ON INEQUALITIES AND LINEAR RELATIONS FOR 7-CORE PARTITIONS

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Dedicated to my advisor, Bruce C. Berndt on the occasion of his seventieth birthday.

Abstract. Recently, Ramanujan's modular equations have been applied by N.D. Baruah and B.C. Berndt to obtain a linear relation for 5-core partitions and by A. Berkovich and H. Yesilyurt to obtain inequalities for 7-core partitions. In this paper, we generalize their results by using the theory of modular forms. In particular, we prove conjectures of Berkovich and Yesilyurt.

1. Introduction and statement of results

A partition λ of n is a non-increasing sequence of natural numbers whose sum is n. A partition λ is said to be a t-core if there are no hook numbers that are multiples of t. For more information on t-core partitions and their roles in representation theory, consult [5]. For example, in Figure 1, λ is a 5-core partition. Let $a_t(n)$ be the number of t-core partitions

8	6	4	3	1
6	4	2	1	
3	1			
1				

Figure 1. a partition λ with hook numbers.

of n. Then, it is well-known that

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}. \tag{1}$$

Here and in the sequel, we use the following standard q-series notation:

$$(a;q)_0 := 1, \ \ (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \ n \geq 1,$$

and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \ |q| < 1.$$

Recently, Ramanujan's modular equations have been applied by N.D. Baruah and B.C. Berndt [1] to obtain a linear relation for 5-core partitions and by A. Berkovich and H.

Yesilyurt [2] to get inequalities for 7-core partitions. In particular, in [1, Theorem 4.2], Baruah and Berndt proved that

$$a_5(4n+3) = a_5(2n+1) + 2a_5(n), \tag{2}$$

and in [2, Theorem 1.1], Berkovich and Yesilyurt proved the inequalities

$$a_7(2n+2) \geq 2a_7(n) \quad \text{and} \quad a_7(4n+6) \geq 10a_7(n),$$

for all $n \geq 0$. In the same paper, Berkovich and Yesilyurt conjectured the stronger inequalities:

$$a_7(2n+2) \geq 3a_7(n), \tag{3}$$

$$a_7(4n+6) \geq 15a_7(n), \tag{4}$$

for all $n \geq 1$.

In this note, we will prove the following theorem.

Theorem 1. *Let k be a nonnegative integer. Then for all $n \geq \frac{2^{2k-2}+2^{3k-1}+2^{4k-2}}{7} \cdot \pi^4$, we have*

$$a_7(2^k n + 2^{k+1} - 2) \geq (4^k - 1)a_7(n).$$

By looking at the cases $k = 1$ and $k = 2$ in Theorem 1, we will show that (3) and (4) are true.

In light of (3), it is natural to seek inequalities for 7-core partitions of the form

$$a_7(pn + p - 2) \geq A(p)a_7(n), \quad \text{for all } n \geq 1, \tag{5}$$

where p is an odd prime and $A(p)$ is an integer depending on p. Though the method we will use to prove Theorem 1 can be applied to find inequalities like (5), we will use Hecke operators to prove the following theorem. This approach might not give an optimal inequality, but we do not need to calculate the first few values of $a_7(n)$ to verify inequalities.

Theorem 2. *Let p be an odd prime. If $p \equiv 1,\ 2,$ or $4 \pmod 7$, then for all $n \geq 1$, we have*

$$a_7(pn + 2p - 2) > \frac{(p^2-1)}{8} a_7(n). \tag{6}$$

If $p \equiv 3,\ 5,$ or $6 \pmod 7$, then for all $n \geq 1$ with $(n, p) = 1$, we have

$$a_7(pn + 2p - 2) > \left(\frac{3(p-1)^2}{4} + 21 \right) a_7(n). \tag{7}$$

For example,

$$a_7(3n+4) > 7a_7(n) \quad \text{and} \quad a_7(13n+24) > 106a_7(n),$$

for $n \geq 1$.

The next goal of this article is to obtain linear relations for 7-core partitions, which are analogous to (2). In Section 4, we will note that (2) can be seen from the fact that the generating function for 5-core partitions is essentially a Hecke eigenform, which implies the following generalization of (2). For all $n \geq 0$ and all primes p, we have

$$a_5(p^2n + p^2 - 1) = \left(p + \left(\frac{p}{5}\right)\right) a_5(pn + p - 1) - p\left(\frac{p}{5}\right) a_5(n). \tag{8}$$

By using a similar argument, we can find linear relations of 7-core partitions which are analogous to (8).

Theorem 3. *For a given prime p, we have*

$$a_7(p^2n + 2p^2 - 2) = \begin{cases} (p^2-1)a_7(pn+2p-2) + p^2 a_7(n), \ \textit{if } p \equiv 3,5,6 \pmod 7, \\ (p^2+1)a_7(pn+2p-2) - (p^2 + b(p)(p^2+1) - b(p)^2)a_7(n), \\ \textit{if } p \equiv 1,2,4 \pmod 7, \\ 2a_7(2n+2) + 11a_7(n), \ \textit{if } p = 2, \end{cases}$$

for all n with $(n,p) = 1$, where $b(p) = 2(x^2 - 7y^2)$ and x, y are positive integers satisfying $x^2 + 7y^2 = p$.

Our final aim is to describe how to obtain inequalities or linear relations for t-core partitions, where t is a prime which is larger than 7. In this direction, we prove the following theorem.

Theorem 4. *Suppose that $t \geq 7$ is prime, and $k \geq 1$. Let $\delta_t = \frac{t^2-1}{24}$ and $\sigma_{\frac{t-1}{2},\chi}(n) := \sum_{d|n} \chi\left(\frac{n}{d}\right) d^{\frac{t-3}{2}}$, where $\chi(n)$ is the usual Legendre symbol $\left(\frac{n}{t}\right)$. Then, for all sufficiently large n with $(n,p) = 1$, we have*

$$a_t(p^kn + \delta_t(p^k - 1)) > \left(\sigma_{\frac{t-1}{2},\chi}(p^k) - 1\right) a_t(n). \tag{9}$$

Throughout this paper, the following relation between t-core partitions and the coefficients of an eta-quotient will play an important role.

$$\frac{\eta^t(tz)}{\eta(z)} = \sum_{n=\delta_t}^{\infty} c_t(n)q^n = q^{\delta_t}\frac{(q^t;q^t)^t}{(q;q)_\infty} = \sum_{n=0}^{\infty} a_t(n)q^{n+\delta_t},$$

where $\delta_t = \frac{t^2-1}{24}$ and $\eta(z)$ is Dedekind's eta function defined by $\eta(z) = q^{\frac{1}{24}}(q)_\infty$, where $q = \exp(2\pi i z)$. Thus, we have

$$a_t(n) = c_t(n + \delta_t), \text{ for all } n \geq 0. \tag{10}$$

This article is organized as follows. In Section 2, we will review basic properties of modular forms. In Sections 3 and 4, we will prove inequalities for 7-core partitions. In Section 5, we will study linear relations for 5-core partitions and 7-core partitions. Next, in Section 6, we will give an outline how to obtain inequalities and linear relations of t-core partitions for larger primes t. Finally, we will conclude this article with possible future projects.

2. Preliminary Facts

This section contains the basic definitions and properties of modular forms that we will use later. For additional basic properties of modular forms, see [6, Chaps. 1, 2, and 3].

Define $\Gamma = SL_2(\mathbb{Z})$, $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}$. Let $\mathcal{M}_k(\Gamma)$ (resp. $\mathcal{S}_k(\Gamma)$) denote the vector space of holomorphic forms (resp. cusp forms) of weight k. Let $\mathcal{M}_k(\Gamma_0(N), \chi)$ (resp. $\mathcal{S}_k(\Gamma_0(N), \chi)$) denote the vector space of holomorphic forms (resp. cusp forms) on $\Gamma_0(N)$ with character χ. It is well known that for primes $t \geq 5$, $\frac{\eta^t(tz)}{\eta(z)} \in \mathcal{M}_{(t-1)/2}(\Gamma_0(t), \chi)$, where χ is the usual Legendre symbol mod t.

For a prime p, we need to define the Hecke operators T_p on $\mathcal{M}_k(\Gamma_0(N), \chi)$. If $f(q)$ has the Fourier expansion $f(q) = \sum a(n)q^n$, then

$$T_p f := \sum \left(a(pn) + \chi(p)p^{k-1}a(\frac{n}{p}) \right) q^n.$$

It is a standard fact that $T_p f \in \mathcal{M}_k(\Gamma_0(N), \chi)$ (resp. $\mathcal{S}_k(\Gamma_0(N), \chi)$) if $f \in \mathcal{M}_k(\Gamma_0(N), \chi)$ (resp. $\mathcal{S}_k(\Gamma_0(N), \chi)$. We say that $f(z)$ is an eigenform of T_p if there is a $\lambda_p \in \mathbb{C}$ such that $T_p f = \lambda_p f$. We call $f(z) \in \mathcal{M}_k(\Gamma_0(N), \chi)$ a Hecke eigenform if $f(z)$ is an eigenform of T_p for all primes $p \nmid N$. The space of cusp form $\mathcal{S}_k(\Gamma_0(N), \chi)$ has a subspace $\mathcal{S}_k^{new}(\Gamma_0(N), \chi)$, and we call the Hecke eigenforms in $\mathcal{S}_k^{new}(\Gamma_0(N), \chi)$ newforms. Throughout this paper, we assume that each newform $g(z)$ is normalized so that the Fourier coefficient of q of $g(z)$ is 1. It is well-known that $\mathcal{S}_k^{new}(\Gamma_0(N), \chi)$ has a basis consisting of newforms and if $g(z) \in \mathcal{S}_k^{new}(\Gamma_0(N), \chi)$ is a newform and $g(z)$ has a Fourier expansion of the form $g(z) = \sum_{n=1}^{\infty} b(n)q^n$, then $|b(p)| \leq 2p^{\frac{k-1}{2}}$ for all primes p, by the famous theorem of Deligne.

3. Proof of Theorem 1

Recall that $c_7(n)$ is defined by $\frac{\eta^7(7z)}{\eta(z)} = \sum_{n=2}^{\infty} c_7(n)q^n$. Then, in light of (10), to prove Theorem 1, it suffices to show that

$$c_7(2^k n) \geq (4^k - 1)c_7(n), \text{ for all } n \geq \frac{2^{2k-2} + 2^{3k-1} + 2^{4k-2}}{7} \cdot \pi^4. \tag{11}$$

Define $f(z) := \frac{\eta^7(7z)}{\eta(z)} \in \mathcal{M}(\Gamma_0(7), \chi)$, where $\chi(n) = \left(\frac{n}{7}\right)$. Because $\dim \mathcal{M}_3(\Gamma_0(7), \chi) = 3$, by a simple calculation, we can easily verify that

$$8f(z) = E_3(z) - g(z),$$

where $E_3 \in \mathcal{M}_3(\Gamma_0(7),\chi)$ is the Eisenstein series defined by

$$E_3(z) = \sum_{n=1}^{\infty} \sigma_{3,\chi}(n) q^n,$$

where $\sigma_{3,\chi}(n) := \sum_{d|n} \chi(n/d) d^2$, and $g \in \mathcal{S}_3(\Gamma_0(7),\chi)$ is the newform defined by $g(z) := \eta^3(7z)\eta^3(z)$.

Before starting the proof, we need some lemmas. Define $b(n)$ by $g(z) = \sum_{n\geq 1} b(n) q^n$. The following lemma can also be seen from the fact that $g(z)$ is the CM form arising from the field $\mathbb{Q}(\sqrt{-7})$.

Lemma 5 (Lemma 2 of [3])**.** *Let $n = 7^c p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_t^{b_t}$ be the prime factorization of n into primes $p_i \equiv 1, 2, 4 \pmod 7$ and $q_j \equiv 3, 5, 6 \pmod 7$. Let $p_i = x_i^2 + 7y_i^2$ for some positive integer x_i, y_i for odd primes and $x_i = y_i = 1/2$ when $p_i = 2$. Then,*

$$b(n) = \begin{cases} 0 \text{ , if some } b_j \text{ is odd,} \\ (-7)^c \prod_{i=1}^{s} \frac{\beta_i^{2k+2} - \overline{\beta_i}^{2k+2}}{\beta_i^2 - \overline{\beta_i}^2} \prod_{j=1}^{t} q_j^{b_j} \text{ , otherwise,} \end{cases}$$

where $\beta_i = x_i + y_i\sqrt{-7}$. In particular, we have $b(p_i) = 2(x_i^2 - 7y_i^2)$.

Remark. It is well known that if p is an odd prime that is congruent to 1, 2, or 4 modulo 7, then there are positive integers x and y satisfying $p = x^2 + 7y^2$.

For later use, we need an upper bound for $b(p^k)$ for the primes that are quadratic residues mod 7.

Lemma 6. *For all k, we have*

$$|b(2^k)| \leq \frac{4}{\sqrt{7}} 2^k. \tag{12}$$

Proof. By Lemma 5, we have $b(2^k) = \frac{\beta^{2k+2} - \overline{\beta}^{2k+2}}{\beta^2 - \overline{\beta}^2}$, where $\beta = \frac{1}{2}(1 + \sqrt{-7})$. Thus, we have

$$|b(2^k)| \leq \frac{|\beta|^{2k+2} + |\overline{\beta}|^{2k+2}}{\sqrt{7}} \leq \frac{4}{\sqrt{7}} 2^k.$$

□

Lemma 7. *For all $k \geq 1$ and all odd primes $p \equiv 1, 2, 4 \pmod 7$, we have*

$$|b(p^k)| \leq p^{k+\frac{1}{2}}.$$

Proof. By Lemma 5, we have $b(p^k) = \frac{\beta^{2k+2} - \overline{\beta}^{2k+2}}{\beta^2 - \overline{\beta}^2}$, where $\beta = x + y\sqrt{-7}$ and $x^2 + 7y^2 = p$. Because $7x^2y^2 = x^2(p - x^2)$ and x is an integer, $xy\sqrt{7} \geq \sqrt{p-1}$. Thus, we have

$$|b(p^k)| \leq \frac{|\beta|^{2k+2} + |\overline{\beta}|^{2k+2}}{4xy\sqrt{7}} = \frac{2p}{4xy\sqrt{7}} p^k \leq \frac{2p}{4\sqrt{p-1}} p^k \leq \sqrt{p} p^k.$$

□

We can easily get a lower bound for $\sigma_{3,\chi}(n)$.

Lemma 8 (Lemma 3 of [4]). *For all $n \geq 1$,*

$$\sigma_{3,\chi}(n) \geq n^2 \prod_{\substack{q|n \\ q \text{ prime}}} \left(1 - \frac{1}{q^2}\right).$$

Let us fix a positive integer k. Since $8f(z) = E_3(z) - g(z)$, to prove Theorem 1, by (11), it is enough to show that

$$F(n,k) := \sigma_{3,\chi}(2^k n) - A_k \sigma_{3,\chi}(n) - b(2^k n) + A_k b(n) > 0, \text{ for all } n \geq \frac{2^{2k-2} + 2^{3k-1} + 2^{4k-2}}{7} \cdot \pi^4,$$

where $A_k = 4^k - 1$.

Proof of Theorem 1. Let ℓ be the largest integer satisfying $2^\ell | n$ and denote $n = 2^\ell m$. Then

$$F(n,k) = \sigma_{3,\chi}(2^{k+\ell})\sigma_{3,\chi}(m) - A_k \sigma_{3,\chi}(2^\ell)\sigma_{3,\chi}(m) - b(2^{k+\ell})b(m) + A_k b(2^\ell)b(m).$$

Note that $\sigma_{3,\chi}(2^\ell) = 1 + 4 + \cdots + 4^\ell = \frac{4^{\ell+1}-1}{3}$. Thus, we have

$$\sigma_{3,\chi}(2^{k+\ell}) = (1 + A_k)\sigma_{3,\chi}(2^\ell) + A_k/3.$$

In summary,

$$F(n,k) = \sigma_{3,\chi}(2^\ell)\sigma_{3,\chi}(m) + (A_k/3)\sigma_{3,\chi}(m) - b(2^{k+\ell})b(m) + A_k b(2^\ell)b(m)$$

By Lemma 8,

$$\sigma_{3,\chi}(m) \geq m^2 \prod_{\substack{q|m \\ q \text{ prime}}} \left(1 - \frac{1}{q^2}\right) \geq m^2 \prod_{q \text{ prime}} \left(1 - \frac{1}{q^2}\right) = \frac{6}{\pi^2} m^2.$$

And by Lemma 7, $|b(m)| \leq m^{3/2}$. Therefore,

$$\begin{aligned} F(n,k) &\geq \frac{4^{\ell+1}-1}{3}\frac{6}{\pi^2}m^2 + (A_k/3)\frac{6}{\pi^2}m^2 - \frac{4}{\sqrt{7}}2^{k+\ell}m^{3/2} - A_k\frac{4}{\sqrt{7}}2^\ell m^{3/2} \\ &\geq 4^\ell m^2 \left(\frac{8}{\pi^2} - \frac{4(2^k + A_k)}{\sqrt{7m}2^\ell}\right) \geq 4^{\ell+1}m^2 \left(\frac{2}{\pi^2} - \frac{2^k + 4^k}{\sqrt{7n}}\right). \end{aligned}$$

Thus, if $2\sqrt{7n} > (2^k + 4^k)\pi^2$, then $F(n,k) \geq 0$. This completes the proof of Theorem 1. □

Recall that the conjecture of Berkovich and Yesilyurt ((3) and (4)) are the cases $k = 1$ and $k = 2$ in Theorem 1, respectively. Therefore, to verify them, it suffices to check the first 126 and 1392 values of n, respectively.

Corollary 9. *For all $n \geq 1$, we have*

$$a_7(2n+2) \geq 3a_7(n),$$
$$a_7(4n+6) \geq 15a_7(n).$$

4. Proof of Theorem 2

In fact, we can obtain inequalities of the form $a_7(pn+p-2) > A(p)a_7(n)$ by using the same argument as in Section 3. However, in this section, we will use a different approach to obtain inequalities. This approach might not give an optimal inequality as in Section 3, but we do not need to calculate values of $a_7(n)$ case by case. First, note that

$$\left\{ \frac{\eta^7(z)}{\eta(7z)} = 1 + \cdots, \quad g(z) = \eta^3(z)\eta^3(7z) = \sum_{n=1}^{\infty} b(n)q^n, \quad f(z) = \frac{\eta^7(7z)}{\eta(z)} = \sum_{n=2}^{\infty} c_7(n)q^n \right\}$$

is a basis of $\mathcal{M}_3(\Gamma_0(7), \chi)$. Throughout this section, we will denote $c_7(n)$ by $c(n)$. For the odd primes p, we have

$$T_p f = c(p)g + (c(2p) + 3c(p))f, \tag{13}$$

because $T_p f \in \mathcal{M}_3(\Gamma_0(7), \chi)$. Therefore, we see that

$$T_p f - (c(2p) - 2c(p))f = c(p)(f+g). \tag{14}$$

Recall that $8(f(z)+g(z)) = E_3(z) + 7g(z) = \sum_{n=1}^{\infty}(\sigma_{3,\chi}(n) + 7b(n))q^n$.

Lemma 10. *For all $n \geq 3$, we have*

$$\sigma_{3,\chi}(n) + 7b(n) > 0.$$

Proof of Lemma 10. By an argument similar to the one in Section 3, we have

$$\sigma_{3,\chi}(n) + 7b(n) \geq \frac{6}{\pi^2}n^2 - \frac{8}{\sqrt{7}}n^{3/2} \geq \left(\frac{6}{\pi^2} - \frac{8}{\sqrt{7n}}\right)n^2.$$

Thus, if $n > \frac{16\pi^4}{63} \approx 24.7388$ then $\sigma_{3,\chi}(n) + 7b(n) > 0$. By calculating the first 25 values of $\sigma_{3,\chi}(n) + 7b(n)$, we see that $\sigma_{3,\chi}(n) + 7b(n) > 0$ for all $n \geq 3$. □

Suppose that $\chi(p) = -1$. Recall that $8c(n) = \sigma_{3,\chi}(n) - b(n)$ and $b(p) = 0$, by Lemma 5. Thus, we deduce that

$$T_p f - \frac{7(p^2-1)}{8}f = \frac{p^2-1}{8}(f+g). \tag{15}$$

Thus, by Lemma 10, we see that for all prime $p \equiv 3, 5, 6 \pmod 7$ and all $n \geq 3$, we have

$$c(pn) - \frac{7(p^2-1)}{8}c(n) > 0.$$

By (10), this is equivalent to the claimed inequality (6) in Theorem 2.

By a simple calculation, we see that

$$T_2 f = g + 5f. \tag{16}$$

By Lemma 10, (16) implies that for all odd numbers $n \geq 3$,

$$c(2n) > 4c(n). \tag{17}$$

For the primes $p \geq 3$ with $\chi(p) = 1$, by Lemma 10 and (17), we have

$$c(pn) > 6c(p)c(n),$$

for all $n \geq 3$ with $(n, p) = 1$. Since $b(p) = 2(x^2 - 7y^2)$, where x and y are positive integers satisfying $x^2 + 7y^2 = p$, $b(p) \leq 2p - 28$. In summary, we have deduced that

$$c(pn) > \left(\frac{3(p-1)^2}{4} + 21 \right) c(n),$$

for all $n \geq 3$ with $(n, p) = 1$. By (10), this implies the claimed inequality (7) in Theorem 2.

5. Linear relations of 5, 7-cores

In light of (10), Theorem 4.2 of [1] is equivalent to

$$T_2 \frac{\eta^5(5z)}{\eta(z)} = \frac{\eta^5(5z)}{\eta(z)}.$$

Since $\frac{\eta^5(5z)}{\eta(z)}$ is a Hecke eigenform, we can easily deduce the following generalization of Theorem 4.2 of [1]:

$$T_p \frac{\eta^5(5z)}{\eta(z)} = \left(p + \left(\frac{p}{5} \right) \right) \frac{\eta^5(5z)}{\eta(z)},$$

for every prime p, which implies (8).

The same idea can be used to get linear relations for 7-cores because $T_p f$, $T_p(T_p f)$ and f are linearly dependent. For the primes p with $\chi(p) = -1$, by (13), we have

$$T_p^2 f - (c(2p) + 3c(p)) T_p f = 0.$$

From this, we can deduce that for all integers n,

$$c(p^2 n) = \begin{cases} (p^2-1)c(pn) + p^2 c(n), \text{ if } (n,\ p) = 1, \\ (p^2-1)c(pn) + 2p^2 c(n) - p^2(p^2-1)c(n/p) - p^4 c(n/p^2), \text{ otherwise}, \end{cases} \tag{18}$$

where we understand $c(\frac{n}{m}) = 0$ if $\frac{n}{m}$ is not an integer.

For the primes $p \geq 3$ with $\chi(p) = 1$, by (13), we have

$$T_p^2 f = (c(p)b(p) + 8c(p)^2) g + (8c(p))^2 f.$$

Therefore, we arrive at

$$T_p^2 f - (p^2+1)T_p f + (b(p)(p^2+1) - b(p)^2) f = 0,$$

where $b(p) = 2(x^2 - 7y^2)$ and x, y are positive integers satisfying $p = x^2 + 7y^2$. Thus, we see that

(19)

$$c(p^2 n) = \begin{cases} (p^2+1)c(pn) - (p^2 + b(p)(p^2+1) - b(p)^2)c(n), \text{ if } (n,p) = 1, \\ (p^2+1)c(pn) - (2p^2 + b(p)(p^2+1) - b(p)^2)c(n) + (p^4+p^2)c(n/p) - p^4 c(n/p^2), \text{ otherwise,} \end{cases}$$

where we understand $c(\frac{n}{m}) = 0$ if $\frac{n}{m}$ is not an integer.

When $p = 2$, by a simple calculation, we obtain

$$T_2^2 f = 2T_2 f + 15f. \tag{20}$$

Thus, from (18), (19) and (20), we can conclude Theorem 3.

6. t-cores of prime $t > 7$

Now we will outline how to obtain an inequality or a linear relation for t-cores, where t is a prime that is larger than 7. Recall that $\frac{\eta^t(tz)}{\eta(z)} \in \mathcal{M}(\Gamma_0(t), \chi)$, where $\chi(n) = \left(\frac{n}{t}\right)$. Since $\Gamma_0(t)$ has two cusps, namely 0 and ∞, we have

$$h(z) := \frac{\eta^t(tz)}{\eta(z)} = \sum_{n=0}^{\infty} c_t(n) q^n = e_t E_{\frac{t-1}{2}}(z) + g(z),$$

for a positive constant e_t, where $E_{\frac{t-1}{2}}$ is the Eisenstein series defined by

$$E_{\frac{t-1}{2}}(z) = \sum_{n=1}^{\infty} \sigma_{\frac{t-1}{2},\chi}(n) q^n,$$

where $\sigma_{\frac{t-1}{2},\chi}(n) := \sum_{d|n} \chi\left(\frac{n}{d}\right) d^{\frac{t-3}{2}}$, and $g \in \mathcal{S}_{\frac{t-1}{2}}(\Gamma_0(t), \chi)$. Since t is a prime, g is in $\mathcal{S}^{new}_{\frac{t-1}{2}}(\Gamma_0(t), \chi)$.

Remark. In [4], A. Granville and K. Ono noted that G. Almkvist evaluated the constant e_t and proved that $1/e_t$ is always an integer.

Recall that $\mathcal{S}^{new}_{\frac{t-1}{2}}(\Gamma_0(t), \chi)$ has a basis consisting of newforms. Therefore, we have

$$g(z) = \sum_{i=1}^{s} r_i g_i(z), \tag{21}$$

where s is the dimension of $\mathcal{S}^{new}_{\frac{t-1}{2}}(\Gamma_0(t), \chi)$ and the $g_i(z)$ are normalized newforms in $\mathcal{S}_{\frac{t-1}{2}}(\Gamma_0(t), \chi)$. For the coefficients of newforms, we can easily obtain an upper bound by using Deligne's bound.

Lemma 11 (Lemma 2 of [4]). *Let $b(n)$ be the coefficients of a new form $g \in \mathcal{S}_k^{new}(\Gamma_0(N), \chi)$. Let $\Omega(n)$ denote the number of prime divisors of n, counting multiplicity. Then,*

$$|b(n)| \leq n^{\frac{k-1}{2}}(1+\sqrt{2})^{\Omega(n)} \leq n^{\frac{k-1}{2}+\tau},$$

where τ is the number satisfying $(1+\sqrt{2}) = 2^\tau$.

Then, as in Section 3, we can prove that

$$\sigma_{\frac{t-1}{2},\chi} \geq n^{(t-3)/2}\zeta\left(\frac{t-3}{2}\right)^{-1},$$

where $\zeta(z)$ is the Riemann zeta function. Thus, we can expect that $a_t(n)$ is dominated by $\sigma_{\frac{t-1}{2},\chi}(n)$ for sufficiently large n.

For simplicity, we only consider integers n that are coprime to p. For the integers with $p|n$, we also can use the same idea, but we need a careful estimate for $\sigma_{\frac{t-1}{2},\chi}(pn)$ since $\sigma_{\frac{t-1}{2},\chi}(n)$ is not totally multiplicative.

Proof of Theorem 4. Define

$$R_t := \sum |r_i|, \tag{22}$$

where the r_i are the constants appearing in (21). Denote the Fourier expansion of $g_i(z)$ as $\sum_{n=1}^\infty b_i(n)q^n$. Fix a prime p and a positive integer k. Then, for all n with $(n,p) = 1$, by Lemma 11, we have

$$\begin{aligned}
&c_t(p^k n) - (\sigma_{\frac{t-1}{2},\chi}(p^k) - 1)c_t(n) \\
&= e_t\sigma_{\frac{t-1}{2},\chi}(n) + \sum_{i=1}^s r_i b_i(p^k n) - (\sigma_{\frac{t-1}{2},\chi}(p^k) - 1)\sum_{i=1}^s r_i b_i(n) \\
&\geq e_t\sigma_{\frac{t-1}{2},\chi}(n) - R_t(p^k n)^{\frac{t-3}{4}+\tau} - (\sigma_{\frac{t-1}{2},\chi}(p^k) - 1)R_t n^{\frac{t-3}{4}+\tau} \\
&\geq e_t\zeta\left(\frac{t-3}{2}\right)^{-1} n^{\frac{t-3}{2}} - R' n^{\frac{t-3}{4}+\tau} \\
&\geq n^{\frac{t-3}{2}}\left(\frac{e_t}{\zeta(\frac{t-3}{2})} - \frac{R'}{n^{\frac{t-3}{4}-\tau}}\right),
\end{aligned}$$

where $R' = R_t p^{k(\frac{t-3}{4}+\tau)} + (\sigma_{\frac{t-1}{2},\chi}(p^k) - 1)R_t$. Since $t \geq 11$ and $\tau < 2$, by (10), we have $a_t(p^k n + \delta_t(p^k - 1)) > \left(\sigma_{\frac{t-1}{2},\chi}(p^k) - 1\right) a_t(n)$ for sufficiently large n. □

For the linear relations, note that $h(z) := \frac{\eta^t(tz)}{\eta(z)} \in \mathcal{M}(\Gamma_0(t), \chi)$ and $\dim \mathcal{M}(\Gamma_0(t), \chi) \leq \delta_t + 1$. Since $\frac{\eta^t(z)}{\eta(tz)} = 1 + \cdots \in \mathcal{M}(\Gamma_0(t), \chi)$ and the constant terms of the Fourier expansion of $T_p^{\delta_t}h, \ldots, T_p h$, and h are zero, $T_p^{\delta_t}h, \ldots, T_p h$ and h are linearly dependent. Thus, by calculating as in Section 5, we can obtain linear relations for t-cores.

7. Concluding Remarks

The following questions arise naturally. First, is there a partition statistic that can explain inequalities of t-cores? For example, can we find a partition statistic that divides $a_7(2n+2)$ into three classes such that the number of elements in each class $\geq a_7(n)$? Next, can we find an upper bound for the constant R_t in (22), which enables us to get an explicit version of Theorem 4? In a forthcoming paper with J. Rouse, the author will answer the second question.

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