components of the curvature tensor at the origin.*

Corollary 6.17 (to the Proof of Riemann’s Theorem). *Sectional curvatures determine the curvature tensor.*

Of course, the fact that the sectional curvatures determine the curvature tensor can be proved easily directly. See Bishop and Crittenden, Corollary 2, p 164, and the explicit polarization formula of Problem 2, p 165.

We give an outline of the proof of the 2-dimensional case of Riemann’s theorem. We know that the vector field $x\partial_x + y\partial_y$ is radial, so that the normal coordinate parametrization, which is a numerical realization of the exponential map, preserves its square-length $x^2 + y^2$. Thus, we obtain an identity:

$$(10) \quad x^2 E + 2xyF + y^2 G = x^2 + y^2.$$

By Problem 6.13 we know that $y\partial_x - x\partial_y$ is a normal Jacobi field along every radial geodesic. We do not need the Jacobi equation, but the fact that the two vector fields are orthogonal gives:

$$(11) \quad xy(E - G) + (y^2 - x^2)F = 0.$$

The classical formulas for the Christoffel symbols, which can be calculated from the Koszul formulas by plugging in coordinate vector fields give us equations expressing the first derivatives of E, F, G in terms of linear expressions in those Christoffel symbols. (The coefficients of those linear equations are each E, F , or G .) But we have seen that the Christoffel symbols all vanish at the origin. Hence, E, F, G have

linear coefficients 0; of course, they have constant terms 1, 0, 1, respectively. Now match up the fourth order terms in (10) and (11). The result is that at the origin all of the second derivatives vanish except three, which are themselves equal as follows:

$$E_{yy} = -2F_{xy} = G_{xx}.$$

Finally, by the classical equations for the curvature tensor in terms of the Christoffel symbols we calculate that the Gaussian curvature at the origin is $K = -3E_{yy}/2$.

Problem 6.18. For the indefinite case, for which $E(0,0) = -1$ and $G(0,0) = 1$, show that the proof outlined still goes through with only some sign changes.

7. CONJUGATE POINTS

Let $p \in M$ and let γ be a geodesic through p . A point q on γ , $q \neq p$, is a *conjugate point of p along γ* if there exists a Jacobi field $J \neq 0$ on γ such that J vanishes at both p and q . We could equivalently define a conjugate point q of p to be a singular value of \exp_p , for if we had $v \in (M_p)_x$, $v \neq 0$, such that $\exp_{p*} v = 0$, then $J(s) = \exp_{p*} sv$ defines a Jacobi field along the geodesic $s \rightarrow \exp_p sx$ such that $J(0) = 0$ and $J(1) = 0$. (To interpret $sv \in (M_p)_{sx}$ we use the canonical isomorphism of M_p with its tangent spaces.)

Both viewpoints of conjugate points are important. The definition as given shows that the relation “is a conjugate point of” is symmetric, and on the other hand, by applying Sard’s Theorem to \exp_p we obtain that the set of all conjugate points of p forms a set of measure 0. Moreover, the singular value property gives us the following theorem, which is half of the property of local nonminimization beyond a first conjugate point q of p along γ . In a more general setting, such as a locally compact intrinsic metric space, we would take this geometrically significant property as the *definition* of a conjugate point.

Theorem 7.1 (Local Nonminimization Implies Conjugate Point). *Let $\gamma([0, s])$ be a nonselfintersecting geodesic segment such that for every neighborhood U of $\gamma([0, s])$ there is $s' > s$ such that there is a shorter curve in U from $\gamma(0)$ to $\gamma(s')$ than $\gamma([0, s'])$. Then $\gamma([0, s])$ has a conjugate point of $\gamma(0)$.*

Proof. Suppose there are no conjugate points of $p = \gamma(0)$ on $\gamma([0, s])$. Then \exp_p is a diffeomorphism of some neighborhood V of $s\gamma'(0)$ onto a neighborhood of $\gamma(s)$. Moreover, we may assume that V is so small that \exp_p is also a diffeomorphism on the “cone” $\cup\{tV : 0 < t < 1\} = W$. We can also take a central normal ball $B \subset M_p$ so small that $\exp_p B$ intersects $\gamma([0, s])$ only in an initial radial segment of γ . Let $X = B \cup W$. Then $\exp_p : X \rightarrow \exp_p X = U$ is a diffeomorphism onto a neighborhood of $\gamma([0, s])$ which is thereby filled with a field of radial geodesics uniquely connecting points to p . Now we use the proof of the minimization theorem, page 25, to show that if $\gamma(s') \in U$, then $\gamma([0, s'])$ is the shortest curve in U from p to $\gamma(s')$. \square

Remark 7.2. To turn the local nonminimizing property into a characterization of first conjugate points it is necessary to free ourselves of the restriction that $\gamma([0, s])$ be nonselfintersecting. We do this by taking the neighborhoods U in question to be a neighborhood in the space of rectifiable curves starting at p . To topologize the space of curves we parametrize them with constant speed on $[0, 1]$ and use the topology of uniform convergence. We construct W as we did above, and take B

to be any central normal ball, so that for some $\epsilon > 0$, \exp_p is a diffeomorphism of some neighborhood of $t\gamma'(0)$ within X onto the ϵ -ball centered at $\gamma(t)$ for every $t, 0 < t < 1$. Then any rectifiable curve which is uniformly ϵ -close to γ can be lifted uniquely to a curve in X . This means that we can define a unique radial geodesic to each point of the curve, so that we can carry out the comparison of lengths as in the proof of Theorem 5.2.

7.1. Second Variation. We have seen that when we vary a geodesic to nearby curves with the same endpoints, then the first derivative of energy or arclength is 0. Moreover, if there is no conjugate point, then the critical value of energy or length actually is a local minimum. As in calculus, to gain information about what is happening at a critical point we need to look at second derivatives. We will only do the simple case for which the endpoints are fixed; the more general case in which we study the distance to a submanifold would require some notions that we have not defined, *focal points* and the *second fundamental form of a submanifold*. The details of the more general case can be found in Bishop & Crittenden or other references.

We assume that the geodesic base curve and the longitudinal curves of the variations we consider are normalized in their parametrization, so that they have constant speed and are parametrized on $[0, 1]$. Intuitively it is clear that we don't lose anything by such a restriction. It makes energy equal to the square of the length among the curves under consideration, so that there is no difference which we choose to calculate with. Moreover, the restriction is fitting to the conclusion we want to draw: by showing that some second derivative is negative we conclude that the value is not a minimum. Thus, it is some particular special variations that we want to discover from a calculation of second variation, and great generality is not required as long as what we do points to the special variations needed.

So suppose that Q is a smooth rectangle with base geodesic and constant end transversal curves, and longitudinal parameter $s, 0 \leq s \leq 1$. Let X be the longitudinal vector field and Y the transverse vector field. We will use such facts as $D_X Y = D_Y X$ and $R_{XY} = -D_X D_Y + D_Y D_X$ without going through some formal justification. Our starting point is the formula for first variation of energy:

$$\frac{dE}{dt} = 2 \int_0^1 g(D_X Y, X) ds.$$

Differentiating with respect to t again:

$$\begin{aligned} \frac{d^2 E}{dt^2} &= 2 \int_0^1 [g(D_Y D_X Y, X) + g(D_X Y, D_Y X)] ds \\ &= 2 \int_0^1 [g(D_X D_Y Y + R_{XY} Y, X) + g(D_X Y, D_X Y)] ds \\ &= 2 \int_0^1 \left[\frac{\partial}{\partial s} g(D_Y Y, X) - g(D_Y Y, D_X X) - g(R_{XY} X, Y) + g(D_X Y, D_X Y) \right] ds. \end{aligned}$$

Now we evaluate at $t = 0$, so that $D_X X = 0$ because the base curve is a geodesic, and the term which can be integrated drops out because the end transverse curves are constant: $D_Y Y = 0$, at $s = 0, 1$. For the term $g(D_X Y, D_X Y)$ we have an alternative form $\frac{\partial}{\partial s} g(D_X Y, Y) - g(D_X D_X Y, Y)$, and again the direct integration of the derivative drops out because of the fixed end condition. We also simplify the

notation, writing $Y' = D_X Y$ and $Y'' = D_X D_X Y$ when $t = 0$. Thus, we get *Synge's formula for second variation*:

$$\begin{aligned} \frac{d^2 E}{dt^2}(0) &= 2 \int_0^1 [g(Y', Y') - g(R_{XY} X, Y)] ds \\ &= -2 \int_0^1 g(Y'' + R_{XY} X, Y) ds. \end{aligned}$$

There are several interesting features to note. The result only depends on the variation vector field Y along the base, not on other properties of Q . This is an illustration of the rule that when a first derivative vanishes, then the second derivative becomes tensorial. The formula is a quadratic form in Y , so that it makes sense to consider the corresponding bilinear form, which is called the *index form for variations of the base curve*. The formulas for the index form are rather transparent; just replace one of the Y 's in each term by a Z to get $I(Y, Z)$.

From the second expression we see that if we can take Y to be a Jacobi field, so that the two ends of the base geodesic are conjugate points of each other, then the second variation vanishes. In fact, $I(Y, Z) = 0$ for every Z , which tells us that the Jacobi field is in the nullspace of the index form. Conversely, if $I(Y, Z) = 0$ for every Z , then we must have $Y'' + R_{XY} X = 0$, that is Y is a Jacobi field. Hence we have identified the nullspace of I : it consists of Jacobi fields which vanish at both ends.

Theorem 7.3 (Nonminimization Beyond Conjugate Point). *If a geodesic segment γ has an interior conjugate point, then it is not minimal.*

Proof. Make the parametrization be such that $\gamma(1)$ is a conjugate point of $\gamma(0)$, and γ extends beyond to some $\gamma(s)$, $s > 1$. Using the nontrivial Jacobi field which vanishes at $\gamma(0)$ and $\gamma(1)$ as a variation field, we obtain a variation of that first part of γ for which the change in arclength is $O(t^3)$. These varying curves form angles at $\gamma(1)$ with γ looking backwards proportional to t , up to higher order. If we cut across the obtuse angles formed, the saving in length is on the order of t^2 , so that saving is greater in magnitude than the gain in length $O(t^3)$ that we have from the variation. Hence there must be shorter curves nearby γ connecting $\gamma(0)$ and $\gamma(s)$. \square

The claim in the proof that “the saving in length is on the order of t^2 ” requires a proof, for which we introduce some second variations of vector fields which are not 0 at the ends. For a vector field Y along γ we still define an index form

$$(12) \quad I(Y) = 2 \int_0^1 [g(Y', Y') - g(R_{XY} X, Y)] ds.$$

To interpret $I(Y)$ as a second derivative of the energy of some rectangle we note that in leading up to (12) when we had a fixed-end variation, we integrated terms $\frac{\partial}{\partial s} g(X, D_Y Y)$. Thus, the weaker condition $D_Y Y = 0$ at $s = 0$ and 1 would also suffice to derive (12). For any given Y this condition can be attained by letting the initial and final transverse curves be geodesics. In fact, that was how we showed the existence of a variation attached to a given variation field anyway.

Moreover, we also want to interpret (12) as a second derivative in the case where Y is only continuous and piecewise C^1 . We can do so if we require that at the finite

number of places where $Y' = D_X Y$ does not exist we have $D_Y Y = 0$; in particular, it is again enough to make the transverse curves at those places be geodesics.

Theorem 7.4 (The Basic Inequality). *Suppose that there are no conjugate points of $\gamma(0)$ on $\gamma((0, 1])$. Let V be a piecewise C^1 vector field on γ orthogonal to γ such that $V(0) = 0$. Let Y be the unique Jacobi field such that $Y(0) = 0$ and $Y(1) = V(1)$. Then $I(V) \geq I(Y)$ and equality occurs only if $V = Y$.*

Proof. The inequality $I(V) \geq I(Y)$ is actually an immediate consequence of the minimization of energy by geodesics: if we take a rectangle Q such that $V = \frac{\partial Q}{\partial t}$ and the final transversal $t \rightarrow Q(1, t)$ is a geodesic, then we can define another rectangle \hat{Q} by making $\hat{Q}_t : s \rightarrow \hat{Q}(s, t)$ be the geodesic from $\gamma(0)$ to $Q(1, t)$. Then $E(\hat{Q}_t) \leq E(Q_t)$, so also $I(Y) \leq I(V)$.

To show that the case of equality requires $V = Y$ we make a calculation involving a basis of Y_1, \dots, Y_{n-1} of the Jacobi fields which are 0 at $s = 0$ and perpendicular to γ at $s = 1$. We can write $V = \sum f_i Y_i$, where the coefficients f_i are piecewise C^1 . We let

$$V' = \sum f'_i Y_i + \sum f_i Y'_i = W + Z.$$

Lemma 7.5. *If Y and Z are Jacobi fields along a geodesic γ , then $g(Y', Z) - g(Y, Z')$ is constant. If Y and Z vanish at the same point, then the constant is 0, so that $g(Y', Z) = g(Y, Z')$.*

Proof. Let $X = \gamma'$. Differentiating the difference we have

$$g(Y'', Z) + g(Y', Z') - g(Y', Z') - g(Y, Z'') = -g(R_{XY} X, Z) + 0 + g(Y, R_{XZ} X) = 0,$$

by the symmetry in pairs of R . \square

Now we have

$$\begin{aligned} (13) \quad g(V, \sum f'_j Y'_j) &= \sum f_i f'_j g(Y_i, Y'_j) \\ &= \sum f_i f'_j g(Y'_i, Y_j), \quad \text{by Lemma 7.5,} \\ &= g(W, Z). \end{aligned}$$

$$\begin{aligned} (14) \quad g(V, Z)' &= g(V', Z) + g(V, Z') \\ &= g(W, Z) + g(Z, Z) + g(V, \sum f'_i Y'_i) + g(V, \sum f_i Y''_i) \\ &= g(W, Z) + g(Z, Z) + g(W, Z) - g(V, \sum f_i R_{XY_i} X) \\ &= 2g(W, Z) + g(Z, Z) - g(V, R_{XV} X). \end{aligned}$$

The integrand in $I(V)$ is thus

$$\begin{aligned} g(V', V') - g(R_{XV} X, V) &= g(W + Z, W + Z) - g(R_{XV} V, V) \\ &= g(W, W) + g(V, Z)', \quad \text{by (14),} \end{aligned}$$

Note that V and Z are continuous, piecewise C^1 , so that we can integrate to get

$$\begin{aligned} (15) \quad I(V) &= \int_0^1 g(W, W) ds + g(V(1), Z(1)) - g(V(0), Z(0)) \\ &= \int_0^1 g(W, W) ds + g(V(1), Z(1)). \end{aligned}$$

We make the same calculation with $Y = \sum f_i(1)Y_i$, wherein the field corresponding to W is 0 and $V(1) = Y(1)$, $Z(1) = \sum f_i(1)Y_i'(1)$ are the same. Thus,

$$I(V) - I(Y) = \int_0^1 g(W, W) ds \geq 0,$$

and the condition for equality is that the piecewise continuous field $W = 0$, hence the f_i are constant and $V = Y$. This completes the proof of the Basic Inequality. \square

Now we can give a precise version of the proof that geodesics do not minimize beyond conjugate points.

Let $q = \gamma(s_2)$ be a conjugate point of $p = \gamma(0)$, and let Y be a nonzero Jacobi field vanishing at p and q . If we extend Y by a segment of 0-vectors beyond q , we will still have $I(Y) = 0$. Let ϵ be a positive number such that there is no conjugate point of $\gamma(s_2 - \epsilon)$ on $\gamma((s_2 - \epsilon, s_2 + \epsilon])$. Reparametrize γ (and scale ϵ, s_2 correspondingly) so that $s_2 + \epsilon = 1, s_2 - \epsilon = s_1$. We will indicate vector fields along γ by a triple designating what they are on each subinterval $[0, s_1], [s_1, s_2], [s_2, 1]$. For example, $(Y, Y, 0)$ denotes the vector field which is Y on $[0, s_1]$, Y on $[s_1, s_2]$, and 0 on $[s_2, 1]$. As long as we force the transverse curves at the break points s_1, s_2 to be geodesics, the summing of index forms for subintervals will represent a sum of second derivatives of energies which gives the second derivative of the whole. Let W be the Jacobi field such that $W(s_1) = Y(s_1)$ and $W(1) = 0$. Then we satisfy the condition for the basic inequality on the interval $[s_1, 1]$, so that

$$0 = I((Y, Y, 0)) > I((Y, W, W)).$$

Since there is a vector field (Y, W, W) having negative second variation, the longitudinal curves of any rectangle fitting it are shorter than $\gamma([0, 1])$ on the order of t^2 . This completes the proof of the nonminimization beyond conjugate points.

Remark 7.6. In Problem 5.9 the condition that the field be a Jacobi field along every geodesic should be derived from the fact that it is a Killing field independently of the other parts of the problem.

Example 7.7. The following kind of metric comes up in classical mechanics because the space of positions of a rigid body rotating about a fixed point is $SO(3)$, and the kinetic energy of the rigid body can be viewed as a Riemannian metric having the left-invariance considered. A general theorem of mechanics of a freely moving conservative system says that the free, unforced motions are geodesics of the kinetic energy metric. See Bishop & Goldberg, chapter 6.

On $SO(3)$ and S^3 we have parallelizations by left invariant vector fields X_1, X_2, X_3 satisfying

$$[X_1, X_2] = X_3, [X_2, X_3] = X_1, [X_3, X_1] = X_2.$$

The dual basis $\omega^1, \omega^2, \omega^3$ satisfies the Maurer-Cartan equations

$$d\omega^1 = -\omega^2 \wedge \omega^3, d\omega^2 = -\omega^3 \wedge \omega^1, d\omega^3 = -\omega^1 \wedge \omega^2.$$

We introduce a left-invariant metric for which the X_i are orthogonal and have constant lengths a, b, c :

$$\begin{aligned} ds^2 &= a^2(\omega^2)^2 + b^2(\omega^3)^2 + c^2(\omega^1)^2 \\ &= (\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2. \end{aligned}$$

Hence $d\eta^1 = -a\omega^2 \wedge \omega^3 = -\frac{a}{bc}\eta^2 \wedge \eta^3 = -\varphi_2^1 \wedge \eta^2 - \varphi_3^1 \wedge \eta^3$, where the φ_j^i are the Levi-Civita connection forms for our metric. By Cartan's Lemma, φ_2^1 and φ_3^1 are linear combinations of η^2 and η^3 , hence have no η^1 term. By cyclic permutations we conclude that each connection form is a multiple of a single basis element:

$$\varphi_2^1 = -C\eta^3, \varphi_3^2 = -A\eta^1, \varphi_1^3 = -B\eta^3.$$

Then the first structural equations give us what A, B, C are, using a little linear algebra:

$$A = \frac{b^2 + c^2 - a^2}{2abc},$$

and the cyclic permutations. We continue by calculating the curvature by using the second structural equations.

$$\Phi_2^1 = (AC + BC - AB)\eta^1 \wedge \eta^2 = K_{12}\eta^1 \wedge \eta^2,$$

and again cyclicly. Thus, the curvature as a symmetric operator on bivectors is diagonal with respect to the assumed basis. We reduce the numerator to a sum and difference of squares to see whether the curvatures can be negative:

$$K_{12} = \frac{3(u-v)^2 + (u+v)^2 - (3w-u-v)^2}{12uvw},$$

where $u = a^2, v = b^2, w = c^2$.

7.2. Loops and Closed Geodesics. [This material is from Bishop & Crittenden, Section 11.7, with little change.] A *closed geodesic*, or *geodesic loop* is a geodesic segment for which the initial and final points coincide. A *smooth* closed geodesic, or *periodic* geodesic is a geodesic loop for which the initial and final tangents coincide.

In a compact Riemannian manifold we can get convergent subsequences of a family of constant speed curves uniformly bounded in length, using the Arzelà-Ascoli theorem. If we have a continuous loop with base point p , we can obtain a rectifiable curve in the same homotopy class by replacing uniformly confined subsegments (which exist by uniform continuity) by unique minimal geodesic segments. Then we can reparametrize the resulting piecewise smooth curve by its constant-speed representative (all parametrized on $[0, 1]$). Applying the Arzelà-Ascoli Theorem to a sequence of such homotopic constant-speed curves with length descending to the infimum length produces a loop of *minimum* length in the homotopy class. It is obviously a geodesic loop. More generally, the same procedure works in a *complete* Riemannian manifold. (Completeness is next on the agenda.)

Proposition 7.8. *In a compact Riemannian manifold each pointed homotopy class of loops contains a minimal geodesic loop.*

If we allow the base point to float during homotopies, then we get *free* homotopy classes of loops. If we connect a loop back and forth to a base point by a curve, then the resulting pointed loops, as we vary the connecting curve, give conjugate elements of the fundamental group. In this way we associate uniquely to a free homotopy class of loops a conjugate class in the fundamental group.

Using the same trick, in a compact manifold we can extract a periodic geodesic representative of a free homotopy class of loops. However, in the complete, but noncompact, case the trick does not necessarily work because the length-decreasing sequence can go off to infinity. For pointed loops that does not happen because

closed bounded sets are compact, and everything takes place in a closed ball about the base point.

Theorem 7.9 (Synge's Theorem on Simple Connectedness). *If M is compact, orientable, even-dimensional, and has positive sectional curvatures, then M is simply connected.*

Proof. The idea is to use second variation to show that a nontrivial periodic geodesic cannot be minimal in its homotopy class. By parallel translation once around such a periodic geodesic, we get a map of the normal space to the geodesic at the initial point into itself. Because that map has determinant 1 (Why?) and the normal space is odd-dimensional, there must be an eigenvalue equal to 1, hence a fixed vector v . This fixed vector v generates a parallel field V , joining up smoothly at the ends since v is fixed by parallel translation. Thus, the second variation of an attached rectangle is

$$I(V) = \int_0^1 [g(V', V') - g(R_{XV}X, V)] ds = - \int_0^1 g(R_{XV}X, V) ds < 0,$$

so there are nearby homotopic shorter curves. \square

We give some variations on the same theme as problems.

Problem 7.10. Let M be compact, even-dimensional, nonorientable, and have positive curvature. Show that the fundamental group of M is Z_2 .

Problem 7.11. Let M be compact, odd-dimensional, and have positive curvature. Show that M is orientable.

A method of Klingenberg gives us either geodesic loops or conjugate points. The hypothesis can be weakened to completeness and an assumption that there is a cut point. The definition of cut points and the facts used about them will be given after we discuss completeness.

Theorem 7.12 (Klingenberg's Theorem). *Let M be compact, $p \in M$ and m a point of the cut locus of p which is nearest to p . If m is not a conjugate point of p , then there is a unique geodesic loop based at p through m and having both segments to m minimal.*

Proof. . If m is not a conjugate point, then there are at least two minimal segments from p to m . We show that there are just two and that they match smoothly at m . Let γ and σ be any two. By matching smoothly at m we mean that $\gamma'(1) = -\sigma'(1)$. Otherwise, there will be local distance functions f and g , giving the distance from p in neighborhoods of γ and σ , respectively. In a neighborhood of m , the equation $f = g$ defines a smooth hypersurface not perpendicular to γ or σ . (The proof given in B & C has an error at this point.) In a direction on this hypersurface making acute angles with both geodesics there are points which are nearer to p which can be reached by distinct geodesics, one near γ , one near σ . That is, there are cut points of p closer than m . \square

Corollary 7.13. *Let M be compact and let (p, m) be a pair which realizes the minimum distance from a point to its cut locus. Then either p and m are conjugate to each other, or there is a unique periodic geodesic through p and m such that both segments are minimal.*

Corollary 7.14. *Let M be compact, even-dimensional, orientable, with positive curvature, and let p, m be as in corollary 7.13. Then p and m are conjugate.*

8. COMPLETENESS

In a topological metric space X , a sequence (x_n) is a *Cauchy sequence* if for every $\epsilon > 0$ there is n_0 such that for all $n > n_0$, $m > n_0$, we have $d(x_n, x_m) < \epsilon$. A convergent sequence is a Cauchy sequence. If every Cauchy sequence is convergent, then X is a *complete* metric space. If some subsequence of a Cauchy sequence converges, then the sequence itself converges. Hence a compact metric space is complete.

A noncomplete metric space X has an essentially unique *completion*, a complete metric space in which X is isometrically imbedded and no unnecessary identifications are made among the points added to make the space complete or among those and the original points of X . There is a standard way of constructing the completion: start with the set of all Cauchy sequences; put an equivalence relation on that set, requiring two sequences to be equivalent if the sequence which results from interleaving them is still Cauchy; the metric is extended to the set of equivalence classes by taking the limit of distances $d(x_n, y_n)$; we imbed X isometrically in this set of equivalence classes as the classes represented by constant sequences.

The completion of a Riemannian manifold does not have to be a manifold. For example, we can start with the Euclidean plane, make it noncomplete by puncturing it; then we can take the universal simply connected covering manifold X . As a manifold X is just diffeomorphic to the plane. However, as a metric space there is only one equivalence class of Cauchy sequences which does not converge: the projection of a representative converges to the point we removed. In terms of Riemann surface theory X is the Riemann surface of the complex logarithm function; that is, the locally defined log can be defined as a single-valued complex-analytic function on X . So we call X the *logarithmic covering* of the punctured plane.

Problem 8.1. Discover a topological property of the completion of the logarithmic covering which shows that it is not a manifold.

Recall that a metric space is *intrinsic* if the distance between two points is the infimum of lengths of curves between the points. We say that a metric space is a *geodesic* metric space if it is intrinsic and for every pair of points there is a curve between the points whose length realizes the distance between the points. A space is *locally geodesic* if every point has a neighborhood in which distances between pairs from that neighborhood are realized by curve lengths. Thus, we have seen that a Riemannian manifold is a locally geodesic space.

If we parametrize a geodesic in a Riemannian manifold by the arc length measured from some point on it, then it becomes an isometric immersion from an interval to the Riemannian manifold. We have realized the geodesics as the projections of the integral curves of the vector field E_1 on FM . In particular we will

have *maximal* geodesics, ones which cannot be extended as geodesics to a larger interval; the domain is always an open interval. If that interval is not all of \mathbf{R} , then there will be a Cauchy sequence in the interval converging to a finite end. The image distances are no farther apart, hence also a Cauchy sequence. Thus, if the Riemannian manifold is complete, the limit point of that sequence can be used to extend the geodesic to one more point. We have called this ability to always extend a geodesic to a finite end of its domain *geodesic completeness*. We have therefore proved:

Proposition 8.2. *A complete Riemannian manifold is geodesically complete. That is, maximal geodesics are defined on the whole real line.*

The converse of this theorem is known as the Hopf-Rinow Theorem, proved in its original form by H. Hopf and W. Rinow (On the concept of complete differential-geometric surfaces, *Comment. Math. Helv.* **3** (1931), 209-225.) It was recognized later, by de Rham, that it was important to weaken the hypothesis a little more: if we assume that just those geodesics extending from a single point can be extended infinitely, then the Riemannian space is complete. It was that form which has become accepted as the classical form of the Hopf-Rinow Theorem. Strangely, an even better version was proved in 1935: S. Cohn-Vossen, Existenz Kurzester Wege. *Doklady SSSR* 8 (1935), 339-342. The Cohn-Vossen version is applicable to locally geodesic spaces which have maximal shortest paths of finite length; the de Rham improvement, requiring the extendibility of only those geodesics which originate from some single point, is also valid for Cohn-Vossen's version

Theorem 8.3 (The Hopf-Rinow-CohnVossen Theorem). *If X is a locally geodesic space such that there is a point p for which every maximal geodesic through p is defined on a closed interval, then X is a complete metric space.*

Of course, for locally compact spaces, the assumption that the space is locally geodesic can be derived from the metric being intrinsic.

In the same context, of locally compact intrinsic complete metric spaces, it then follows that the space is globally geodesic.

The modern form of the H-R-CV theorem specifies several equivalent conditions and a consequence. This facilitates the proof of the main implication, which was stated as the H-R-CV theorem above.

Theorem 8.4 (The Hopf-Rinow-CohnVossen Theorem). *In a locally compact intrinsic metric space M , the following are equivalent:*

- (i): *Every halfopen minimizing geodesic from a fixed base point extends to a closed interval.*
- (ii): *Bounded closed subsets are compact. (This is often said: M is finite-compact.)*
- (iii): *M is complete.*
- (iv): *Every halfopen geodesic extends to a closed interval.*

Any of these implies: M is a geodesic space (i.e., any two points may be joined by a shortest curve).

Proof. We start with an outline of proof. We establish a cycle (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i), and then the final assertion is clear from (ii).

The implication $(ii) \Rightarrow (iii)$ is true in any metric space and we have already noted the truth of $(iii) \Rightarrow (iv)$ above; $(iv) \Rightarrow (i)$ is trivial. Thus, the more difficult part, $(i) \Rightarrow (ii)$ is left.

So let us assume (i) with base point p and define

$$E_r = \{q : d(p, q) \leq r \text{ and there is a minimizing geodesic from } p \text{ to } q\},$$

$$\bar{B}_r = \text{the closed metric ball of radius } r \text{ with center } p,$$

$$A = \{r : E_r = \bar{B}_r\}.$$

The idea is to show that A is both open and closed in $[0, \infty)$.

(In the Riemannian case E_r is the image under \exp_p of the intersection of the domain of \exp_p with the closed ball in M_p of radius r . The proof of the fact that E_r is compact then follows by using the continuity of \exp_p . From there the pattern of proof was given by deRham, copied into Bishop & Crittenden.)

Now we give the details of the proof of the H-R-CV theorem.

(1) E_r is compact for all r .

The set I of r for which E_r is compact is an interval: for if $s < r$ and E_r is compact, let (q_n) be a sequence in E_s . A sequence of minimizing segments from p to q_n lies in the compact set E_r , so by the Arzelà-Ascoli Theorem has a convergent subsequence. The limit curve is necessarily a minimizing geodesic, and its endpoint is the limit of the corresponding subsequence of (q_n) and is in E_s .

The interval I is open: for if E_r is compact, then by local compactness we can cover it by a finite number of open sets having compact closure. The union of these closures will contain all points within distance $\epsilon > 0$ of E_r , so it will be a compact set K containing $E_{r+\epsilon}$. Repeating the above argument with E_r replaced by K and E_s by $E_{r+\epsilon}$ shows that the latter is compact.

Suppose that $I = [0, r)$ where r is finite. Let (q_n) be a sequence in E_r and σ_n a minimizing geodesic from p to q_n . Using the compactness of E_s , for $s < r$, we construct successive subsequences σ_n^m for each m which converge on $[0, s_m]$ where $s_m \rightarrow r$. Then the diagonal subsequence σ_n^n converges to a minimizing halfopen geodesic segment on $[0, r)$. By hypothesis, this halfopen geodesic has an extension to r , which provides a limit point of (q_n) and a minimizing geodesic to it from p . Thus, E_r is compact, contradicting the assumption that $r \notin I$. Hence, $I = [0, \infty)$, that is, all E_r are compact.

(2) The set A of all r for which $E_r = \bar{B}_r$ is an interval.

Indeed, if every point at distance r or less can be reached by a minimizing geodesic from p , and if $s < r$, then certainly every point of distance s or less can be so reached.

(3) A is closed. If r is a limit point of elements s in A , then $q \in \bar{B}_r$ is certainly a limit of points $q_s \in \bar{B}_s \subset E_r$. Since E_r is compact, $q \in E_r$.

(4) A is open. If $E_r = \bar{B}_r$, then we can cover \bar{B}_r by a larger compact set including some $\bar{B}_{r+2\epsilon}$, so the latter is compact. For $q \in \bar{B}_{r+\epsilon}$ a sequence of curves from p to q with lengths converging to $d(p, q)$ will eventually be in $\bar{B}_{r+2\epsilon}$. By the Arzelà-Ascoli Theorem there will be a subsequence converging to a minimizing segment from p to q , so that $q \in E_{r+\epsilon}$. Hence, $E_{r+\epsilon} = \bar{B}_{r+\epsilon}$, showing that A is open. \square

8.1. Cut points. In this section we consider Riemannian manifolds. The notion of a cut point in a more general space, such as a complete locally compact interior metric space, or even a complete Riemannian manifold with boundary, is difficult to formulate so as to retain the nice properties that cut points have in complete Riemannian manifolds. Also, they don't make much sense in noncomplete Riemannian manifolds.

If γ is a geodesic, $p = \gamma(0)$, then $q = \gamma(s)$ is a *cut point* (or *minimum point*) of p along γ if γ realizes the distance $d(p, q)$, but for every $r > s$, γ does not realize the distance $d(p, \gamma(r))$.

Theorem 8.5. *If q is a cut point of p , then either there are two minimizing segments from p to q or q is a first conjugate point of p along a minimizing segment.*

Proof. . Let $q = \gamma(s)$ and suppose that $s_n > s$ is a sequence converging to s and that σ_n is a curve from p to $\gamma(s_n)$ such that the length of σ_n is less than the length of the segment of γ from p to $\gamma(s_n)$. Since we have assumed that the space is complete, we may as well suppose that σ_n is a minimizing segment. We take a convergent subsequence, the limit of which will be a minimizing segment from p to q . If the limit is γ , then \exp_p is not one-to-one in a neighborhood of $s\gamma'(0)$; this shows that q is a conjugate point of p along γ . \square

Corollary 8.6. *The relation “is a cut point of” is symmetric.*

This follows from the fact that both relations “has two or more minimizing geodesic to” and “is a first conjugate point of” are symmetric. The symmetry of the latter follows from the characterization of conjugate points by Jacobi fields.

It is not hard to show that the distance to a cut point from p along the geodesic with direction $v = \gamma'(0)$ is a continuous function of v to the extended positive reals. The manifold is compact if and only if there is a cut point in every direction. If we remove the cut locus of p from the manifold, then the remaining subset is diffeomorphic to a star-shaped open set in the tangent space via \exp_p . In particular, the noncut locus is homeomorphic to an open n -ball. This shows that all the nontrivial topology of the manifold is conveyed by the cut locus and how it is glued onto that ball. The cut locus is the union of some conjugate points, which have measure 0 by Sard's Theorem, and subsets of hypersurfaces obtained by equating “local distance functions from p ”, the localization being in neighborhoods of minimizing geodesics to the cut points. Thus, the cut locus has measure 0. Other than that it can be very messy; H. Gluck and M. Singer have constructed examples for which the cut locus of some point is nontriangulable.

9. CURVATURE AND TOPOLOGY

9.1. Hadamard manifolds. A complete Riemannian manifold with nonpositive curvature is called an *Hadamard manifold*. The two-dimensional case was studied by Hadamard, and then his results were extended to all dimensions by Cartan. There is a further generalization to manifolds for which some point has no conjugate points. We first show that nonpositive curvature implies that there are no conjugate points.

Theorem 9.1. *If M is a Riemannian manifold with nonpositive curvature, then there are no conjugate points.*

Proof. We have to show that if Y is a nonzero Jacobi field such that $Y(0) = 0$, then Y is never 0 again. Note that $Y'(0) \neq 0$, since $Y(0)$ and $Y'(0)$ are deterministic initial conditions for the second-order Jacobi differential equation. Now differentiate $g(Y, Y)$ twice:

$$g(Y, Y)' = 2g(Y, Y'),$$

which is 0 at time 0, and

$$g(Y, Y)'' = 2g(Y', Y') + g(Y, -R_{XY}X),$$

which is positive at time 0. But we may assume that Y is perpendicular to the base geodesic, and that the velocity field of that geodesic is a unit vector field X . Then $g(Y, R_{XY}X) = K_{XY}g(Y, Y) \leq 0$. Hence, $g(Y, Y)'' \geq 0$, so that $g(Y, Y)$ only vanishes at time 0. \square

Theorem 9.2 (The Hadamard-Cartan Theorem). *If M is a complete Riemannian manifold having a point p such that p has no conjugate points, then \exp_p is a covering map. If curvature is nonpositive, then that is true for every p and every fixed-end homotopy class of curves contains a unique geodesic segment. If M is simply connected, then it is diffeomorphic to \mathbf{R}^n via \exp_p .*

Proof. We are given that \exp_p is a local diffeomorphism everywhere. From covering space theory it is sufficient to show that it has the path-lifting property: given a curve $\gamma : [0, 1] \rightarrow M$ and a point v such that $\exp_p(v) = \gamma(0)$ we must show that there is a curve $\bar{\gamma}$ such that $\exp_p \circ \bar{\gamma} = \gamma$ and $\bar{\gamma}(0) = v$.

We can pullback the metric on M to get a metric \exp_p^*g on M_p . The radial lines from the origin are geodesics in this metric, so that by the H-R-CV Theorem the new metric on M_p is complete. We can use the local regularity of \exp_p to lift an open arc of a curve about any point where we have it lifted already. Thus, it becomes a matter of extending from a halfopen interval to the extra point. On a large closed ball in M_p , which is compact, there will be a positive lower bound on the amount distances will be stretched by \exp_p ; hence a Cauchy sequence approaching the open end in M will be lifted to a Cauchy sequence in M_p , providing the point to continue the lift. \square

Theorem 9.3 (Myers' Theorem). *Let M be a complete Riemannian manifold such that there is a positive number c for which $\text{Ric}(v, v) \geq (n-1)cg(v, v)$ for all tangent vectors v . Then M is compact with diameter at most π/\sqrt{c} .*

Proof. We show that along every geodesic there must be a conjugate point within distance π/\sqrt{c} . Let γ be a geodesic parametrized by arc length, and let (E_i) be a parallel frame field along γ with $E_n = \gamma'$. We define fields along γ which would be Jacobi fields vanishing at 0 and π/\sqrt{c} if M had constant sectional curvature c :

$$V_i(s) = \sin \sqrt{cs} E_i(s),$$

for $i = 1, \dots, n-1$.

Then the index form has value

$$I(V_i) = \int_0^{\pi/\sqrt{c}} (c \cdot \cos^2 \sqrt{cs} - K_{\gamma'} V_i \sin^2 \sqrt{cs}) ds.$$

We are given that $\sum K_{\gamma'V_i} = Ric(\gamma', \gamma') \geq (n-1)c$, so that if we add the index forms we get

$$\sum_{i=1}^{n-1} I(V_i) \leq c \cdot \int \cos 2\sqrt{c}s \, ds = 0.$$

If there were no conjugate point on the interval in question, then the basic inequality tells us that the index form would be positive definite. Hence there is a conjugate point and there can be no point at distance from $\gamma(0)$ greater than π/\sqrt{c} . \square

Remark 9.4. We have proved a slightly better result than claimed. We don't have to assume that *all* Ricci curvatures are positively bounded below, but only those for tangents along geodesics radiating from a single point. Then we still get compactness, but not the estimate on the diameter.

Problem 9.5. The *radius* of a Riemannian manifold M is the greatest lower bound of radii of metric balls which cover M . Prove that in general $radius \leq diameter \leq 2 \cdot radius$. Moreover, the upper bound $radius \leq \pi/\sqrt{c}$ can be obtained from completeness and the assumption that the Ricci curvatures of tangents along geodesics radiating from a single point have the lower bound assumed in Myers' Theorem.

When $n = 2$ the condition on Ricci curvature reduces to a lower bound $K \geq c$ on the Gaussian curvature; the conclusion of Myers' Theorem was known for this case much earlier and this result is called Bonnet's Theorem.

9.2. Comparison Theorems. . There is an improved method of doing comparison theorems, refining the technique of using Jacobi fields as in Bishop & Crittenden, employing Riccati equations as well. A good reference for this approach is

J.-H. Eschenburg, Comparison Theorems and Hypersurfaces, *Manuscripta Math.* 59(1987), 295-323.

One of the starting points of modern comparison theory is the Rauch Comparison Theorem. It says that if we have an inequality on sectional curvatures at corresponding points of two geodesics, then the opposite inequality holds for corresponding exponentiated tangent vectors. In effect we compare growth of Jacobi fields when we are given a curvature comparison and the same initial conditions for the Jacobi fields.

The reason that Riccati equations are sometimes more convenient is that they come close to conveying just the right amount of information, while the Jacobi equation has too much detail. If we are concerned with estimating the distance to a conjugate point, we are interested in whether there is a one-dimensional subspace of Jacobi fields with zeros at two points; the Jacobi equation determines individual members of that subspace, while the Riccati equation is aimed at the subspace itself. For the two-dimensional case the interest centers on the fields orthogonal to a geodesic, so that the equations are given in terms of one scalar coefficient f . The Jacobi equation is $f'' + Kf = 0$, where K is the Gaussian curvature along the geodesic. The Riccati equation is $h' + h^2 + K = 0$, where h is related to f by $h = f'/f$. The distance between conjugate points is the distance between two singularities of h . Those singularities are all of the same sort, with h approaching $-\infty$ from the left, $+\infty$ from the right, and asymptotically $h(s)$ behaves like $\pm(s -$

$c)^{-1}$ at a singularity c . For the higher dimensional case we deal with the vector Jacobi equation, while the Riccati equation is a matrix Riccati equation whose solutions package all the solutions of the Jacobi equation.

Geometrically the matrix of the Riccati equation represents the second fundamental forms of a wave front, so the Riccati equation itself expresses how the relative geometry of those wave fronts evolve as one moves orthogonally to them. When we radiate from a point the wave fronts are metric spheres, but the equations have the same form for wave fronts radiating from any submanifold.

To describe a *wave front*, or family of parallel hypersurface all that is required is a real-valued function having gradient of unit length everywhere: $f : M \rightarrow \mathbf{R}$, such that $g(df, df) = 1$. Then the hypersurfaces are $S_t = \{p : f(p) = t\}$, the level hypersurfaces of f . We let $X = \text{grad } f$, the metric dual of df ; that is, the vector field such that $g(X, Y) = df(Y)$ for all vector fields Y . Then we calculate for any tangent vector y :

$$0 = y g(X, X) = 2g(D_y X, X).$$

We can take an extension Y of y such that $[X, Y] = 0$ and $g(X, Y)$ is constant, hence $0 = Xg(X, Y) = g(D_X X, Y) + g(X, D_Y X)$, so that

$$D_X X = 0.$$

Thus, the integral curves of X are geodesics. The distances between the hypersurfaces S_t are measured along these geodesics.

A case of particular importance is $f(q) = d(p, q)$, the distance function defined on a deleted normal neighborhood of p , for which the wave fronts are the concentric spheres about p .

Let $B = DX$, the Hessian tensor of f . Then X is in the nullspace of B , so that we will be mainly concerned with the restriction of B to the normal space X^\perp . As defined by $B = DX$, B is a linear map $Y \rightarrow D_Y X$, which is the shape operator (Weingarten map) of the hypersurfaces S_t . But we also can view B as the second fundamental form, the symmetric bilinear form $(Y, Z) \rightarrow B(Y, Z) = g(D_Y X, Z)$. Usually it will be the operator version that occurs here.

Suppose that J is a vector field orthogonal to X such that $[X, J] = 0$. Then $D_J X = D_X J = BJ$. Applying D_X again we get

$$\begin{aligned} D_X(BJ) &= (D_X B)J + BD_X J = (D_X B)J + B^2 J \\ &= D_X D_J X = D_J D_X X - R_{XJ} X = -R_{XJ} X. \end{aligned}$$

Define the symmetric linear operator R_X by $R_X J = R_{XJ} X$. Since the vector field J can have arbitrary pointwise values perpendicular to X , we get the following operator Riccati equation for B :

$$D_X B + B^2 + R_X = 0.$$

Still assuming that $[X, J] = 0$, we get

$$D_X D_X J = D_X D_J X = D_J D_X X - R_{XJ} X = -R_X J,$$

which is the Jacobi equation; so such a J must be a Jacobi field along the integral curves of X . We can take $n - 1$ such fields J_1, \dots, J_{n-1} which are orthogonal to X at some point, and use them to make a linear isomorphism $\mathbf{R}^{n-1} \rightarrow X^\perp$ which we denote by J . The row of derivative fields $D_X J = J'$ can also be regarded as such a linear map, so that $\tilde{B} = J' J^{-1}$ makes sense as a linear operator on X^\perp .

Writing the definition of \tilde{B} as $J' = \tilde{B}J$, the fact that \tilde{B} satisfies the same Riccati equation is just a repeat of the previous calculation. Finally, we can rig J and J' at one point so that \tilde{B} coincides with B at that point, hence everywhere. This establishes the usual relation between the Riccati equation and the corresponding second-order linear equation, here the Jacobi equation.

9.3. Reduction to a Scalar Equation. If we assume that B has a simple eigenvalue λ , then locally λ will be a smooth function and will have a smooth unit eigenvector field U . We show that λ satisfies a scalar Riccati equation of the same form, where the driving operator R_X is replaced by a sectional curvature function.

$$\begin{aligned} X\lambda = \lambda' &= g(BU, U)' = g(B'U + BU', U) + g(BU, U') \\ &= -g(B^2U, U) - g(R_XU, U) = -\lambda^2 - K_{XU}. \end{aligned}$$

We have used the symmetry of B and the fact that $g(BU, U') = \lambda g(U, U') = 0$, since the derivative of a unit vector is always perpendicular to the unit vector itself.

We cannot generally assume that the eigenvalues of B will be simple. However, we can always perturb an initial value of B so that the perturbed solutions of the Riccati equation will have simple eigenvalues locally. Then an upper or lower bound on the eigenvalues derived from the scalar equation can be applied to the eigenvalues of the matrix equation by taking a limit as the perturbations go to 0. This will serve our purposes even in case B has multiple eigenvalues.

9.4. Comparisons for Scalar Riccati Equations. We consider the Riccati equations of the form $f' = -f^2 - H$, where f, H are real-valued functions of a real variable. In our geometric applications H will be the sectional curvature of a section tangent to a geodesic, and f has interpretations as a principal normal curvature (eigenvalue of the second fundamental forms) of a wave front, or a connection coefficient for a frame field adapted to the setting.

A basic trick in dealing with the scalar Riccati equation is the change to the corresponding linear homogeneous second order equation. We let $f = j'/j$, and then easily calculate $j'' = -Hj$. Conversely, a solution of the second order equation leads to a solution of the Riccati equation. Of course, j is only determined up to a ratio. The trick may be viewed as splitting the second order equation into two first order steps, the Riccati equation and the linear equation $j' = -fj$. We assume that H is continuous.

Lemma 9.6. *A solution f on $(0, a)$ either extends continuously to a solution in a neighborhood of 0, or $\lim_{t \rightarrow 0+} tf(t) = 1$. In either case f is uniquely determined on $(0, a)$ by its value $f(0+)$, whether finite or $+\infty$. Similarly, f on $(-a, 0)$ is uniquely extendible to $f : (-a, 0] \rightarrow [-\infty, +\infty)$.*

Proof. . We must have a Taylor expansion $j(t) = c + bt + O(t^2)$ for the corresponding linear equation solution. If $c \neq 0$, $f(t) = \frac{b + O(t)}{c + bt + O(t^2)}$ gives us the continuous extension $f(0) = b/c$. If $c = 0$, then $f(t) = \frac{b + O(t)}{bt + O(t^2)} \rightarrow +\infty$ as $t \rightarrow 0+$ and $tf(t) \rightarrow 1$. The second order equation determines a solution j such that $j(0) = 0$ up to a constant multiple, so that f is uniquely determined when 0 is a singularity. \square

Lemma 9.7. *If we determine a unique solution f_r for $r \neq 0$ by $f_r(r) = 1/r$, then the unique solution singular f at 0 is given by $f(t) = \lim_{r \rightarrow 0+} f_r(t)$.*

Proof. . Take the limit of the corresponding solution j_r for j , which satisfies initial conditions $j_r(r) = r$, $j'_r(r) = 1$. \square

Theorem 9.8 (Driving Function Comparison Theorem). *If $f' = -f^2 - H$, $g' = -g^2 - K$, $f(0) = g(0)$ [which may be $+\infty$], and $H \geq K$, then g exists on at least as great an interval $[0, a)$ as does f , and $f \leq g$ on that interval.*

[Note that f continues until $f(t) \rightarrow -\infty$ as $t \rightarrow a-$.]

Proof. . Let $h = g - f$. Then $h' + (g + f)h = -K + H \geq 0$. If $f(0)$ is finite, we multiply both sides of this inequality by $\exp(\int (f + g)(u)du = k$ to get $(kh)' \geq 0$. Since $h(0) = 0$ and $k \geq 0$, we conclude that $h(t) = g(t) - f(t) \geq 0$ on $[0, a)$. But g can only become singular by going to $-\infty$, so that g must exist on $[0, a)$ too. \square

If $f(0) = +\infty$, then we set $f_r(r) = g_r(r) = 1/r$, use the result just proved, and take a limit as $r \rightarrow 0+$.

Theorem 9.9 (The Sturm Comparison Theorem). *If $j'' = -Hj$, $k'' = -Kk$, $j(0) = k(0) = 0$, $H \geq K$, and these solutions are not trivial, then the next 0 of j occurs at or before the next 0 of k .*

Theorem 9.10 (Value Comparison Theorem). *If $f' = -f^2 - H$, $g' = -g^2 - H$, and $f(0) \leq g(0)$, then $f \leq g$ on the maximal interval $[0, a)$ on which f exists.*

Proof. Again let $h = g - f$, so that $h' + (g + f)h = 0$. Clearly $h \geq 0$ on $[0, a)$. \square

Theorem 9.11 (Rauch Comparison Theorem). *Let M and N be Riemannian manifolds, γ and σ unit speed geodesics in each, $X = \gamma'$ and $Y = \sigma'$ their unit tangent vector fields. Suppose that for every pair of vector fields Z and W orthogonal to γ and σ , respectively, we have an inequality on sectional curvatures at corresponding points: $K_{XZ} \leq K_{YW}$. Let J and L be nonzero Jacobi fields orthogonal to γ and σ , respectively, such that $J(0) = 0$, $L(0) = 0$, and $J'(0)$, $L'(0)$ have the same length. Then $g_M(J, J)/g_N(L, L)$ is nondecreasing for $s > 0$; in particular, J is at least as long as L .*

Proof. We start by calculating the logarithmic derivative of the length of a Jacobi field J :

$$(\log g(J, J))' = 2g(J', J)/g(J, J) = g(BJ, J)/g(J, J).$$

For Jacobi fields vanishing at an initial point the operators B are the shape operators of the spherical wave front about that point. On both manifolds these operators have a simple pole $B \sim (1/s)I$ as their initial conditions. By the driving function comparison theorem the eigenvalues of the operators for the spheres on M and N are related oppositely to the relation for curvature. That inequality on eigenvalues is then passed on to an inequality for the logarithmic derivatives, and we have supposed that J and L are asymptotically the same at $s = 0$. \square

Note that we did not have to assume that the dimensions are the same. The most common application is a comparison to constant curvature spaces, which can be stated conveniently as follows.

Theorem 9.12 (Constant Curvature Comparison Theorem). *Suppose that the sectional curvatures of M are bounded by constants: $a \leq K \leq b$. Let $S(a)$ and $S(b)$ be the simply connected complete Riemannian manifolds of constant curvatures a and b , of the same dimension as M . Let \exp_a , \exp_b , and \exp_p be exponential maps for*

$S(a)$, $S(b)$, and M , respectively, each restricted to a normal neighborhood. Then $\exp_p \circ \exp_a^{-1}$ is length nonincreasing and $\exp_p \circ \exp_b^{-1}$ is length nondecreasing. (For convenience we have identified the three tangent spaces by some Euclidean isometry.)

To keep the directions of the inequalities correct you should always bear in mind particular comparisons, say of the Euclidean plane with the unit sphere: it is easy to visualize that the Euclidean lines spread apart faster than great circles making the same initial angle.

An equivalent way of viewing the constant curvature comparison is in terms of triangles. The triangles compared should be sufficiently small so that they lie in a normal coordinate neighborhood and in the sphere are uniquely determined by the three side lengths. For a given triangle in M the *comparison triangle* is the triangle in the constant curvature surface $S(a)$ having the same side lengths. Then an inequality on curvatures, say, $a \leq K$, is conveyed by inequalities between the angles and corresponding distances across the two triangles: the angles are smaller and the distance shorter in the comparison triangle than in the given triangle.

Alternatively, instead of making the three sides the same, one can make two sides and the included angle the same in the given triangle and the comparison triangle, with obvious consequent inequalities between the other corresponding “parts” of the triangles. This called *hinge comparison*.

Alexandrov has turned these triangle comparisons into definitions, for geodesic metric spaces, of what it means for the space to have curvature bounded above or below by a constant. This allows an extension of many ideas of Riemannian geometry to “singular” spaces. For example, he proves that if curvature is bounded above, then the angle between two geodesic rays with a common starting point is well-defined and satisfies many of the usual properties. However, an angle and its supplementary angle has sum $\geq \pi$, but equality may fail. The metric completion of the logarithm spiral surface covering the punctured Euclidean plane has curvature ≤ 0 , but geodesic rays starting at the singular point can have arbitrarily large angle between them. Generally, in spaces with curvature bounded above geodesics may bifurcate (which is an indication of some infinitely negative curvature), but locally a geodesic segment is uniquely determined by its ends.

The opposite case of spaces with curvature bounded below has also been studied. Here again angles are meaningful; for the two-dimensional case, the sum of angles about a point can be at most 2π , and if it is less, the point is regarded as having positive curvature measure. Geodesics cannot bifurcate, but local bipoint uniqueness may fail and indicate positive infinite curvature. Examples of this sort are obtain by gluing two copies of a convex Euclidean set along their boundaries (the *double* of the set).

When a locally compact metric space has curvature bounded both above and below then it is very close to being a manifold. To make it be a manifold we only have to assume one further very natural property: geodesics must be locally extendible. With this hypothesis, Nikolaev proved that there is a $C^{3,\alpha}$ manifold structure and the metric is given by a $C^{1,\alpha}$ Riemannian metric. The number α is a Hölder exponent for the last derivatives, and can be any number between 0 and 1.

Curvature bounds defined in Alexandrov's way are easily seen to be inherited by the limits of spaces, for some reasonable notions of such limits. Thus, Nikolaev's theorem has an important consequence that some limits of Riemannian manifolds are actually Riemannian manifolds. There is a general compactness theorem of Gromov which shows that many such limits exist.

Another kind of result which Alexandrov was able to abstract to spaces with curvature bounded above was the proof that one could compare certain global triangles, given that local ones could be compared. For example, he proved a generalization of the Hadamard-Cartan Theorem to locally compact complete geodesic metric spaces with curvature bounded above by 0. Recently S. Alexander and I generalized this even more, using instead the weaker assumption of geodesic convexity and eliminating the hypothesis of local compactness.

9.5. Volume Comparisons. For a Riemannian manifold we can get comparisons between the volume of balls and spheres (and more generally, tubes) and corresponding volumes in constant curvature spaces founded on curvature inequalities. For *lower* bounds on volume it is hard to do much better than to assume upper bounds on *sectional* curvature and apply the length nondecreasing maps that we get from the Rauch comparison theorem. The more interesting case is to get *upper* bounds on volumes from the weaker assumption of lower bounds on *Ricci* curvature.

Theorem 9.13 (Bishop's Volume Comparison Theorem). *If $\text{Ric}(X, X) \geq (n-1)K$ for all unit vectors $X = \text{grad}(d(p, \cdot))$, then for each ball $B_p(r)$ and sphere $S_p(r) = \partial B_p(r)$ the Riemannian volume, n -dimensional and $(n-1)$ -dimensional, respectively, is less than or equal to that for a ball or sphere of the same radius in a space of constant curvature K .*

For $K > 0$ the result holds for all $r \leq \pi/\sqrt{K}$; for $K \leq 0$ the result holds for all r . In either case it is permissible to let \exp_p be noninjective while its counterpart in the constant curvature space is injective, since counting parts of volume more than once enhances the inequality. This refinement is now attributed to Gromov, but I knew it and thought it was so trivial as to be unnecessary to say explicitly. However, it turned out to be important in applications.

The method of proof is to estimate the Jacobian determinant of the exponential map, by calculating the logarithmic derivative. That much is similar to the proof of the Rauch theorem. We express that Jacobian determinant in terms of the length of an $(n-1)$ -vector:

$$J_1 \wedge \dots \wedge J_{n-1} = j \cdot E_1 \wedge \dots \wedge E_{n-1}.$$

This was done directly in the original proof, given in Bishop & Crittenden, Chapter 11. Now the fashion is to use an operator Riccati equation as intermediary, converting it to a scalar Riccati equation for j'/j by taking the trace.

INDEX

- AA – Arzelà-Ascoli, 10
- adapted frame field, 42
- affine space, 12
- Alexander, S., 61
- Alexandrov space, 60
- Alexandrov, A.D., 60
- Ambrose, W., 20
- Arzelà, C., 9
- Arzelà-Ascoli Theorem, 9, 32, 49, 53
- Ascoli, G., 9
- autoparallel curve, 18

- Banach space, 8
- barrier, 29
- Bartle, R.G., 9
- basic inequality, 47
- basic vector fields, 16
- Bianchi identity, 36
- Bishop’s Volume Comparison Theorem, 61
- Bonnet’s Theorem, 56
- bounded curvature, 59
- bundle of bases, 12, 14
- bundle of frames, 23, 26

- calculus of variations, 30
- Cartan Local Isometry Theorem, 37, 41
- Cartan’s Lemma, 25, 49
- Cartan, E., 40
- Cauchy sequence, 51
- Cayley-Hamilton theorem, 6
- Christoffel symbols, 25, 41
- Clairaut’s Theorem, 29
- cobasis
 - universal, 1, 14
- coframe, 5, 25, 37, 41
 - universal, 27, 32
- Cohn-Vossen, S., 52
- complete
 - geodesically, 19, 21, 22
- completeness, 51
 - development-, 21
 - geodesic, 52
- completion, 51
- concentric spheres, 57
- conjugate homotopy class, 49
- conjugate point, 44, 54
- connection, 10, 15, 17
 - 1-forms, 11, 16
 - existence, 11, 16
 - induced, 15
 - isometric imbedding, 42
 - law of change, 13
 - Levi-Civita, 23
 - metric, 23
 - of parallelation, 12
 - of parallelization, 21
 - on curve, 12
 - on map, 12, 32
 - pullback to curve, 15
 - Riemannian, 23
 - symmetric, 13
 - torsion of parallelization, 13
- conservation of energy, 29
- conservative system, 48
- covariant derivative, 10, 17
 - exterior, 36
 - tensors, 22
- covariant differential, 36
- critical energy, 31
- critical length, 31
- Crittenden, R.J., 43, 49, 53, 61
- cross-section, 14
- curvature, 13, 17
 - 2-form, 13, 17
 - bounded, 59
 - conformal tensor, 40
 - constant, 27, 36, 38
 - decomposition of tensors, 40
 - nonpositive, 54
 - operator, 13
 - pointwise realization, 43
 - positive, 50, 55, 60
 - Ricci, 39, 40, 56, 61
 - scalar, 39
 - sectional, 37
 - space of tensors, 38
 - symetries, 27
 - symmetries, 35
 - Weyl tensor, 40
- cut locus, 50
 - nontriangulable, 54
- cut point, 50, 54

- de Rham, G., 52
- derivative
 - Lie, 28
- development of curve, 21
- Dirichlet principle, 32
- distance, 7
- divergence, 39

- elapsed time, 31
- energy, 30
- energy-critical curve, 32
- equivariant form, 16
- Eschenburg, J.-H., 56
- Euler characteristic, 4
 - Lorentz manifold, 4
- Euler equations, 30
- Euler method, 9
- exponential map, 22, 26

- Finsler metric, 7

- Finsler, P., 7
- first structural equation, 17
- first variation
 - arclength, 32
 - energy, 32
- focal point, 45
- form
 - equvariant, 17
 - horizontal, 17
- frame, 23
 - bundle, 23
- frame field
 - local, 5
- fundamental group, 49, 50
- Fundamental Theorem of Riemannian Geometry, 24
- fundamental vector fields, 15
- Gallilean boost, 38
- Gauss's Lemma, 32
- Gaussian curvature, 37, 56
- general linear group, 14
- general relativity, 4
- geodesic, 9, 32
 - closed, 49
 - existence, 19
 - loop, 49
 - maximal, 52
 - of connection, 18
 - parallelogram, 19
 - periodic, 49
 - pre-, 22
 - surface of revolution, 29
- geodesic loop, 50
- geodesic space, 51
 - locally, 51
- Gluck, H., 54
- Goldberg, S.I., 12, 30
- Gramm-Schmidt procedure, 5
- Grassmann algebra, 6
- Gromov Compactness Theorem, 61
- Gromov, M., 61
- Hadamard manifold, 54
- Hadamard-Cartan Theorem, 55, 61
- Hausdorff separation axiom, 8
- heat equation, 3
- Hessian, 30, 36, 57
- hinge comparison, 60
- Hodge star operator, 40
- holonomy, 20
- holonomy, 19
 - group, 20
- homotopy class, 49
 - free, 49
- Hopf index theorem, 4
- Hopf, H., 52
- Hopf-Rinow Theorem, 52
- Hopf-Rinow-CohnVossen Theorem, 52
- horizontal lift, 15
- imbedding
 - isometric, 25
- index form, 46
- isometry, 25
- Jacobi equation, 34, 56
 - of sphere, 34
- Jacobi field, 33
- Killing field, 28, 48
- Klingenberg's Theorem, 50
- Klingenberg, W., 50
- Koszul formula, 25
- Laplace-Beltrami, 3
- length, 7, 8
- Levi-Civita, T., 10
- Lie algebra
 - orthogonal group, 23
- logarithmic covering, 51
- logarithmic spiral surface, 60
- longitudinal curve, 30
- Lorentz manifold, 31
- Lorentz structure, 4
- Maurer-Cartan equations, 18, 48
 - orthogonal group, 27
- meridian, 30
- metric
 - intrinsic, 7
- metric space
 - complete, 51
 - intrinsic, 44, 51
- minimal locus, *see also* cut locus
- minimizer, 9
 - existence, 9
 - product space, 10
- Minkowski space, 26
- Morse theory, 30
- Morse, M., 30
- Myers' Theorem, 55
- Myers, S., 55
- Nikolaev, I.G., 60
- nonminimization
 - beyond conjugate point, 46
- normal coordinates, 22
 - Taylor expansion, 41
- nullspace
 - index form, 46
- O'Neill, B., 4, 32
- orientable, 4, 39, 50, 51
- orthogonal group, 23
- Palais, R.S., 20
- parallel translation, 11, 15
 - tensors fields, 22

- parallelizability
 - bundle of basis, 16
- parallelization, 12
 - connection of, 21
 - geodesics of connection, 21
- parallelogram law, 7
- parametrization
 - normalized, 45
- partition of unity, 4, 11, 13
- polarization, 7, 43
- pregeodesic, 22, 32
- product metric, 10

- quadric hypersurface, 28

- radius of Riemannian manifold, 56
- Rauch Comparison Theorem, 56, 59
- Rauch, H., 56
- rectangles - smooth, 30
 - existence, 31
- Riccati equation, 56, 61
 - matrix, 57
 - scalar, 58
- Ricci curvature, 39, 61
- Ricci tensor, 38
- Riemann tensor, 41
- Riemann's Theorem, 43
- Riemann, G.B., 41
- Riemannian metrics, 4
- right action, 14
- Rinow, W., 52

- Sard's Theorem, 44, 54
- Sard, Arthur, 44
- scalar curvature, 39
- second fundamental form, 45, 57
- second structural equation, 13, 17
- second variation, 45
- sectional curvature, 37
- semi-Euclidean motion group, 26
- semi-Euclidean spaces, 26
- semi-Riemannian metric, 4
- simply connected, 50
- Singer, I.M., 20
- Singer, M., 54
- singular space, 60
- solder form, 1
- space-like, 31
- spheres
 - constant curvature, 27
- sprays, 20
- Sturm Comparison Theorem, 59
- subbundle, 1
 - horizontal, 15
 - vertical, 15
- Synge's formula
 - 2nd variation, 46
- Synge's Theorem, 50
- Synge, J.L., 46

- taxicab metric, 10, 32
- time-like, 31
- topology of Finsler manifold, 7
- torsion, 12, 17
 - 2-form, 13
- transverse curve, 30
- triangle comparison, 60
- triangle inequality
 - reversed, 32

- uniform equicontinuity, 9
- universal cobasis, 1, 14

- wave front, 57
 - spherical, 59
- Weingarten map, 57
- Weyl tensor, 40

- Yamabe problem, 32
- Yang-Mills extremals, 40