COMBINATORIAL PROOFS OF CERTAIN IDENTITIES INVOLVING PARTIAL THETA FUNCTIONS

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Abstract. In this brief note, we give combinatorial proofs of two identities involving partial theta functions. As an application, we prove an identity for the product of partial theta functions, first established by G.E. Andrews and S.O. Warnaar. We also provide a generalization of the first two identities and give a combinatorial proof of the generalized identities.

1. Introduction

A partial theta function is a sum of the form

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n-1)/2}x^n.$$

We can find many identities involving partial theta functions in Ramanujan's lost notebook [2]. Typically, Ramanujan did not record any proof or motivation for his study of partial theta functions. Thus, we do not know what led Ramanujan to study partial theta functions. However, combinatorially, identities containing partial theta function are very interesting since they indicate what remains after numerous cancellations of certain kinds of partitions.

In a recent paper [4], G.E. Andrews and S.O. Warnaar prove an interesting identity about the product of two partial theta functions, namely,

$$\left(\sum_{n=0}^{\infty}(-a)^n q^{n(n-1)/2}\right)\left(\sum_{n=0}^{\infty}(-b)^n q^{n(n-1)/2}\right) = (q)_\infty (a)_\infty (b)_\infty \sum_{n=0}^{\infty}\frac{(abq^{n-1})_n q^n}{(q)_n (a)_n (b)_n}. \tag{1.1}$$

Here and in the sequel, we use the customary notation for q-series:

$$(a)_0 := (a;q)_0 := 1,$$
$$(a)_n := (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),\ n \geq 1,$$

and

$$(a)_\infty := (a;q)_\infty := \lim_{n\to\infty}(a;q)_n,\ |q|<1.$$

As an immediate corollary, they prove the generalized Jacobi triple product identity which was first proved by Warnaar [12]. In the proof of (1.1), the following two identities involving partial theta functions,

$$\sum_{n=0}^{\infty}(-a)^n q^{n(n-1)/2} = (a)_\infty (q)_\infty \sum_{n=0}^{\infty}\frac{q^n}{(a)_n (q)_n}, \tag{1.2}$$

$$\sum_{n=0}^{\infty}(-a)^n q^{n(n-1)/2} = (a)_\infty \sum_{m=0}^{\infty}\frac{a^m q^{m^2}}{(a)_m (q)_m}, \tag{1.3}$$

play a important role. In [4], Andrews and Warnaar prove (1.2) and (1.3) by using Heine's transformation. In Section 2, we prove (1.2) combinatorially. For (1.3), we prove combinatorially the equivalent identity

$$(q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(a)_n (q)_n} = \sum_{m=0}^{\infty} \frac{a^m q^{m^2}}{(a)_m (q)_m}. \tag{1.4}$$

In Section 3, we give a combinatorial proof of (1.1).

The identities (1.2) and (1.4) admit the beautiful generalizations,

$$\sum_{n=0}^{\infty} \frac{q^n}{(aq)_n (bq)_n} = \frac{1}{(aq)_\infty (bq)_\infty} \sum_{n=0}^{\infty} a^{-n-1}(-b)^n q^{n(n+1)/2} + (1-\frac{1}{a}) \sum_{n=0}^{\infty} \frac{a^{-n}(-b)^n q^{n(n+1)/2}}{(bq)_n}, \tag{1.5}$$

$$(a)_\infty \sum_{n=0}^{\infty} \frac{a^n}{(q)_n (bq)_n} = \sum_{m=0}^{\infty} \frac{a^m b^m q^{m^2}}{(q)_m (bq)_m}, \tag{1.6}$$

respectively. Note that we can obtain (1.2) by putting $a = 1$ and $b = aq^{-1}$ in (1.5) and we can deduce (1.4) by putting $a = q$ and then setting $b = a/q$ in (1.6). Both (1.5) and (1.6) can be found in Ramanujan's lost notebook [3, p. 40]. In [10], I. Pak suggests finding a combinatorial proof of (1.5). In Section 4, we also give combinatorial proofs of (1.5) and (1.6). We conclude this note by suggesting further study.

Before starting our proofs, let us define some notation. For a partition $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n, we denote the sum of all parts of π by $|\pi|$ and the number of parts of π by $\ell(\pi)$. Here and in the sequel, we always denote partitions in weakly decreasing order. Let

$$\begin{aligned}
\mathcal{P} &= \{ \text{ the set of all partitions } \}, \\
\mathcal{P}_n &= \{ \pi \in \mathcal{P} \mid \pi_1 = n \}, \\
\mathcal{P}_{\le n} &= \{ \pi \in \mathcal{P} \mid \pi_1 \le n \}, \\
\tau_n &= \text{ partition of } n(n+1)/2 \text{ into } (n, n-1, \ldots, 1) , \\
\Delta &= \{ \tau_n \mid n \ge 0 \},
\end{aligned}$$

where τ_0 is the empty partition. In the identities we want to prove, there are two variable generating functions for partitions. For example, $\frac{1}{(aq)_\infty}$ generates all partitions π, where the exponent of a is the number of parts in π. To denote the number of parts counted by the exponent of a, we set $\pi^a = (\pi_1^a, \pi_2^a, \ldots, \pi_\ell^a)$ and we will say the partition π^a has the color a. We will define $\mathcal{P}^a$ as the set of partitions π^a. We will similarly define the color of a partition for other sets of partitions. For example, $\mathcal{P}_n^a$ denotes the set of all partitions of color a with the largest part n.

2. Proofs of (1.2) and (1.4)

In [13], A.J. Yee gives a combinatorial proof of (1.2). Here, we give a new and more direct combinatorial proof. This proof clearly shows how cancellation works so that we can use the same idea to prove its generalization (1.5). Since our combinatorial proof for (1.5) is complicated, we will give a complete proof, as motivation, of this special case.

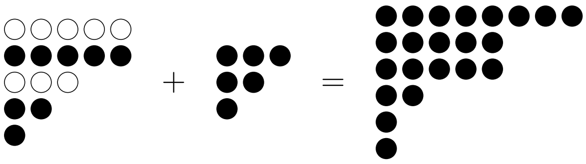

Figure 1. $(5, 5^a, 3, 2^a, 1^a) + (3^a, 2^a, 1^a) = (8^a, 5^a, 5^a, 2^a, 1^a, 1^a)$ of weight $(-1)^3 a^6 = -a^6$.

Proof of (1.2). Replacing a by aq in (1.2) and dividing both sides by $(aq)_\infty (q)_\infty$, we find that

$$\sum_{n=0}^{\infty} \frac{q^n}{(aq)_n (q)_n} = \frac{1}{(q)_\infty (aq)_\infty} \sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2}. \tag{2.1}$$

Note that

$$\frac{q^n}{(aq)_n (q)_n}$$

generates partitions of $\mathcal{P}_n \times \mathcal{P}^a_{\leq n}$. Thus, the left side of (2.1) is the generating function for $\cup_{n\geq 0} \mathcal{P}_n \times \mathcal{P}^a_{\leq n}$, where the exponent of a equals the number of parts in the partition in $P^a_{\leq n}$. On the right side,

$$\frac{1}{(q)_\infty (aq)_\infty}$$

generates partitions of $\mathcal{P} \times \mathcal{P}^a$ and

$$\sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2}$$

generates partitions of Δ^a with weight $(-a)^n$.

Define a function f from $\mathcal{P} \times \mathcal{P}^a \times \Delta^a$ to $\mathcal{P} \times \mathcal{P}^a$ as follows. For given $(\pi, \nu^a, \tau^a_n) \in \mathcal{P} \times \mathcal{P}^a \times \Delta^a$, attach each part of τ^a_n to π, beginning with the largest part of π, to the n-th largest part of π and assign color a to these parts. If the number of parts in π is less than n, append the remaining parts of τ^a_n below ν^a. Finally, we gather parts with color a in weakly decreasing order. The new partition is in $\mathcal{P} \times \mathcal{P}^a$, and we define its weight to be $(-1)^n a^{l(\nu)+n}$. For example, examine Figure 1. In Figure 1, we have $\pi = (5, 3)$, $\nu^a = (5^a, 2^a, 1^a)$ and τ^a_3. We attach 3^a in τ^a_3 to 5 in π and assign color a. Similarly, we attach 2^a in τ^a_3 to 3 and assign color a. Finally, we append the remaining part, namely 1, below ν. After gathering parts with color a, we will have the partition $(8^a, 5^a, 5^a, 2^a, 1^a, 1^a)$ of weight $(-1)^3 a^{3+3} = -a^6$. We will say that the triple $(\pi,\nu^a, \tau^a_n) \in \mathcal{P} \times \mathcal{P}^a \times \Delta^a$ is associated with $(\lambda, \sigma^a) \in \mathcal{P} \times \mathcal{P}^a$, if $f(\pi,\nu^a, \tau^a_n)=(\lambda, \sigma^a)$. Note that for every pair of partitions $(\lambda, \sigma^a) \in \mathcal{P} \times \mathcal{P}^a$, there must be an associated triple of partitions $(\pi,\nu^a, \tau^a_n) \in \mathcal{P} \times \mathcal{P}^a \times \Delta^a$, since we have $(\lambda,\sigma^a,\tau^a_0) \in \mathcal{P} \times \mathcal{P}^a \times \Delta^a$, and if τ^a_n is not the empty partition, the largest part of the resulting partition has color a.

The triple $(\pi, \nu^a, \tau^a_n) \in \mathcal{P} \times \mathcal{P}^a \times \Delta^a$ is associated with a pair of partitions $(\lambda, \sigma^a) \in \mathcal{P} \times \mathcal{P}^a$ with weight $(-1)^n a^{l(\nu)+n}$, where σ^a has at least n different parts that are larger than the largest part of λ. So, the right side of (2.1) is the weighted sum of triples $(\pi, \nu^a, \tau^a_n) \in \mathcal{P} \times \mathcal{P}^a \times \Delta^a$ with the weight $(-1)^n a^{l(\nu)+n}$. Note that $l(\nu^a) + n = l(\sigma^a)$ when (π, ν^a, τ^a_n) is associated with $(\lambda, \sigma^a) \in P \times P^a$.

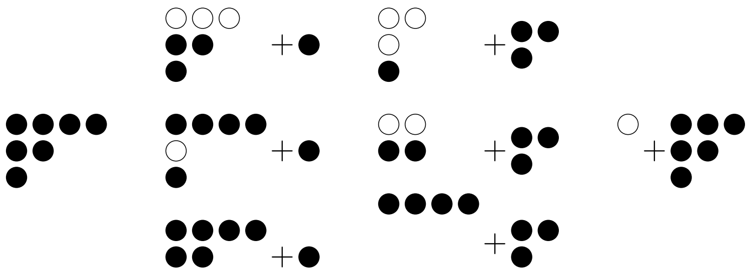

Figure 2. Weighted sum of partitions associated with $(4^a, 2^a, 1^a) = a^3(1-3+3-1) = 0$.

For given $(\lambda, \sigma^a) \in \mathcal{P} \times \mathcal{P}^a$, let n be the number of different parts of σ^a that are larger than the largest part of λ. By construction, on the right side there are $\binom{n}{k}$ associated partition triples $(\pi,\, \nu^a,\, \tau_k^a) \in \mathcal{P} \times \mathcal{P}^a \times \Delta^a$ with weight $(-1)^k a^{l(\sigma^a)}$. Since

$$a^{l(\sigma^a)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0,$$

after cancellation, we are left with the pairs of partitions $(\lambda,\, \sigma^a) \in \mathcal{P} \times \mathcal{P}^a$ with $\sigma_1 \le \lambda_1$, which are generated by the left side of (2.1).

For example, examine Figure 2 to see the case $(4^a, 2^a, 1^a)$. In Figure 2, the first column shows the $\binom{3}{0} = 1$ associated partitions with weight $(-1)^0 a^3 = a^3$. The second column shows the $\binom{3}{1} = 3$ associated partitions with weight $(-1)^1 a^3 = a^3$. Similarly, there are $\binom{3}{2} = 3$ associated partitions with weight $(-1)^2 a^3 = a^3$ in the third column and there is $\binom{3}{3} = 1$ associated partition with weight $(-1)^3 a^3 = -a^3$ in the last column. Therefore, the weighted sum of partitions associated with $(4^a, 2^a, 1^a)$ is zero as we expect.

□

Proof of (1.4). Replacing a by aq in (1.4) and then dividing both sides by $(aq)_\infty$, we find that

$$\sum_{n=0}^{\infty} \frac{q^n}{(aq)_n (q)_n} = \frac{1}{(q)_\infty} \sum_{m=0}^{\infty} \frac{a^m q^{m^2+m}}{(aq)_m (q)_m}. \tag{2.2}$$

As in the proof of (1.2), the left side of (2.2) generates partitions in $\cup_{n\ge 0} \mathcal{P}_n \times \mathcal{P}^a_{\le n}$. The right side of (2.2) generates partitions from $\mathcal{P} \times \mathcal{Q}^a$, where $\mathcal{Q}$ is the set of all partitions that have $k \times (k+1)$ Durfree rectangles. For given $(\pi,\, \nu^a) \in \cup_{n\ge 0} \mathcal{P}_n \times \mathcal{P}^a_{\le n}$, we have three cases.

(Case 1) ν^a is the empty partition. Then $\pi \in \mathcal{P}$.

(Case 2) ν^a has a $k \times (k+1)$ Durfee rectangle. Then $(\pi,\, \nu^a) \in \mathcal{P} \times \mathcal{Q}^a$. Note that $\nu_1^a \le \pi_1$.

(Case 3) ν^a has a Durfee square of side k, but no $k \times (k+1)$ Durfee rectangle. Move each part on the right side of the Durfee square of ν^a one line below its original position. Since

$\pi \in \mathcal{P}_n$, there must be a part of size n in π. Pick a part of size n, then detach $(n-k+1)$ from the part of size n and add it to the first row in the Durfee square, and add 1 to each of the remaining $k-1$ rows on the right side of the Durfee Square. Then the resulting partition has a $k \times (k+1)$ Durfee rectangle, so it is in $\mathcal{Q}^a$. Note that $\nu_1^a > \pi_1$.

Since the above cases are disjoint and each process is easily reversible, we are done.

□

3. Andrews and Warnaar's Product Identity

By (1.2) and (1.4), replacing a and b by aq and bq, respectively, we can see that (1.1) is equivalent to the identity,

$$\sum_{n=0}^{\infty} \frac{q^n}{(aq)_n (q)_n} \sum_{m=0}^{\infty} \frac{b^m q^{m^2+m}}{(bq)_m (q)_m} = \sum_{n=0}^{\infty} \frac{(abq^{n+1})_n q^n}{(q)_n (aq)_n (bq)_n}. \tag{3.1}$$

To prove (3.1) combinatorially, we need to put it in another form

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^n}{(aq)_n (q)_n} \sum_{m=0}^{\infty} \frac{b^m q^{m^2+m}}{(bq)_m (q)_m} &= \sum_{n,m=0}^{\infty} \frac{b^m q^{m^2+m+n}}{(aq)_n (q)_n (bq)_m (q)_m} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{q^k b^{k-n} q^{(k-n)^2}}{(q)_n (aq)_n (q)_{k-n} (bq)_{k-n}},\end{aligned}$$

by collecting terms with $m+n=k$. Thus it is enough to show that

$$\sum_{n=0}^{k} \frac{q^k b^{k-n} q^{(k-n)^2}}{(q)_n (aq)_n (q)_{k-n} (bq)_{k-n}} = \frac{(abq^{k+1})_k q^k}{(q)_k (aq)_k (bq)_k}. \tag{3.2}$$

In (3.2), divide both sides by q^k, multiply both sides by $(q)_k (aq)_k$, and finally replace a by $-aq^{-k}$. Then, we have

$$\sum_{n=0}^{k} \frac{(-1/a)_n a^n b^n q^{n(n+1)/2}}{(bq)_n} \begin{bmatrix} k \\ n \end{bmatrix} = \frac{(-abq)_k}{(bq)_k}, \tag{3.3}$$

where

$$\begin{bmatrix} k \\ n \end{bmatrix} = \frac{(q)_k}{(q)_n (q)_{k-n}}.$$

We thus see that (3.3) is a version of the q-Chu-Vandermonde summation formula. In [9], S. Corteel and J. Lovejoy give a nice bijective proof for (3.3). To show (3.3), Corteel and Lovejoy establish a bijection between certain Frobenius partitions and overpartitions. Here, we give another bijective proof without using Frobenius partitions or overpartitions. Our proof is similar to the proof of Lemma 3.3 in B. C. Berndt and Yee's paper [8]. Especially, we will use the map φ as defined in their paper. Before starting our proof, we need the following lemma.

Lemma 1. *Let* $\{c_n\}_{n=1}^{k}$ *be a sequence of integers satisfying*

$$\infty = c_0 > c_1 > c_2 > \cdots > c_k > c_{k+1} = 0.$$

Then for any integer $m > k$, *there is a* $j \geq 0$ *satisfying* $c_j > m - j > c_{j+1}$.

Proof. Suppose that there is no such j. Then, $m \le c_1$, because if $m > c_1$, then we can take $j = 0$. Similarly, $m - 1 \le c_2$, because if $m - 1 > c_2$, then we can set $j = 1$. By continuing this argument, we reach $m - (k - 1) \le c_k$. Thus, we have $c_k > m - k > c_{k+1}$ by the assumption that $m > k$. This contradicts our assumption that there is no such j. Hence, we are done. □

Proof of (3.3). Since it is well-known that $\begin{bmatrix} k \\ n \end{bmatrix}$ generates all partitions into parts less than or equal to $k - n$, and the number of parts is less than or equal to n [1, Theorem 3.1],

$$q^{n(n+1)/2} \begin{bmatrix} k \\ n \end{bmatrix}$$

generates partitions into n distinct parts of which the largest part is less than or equal to k. Let us denote such partitions by $\mathcal{D}^+(n, \le k)$. Note that $(-1/a; q)_n$ generates partitions into distinct nonnegative parts of which the largest part is strictly less than n. Let us denote such partitions by $\mathcal{D}_{\le n-1}$. Thus, (3.3) is equivalent to

$$\cup_{n=0}^{k} \mathcal{D}^+(n, \le k) \times \mathcal{D}_{\le n-1} \times \mathcal{P}_{\le n} = \mathcal{D}^+_{\le k} \times \mathcal{P}_{\le k}, \tag{3.4}$$

where $\mathcal{D}^+_{\le k}$ is the set of partitions into positive distinct parts that are less than or equal to k.

Let $(\pi, \mu, \nu) \in \mathcal{D}^+(n, \le k) \times \mathcal{D}_{\le n-1} \times \mathcal{P}_{\le n}$. Let $n - m$ be the number of parts in μ. Define σ by

$$\sigma_{n-m+1-i} = \mu_i + \pi_{\mu_i+1} \text{ for } 1 \le i \le n - m,$$

and define δ to be the partition consisting of the remaining m parts of π. Append ν below σ and denote this partition by λ. Then it is clear that $(\delta, \lambda) \in \mathcal{D}_{\le k} \times \mathcal{P}_{\le k}$.

This process is reversible as we now demonstrate. Let $(\delta, \lambda) \in \mathcal{D}_{\le k} \times \mathcal{P}_{\le k}$ be given. Let m be the number of parts of δ. We have two cases.

(Case 1) $\lambda_1 \le m$.

Note that $\delta \in \mathcal{D}^+(m, \le k)$ and $\lambda \in \mathcal{P}_{\le m}$. Thus, $(\delta, \varnothing, \lambda)$ is the inverse.

(Case 2) $\lambda_1 > m$.

Let j be the smallest integer satisfying $\lambda_i \le m + j$ for all $i \ge j$. Let $n = m + j$, $\lambda' = (\lambda_1, \ldots, \lambda_j)$, and let ν be a partition of the remaining parts of λ. Note that the smallest part of λ' is at least equal to n and that the largest part of ν does not exceed n. Let ρ be a partition into l distinct parts. Then, by Lemma 1, for any $M > l$, we can find the smallest j satisfying

$$\rho_j > M - j > \rho_{j+1}.$$

Define a map φ by $\varphi(\rho, m) = (\rho^*, j)$, where

$$\rho_i^* = \begin{cases} \rho_i, & \text{if } 1 \le i \le j, \\ m - j, & \text{if } i = j + 1, \\ \rho_{i-1}, & \text{if } i > j + 1. \end{cases}$$

We apply the map φ a total of $n - m$ times as follows:

$$\varphi(\delta^{(i)}, \lambda_i') = (\delta^{(i+1)}, \mu_{n-m+1-i}), \ 1 \le i \le n - m,$$

where $\delta^{(1)} = \delta$. By definition of the map φ, it is clear that $\mu_1 > \mu_2 > \cdots > \mu_{n-m}$. Let $\pi = \sigma^{(n-m+1)}$ and let μ be a partition consisting of the parts μ_i, where $1 \le i \le n - m$. Then (π, μ, ν) is the desirable inverse. Note that the exponent of a, which equals m, and the exponent of b, which is the number of parts in π plus the number of parts in μ, remain the same throughout this process. Therefore, we are done. $\square$

Example. When $k = 6$ and $n = 4$, let π=(6, 5, 3, 2), μ=(3, 0), ν=(4, 2). Then, $\sigma' = (0+6\ ,3+2)$ and δ=(5, 3). Thus, λ=(6, 5, 4, 2). For the inverse process, let δ=(5, 3) and λ=(6, 5, 4, 2) be given. Then, λ'=(6, 5) and ν=(4, 2). Since $\varphi(\delta, 6)$=((6, 5, 3), 0) and φ((6, 5, 3), 5)=((6, 5, 3, 2), 3), we have π=(6, 5, 3, 2), μ=(3, 0), as we expect.

4. Proofs of (1.5) and (1.6)

To prove (1.5), we will use a similar but more elaborate argument than that used in the proof of (1.2).

Proof of (1.5)*.* Multiply both sides of (1.5) by a. Then, we obtain

(4.1)
$$\sum_{n=0}^{\infty} \frac{aq^n}{(aq)_n(bq)_n} = \frac{1}{(aq)_\infty(bq)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{-n} b^n q^{n(n+1)/2} + (a-1)\sum_{n=0}^{\infty}\frac{(-1)^n a^{-n} b^n q^{n(n+1)/2}}{(bq)_n}.$$

The left side of (4.1) generates pairs of partitions $(\lambda^a,\ \sigma^b) \in \cup_{n\geq 0}\mathcal{P}_n^a \times \mathcal{P}_{\leq n}^b$, where the exponents of a and b equal $l(\lambda)$ and $l(\sigma)$, respectively. To interpret

$$\frac{1}{(aq)_\infty(bq)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{-n} b^n q^{n(n+1)/2}, \tag{4.2}$$

we need to define a map from $\mathcal{P}^a \times \mathcal{P}^b \times \Delta$ to $\mathcal{P}^a \times \mathcal{P}^b$. For a given partition triple $(\pi^a,\ \nu^b,\ \tau_n^b) \in \mathcal{P}^a \times \mathcal{P}^b \times \Delta^b$, we attach the part k of τ_n^b to the k-th largest part of π^a and change its color from a to b. If the number of parts of π^a is less than n, append the remaining parts below ν^b. We gather all parts having color b in weakly decreasing order. Let $(\lambda^a,\ \sigma^b)$ be a pair of partitions such that λ^a and σ^b are partitions consisting of parts having colors a and b, respectively, of the resulting partition. Let us assign a weight to this pair $(\lambda^a,\ \sigma^b)$ as

$$(-1)^n a^{l(\pi)-n} b^{l(\nu)+n}.$$

Then, (4.2) is the weighted sum of triples $(\pi^a,\ \nu^b,\ \tau_n^b) \in \mathcal{P}^a\times\mathcal{P}^b\times\Delta^b$ with weight $(-1)^n a^{l(\pi)-n} b^{l(\nu)+n}$. For a given $(\lambda^a,\ \sigma^b) \in \mathcal{P}^a \times \mathcal{P}^b$, let n be the number of different parts that are larger than the largest part of λ^a. Pick n such different parts and let these parts be denoted by $(b_1,\ b_2,\ldots,\ b_n)$. Since the b_j's are distinct, $b_1 \geq n$. As in the proof of (1.2), there are $\binom{n}{k}$ associated partition triples $(\pi^a,\ \nu^b,\ \tau_k^b) \in \mathcal{P}^a \times \mathcal{P}^b \times \Delta^b$ with the weight $(-1)^n a^{l(\pi)-k} b^{l(\nu)+k}$. Note that $l(\nu^b) + k = l(\sigma^b)$. Among the $\binom{n}{k}$ associated triples, there are two kinds of triples:

(Case 1) $b_1 \notin \nu^b$

There are $\binom{n-1}{k-1}$ associated triples where $b_1 \notin \nu$. Note that there are also $\binom{n-1}{k-1}$ associated triples $(\pi^a,\ \nu^b,\ \tau_{k-1}^b)$ with $b_1 \in \nu^b$. For a given associated triple $(\pi^a,\ \nu^b,\ \tau_{k-1}^b)$ with $b_1 \in \nu$, consider the triple $(\pi^{a*},\ \nu^{b*},\ \tau_k^b)$, where $\pi^{a*} = \pi^a \cup \{b_1 - k\}$, and ν^{b*} is the partition consisting of the parts of ν^b except b_1. Since $l(\pi^a)-k+1 = l(\pi^{a*})-k$, it is clear that they are associated with the same pair of partitions $(\lambda^a,\ \sigma^b)$ with opposite weights unless $b_1 = k$. Thus, the weights of these two pairs of associated triples cancel each other.

(Case 2) $b_1 \in \nu^b$

Note that there are $\binom{n-1}{k}$ such associated triples. Since there are $\binom{n-1}{k}$ associated triples $(\pi^a,\ \nu^b,\ \tau_{k+1}^b)$ with $b_1 \notin \nu^b$, these associated triples cancel each other, unless $b_1 = k+1$ as in Case 1.

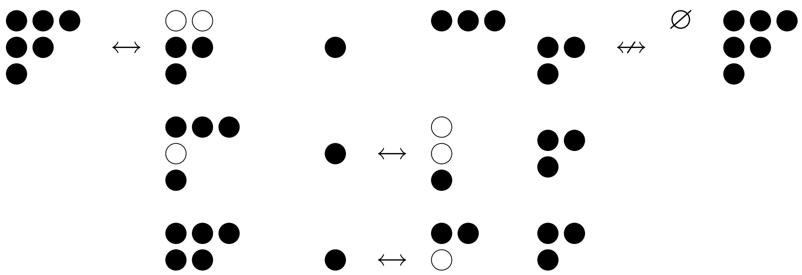

Figure 3. Cancellation of partitions associated with $(3^b,2^b,\ 1^b)$.

Thus, by Case 1 and Case 2, the weighted sum of associated triples of the pair of partitions $(\lambda^a,\ \sigma^b)$ is

$$\begin{aligned} & b^{l(\sigma^b)}a^{l(\pi^a)}\left(\binom{n}{0}-\binom{n-1}{0}\right)+b^{l(\sigma^b)}a^{l(\pi^a)-1}\left(-\binom{n-1}{1}+\binom{n-1}{1}\right) \\ & +\cdots+b^{l(\sigma^b)}a^{l(\pi^a)-n+1}\left((-1)^{n-1}\binom{n-1}{n-1}+(-1)^n\binom{n}{n}\right)=0, \end{aligned} \tag{4.3}$$

unless $b_1 = n$.

Suppose that $b_1 = n$. Then, in (4.3), the last two terms do not cancel each other because the exponents of a are different in their weights. Note that the last term on the right side of (4.3) represents the associated triple $(\pi^a,\ \nu^b,\ \tau_n^b)$ with weight $(-1)^n a^{-n} b^{l(\sigma^b)}$ and the second term from the last on the right side of (4.3) represents the associated triple $(\pi^a,\ \nu^b,\ \tau_{n-1}^b)$ with $b_1 \in \nu$ and weight $(-1)^{n-1}a^{-n+1}b^{l(\sigma^b)}$. Since the b_j's are distinct, when $b_1 = n$, there are no parts with color a and $(b_1,\ldots,b_n)=(n,\ldots,1)$. Thus, these are generated by

$$\frac{(-1)^{n-1}a^{-n+1}(b)^n q^{n(n+1)/2}}{(bq)_n}+\frac{(-1)^n a^{-n}(b)^n q^{n(n+1)/2}}{(bq)_n}.$$

Thus, when $n \geq 1$, the weighted sum of the associated triples is 0. Hence, we are left with pairs of partitions $(\lambda^a,\ \sigma^b) \in \mathcal{P}^a \times \mathcal{P}^b$ with $\sigma_1^a \leq \lambda_1^b$, which are generated by the left side of (4.1). Note that we need $(a-1)$ in the second summation in the right side of (4.1) to correspond to the empty partition.

For example, Figure 3 shows the case $(3^b,2^b,\ 1^b)$. In Figure 3, we can check how cancellations occur in each Case 1 and Case 2.

□

Remark. Now that we know clearly how cancellation works, it is not hard to see that

$$\sum_{n=0}^{\infty}\frac{abq^{2n}}{(aq)_n(bq)_n}=\frac{1}{(aq)_\infty(bq)_\infty}\left(1+\sum_{n=1}^{\infty}(-a/b)^nq^{n(n+1)/2}+\sum_{n=1}^{\infty}(-b/a)^nq^{n(n+1)/2}\right)$$
$$+(a-1)\sum_{n=1}^{\infty}\frac{(-a)^{-n}b^nq^{n(n+1)/2}}{(bq;q)_n}+(b-1)\sum_{n=1}^{\infty}\frac{(-b)^{-n}a^nq^{n(n+1)/2}}{(aq;q)_n}+(ab-1).$$

This is because

$$\frac{1}{(aq)_\infty(bq)_\infty}\sum_{n=1}^{\infty}(-b/a)^nq^{n(n+1)/2}+(b-1)\sum_{n=1}^{\infty}\frac{(-b)^{-n}a^nq^{n(n+1)/2}}{(aq;q)_n}$$

will cancel the pairs of partitions $(\pi^a,\, \nu^b)\in\mathcal{P}^a\times\mathcal{P}^b$ such that $\pi_1^a>\sigma_1^b$. Therefore, we are left with the pairs of partitions $(\pi^a,\, \nu^b)\in\mathcal{P}^a\times\mathcal{P}^b$ with $\pi_1^a=\sigma_1^b$. Note that this gives a generalization of Entry 1.4.11 [3, p. 10].

Unlike the proof of (1.5), we cannot apply the argument used in the proof of (1.4) to prove (1.6). In [7], Berndt and Yee prove the case $b=1$ of (1.6). We will show that their bijection actually implies a stronger result by tracking the exponent of b. Before beginning the proof, we need to introduce Franklin's bijection [11]. In [7], the authors nicely introduce this bijection. Thus we are going to follow their notation. Let us define a map $f_{k,s}$ from a partition $\delta=(\delta_1,\,\delta_2,\ldots,\,\delta_m)$ to a partition $\rho=(\rho_1,\,\rho_2,\ldots,\,\rho_m)$ as follows. If $\delta_1-\delta_{k+1}>s$ (the k-th excess is greater than s in Sylvester's terminology), define $f_{k,s}(\delta)=\rho$, where for $1\le i\le m$,

$$\rho_i=\begin{cases}\delta_{i+1}-1 \text{ for } i<k,\\ \delta_i-s-1 \text{ for } i=k,\\ \delta_i \text{ for } i>k.\end{cases}$$

Otherwise, $f_{k,s}(\delta)=\delta$. Now, we are ready to start our proof.

Proof of (1.6). Divide both sides of (1.6) by $(a)_\infty$. Then, the left side becomes a generating function for $A(n,m,N)$, namely,

$$\sum_{n=0}^{\infty}\frac{a^n}{(q)_n(bq)_n}=\sum_{n,m,N\ge 0}|A(n,m,N)|a^nb^mq^N,$$

where $A(n,m,N)$ is the number of partitions of $N+n^2$ with Durfee square of side n, and the number of rows below the Durfee square is m. The right side becomes a generating function for $B(n,m,N)$, namely,

$$\frac{1}{(a)_\infty}\sum_{m=0}^{\infty}\frac{a^mb^mq^{m^2}}{(q)_m(bq)_m}=\sum_{n,m,N\ge 0}|B(n,m,N)|a^nb^mq^N,$$

where $B(n,m,N)$ is the number of vector partitions (π,ν) such that $|\pi|+|\nu|=N$, π has at most $n-d$ nonnegative parts, ν has Durfee square of side $d\le n$, and the number of parts in ν is m. We will establish a bijection between $A(n,m,N)$ and $B(n,m,N)$. We consider an $n\times n$ Durfee square, and then append π to the right of the Durfee square and append ν below the Durfee square. If the largest part of ν is less than or equal to n, this is the desirable partition λ counted by $A(n,m,N)$. Otherwise, let $\sigma=(\sigma_1,\,\sigma_2,\ldots,\sigma_l)$ be a partition to the right of the Durfee square of side d of ν. For convenience, we define $\sigma_{l+1}=0$. Let

r_1 be the smallest j satisfying $f^j_{1,n-d}(\sigma) = f^{j+1}_{1,n-d}(\sigma)$. Then we add $n-d+1$ to π' as a part r_1 times, where π' is the conjugate of π. For simplicity of notation, we denote $f^{r_1}_{1,n-d}(\sigma)$ by σ. Next, let r_2 be the smallest j such that $f^j_{2,n-d}(\sigma) = f^{j+1}_{2,n-d}(\sigma)$. Then we add $n-d+2$ to π' as a part r_2 times and denote $f^{r_2}_{2,n-d}(\sigma)$ by σ. Repeat this step $l-2$ more times with the k-th excess, where $k = 3,\dots,l$. Since this step terminates when $\sigma_1 - \sigma_{l+1} \le n-d$, the resulting σ has its largest part less than or equal to $n-d$. Since $l \le d$, the added part to π' is less than or equal to n. Note that there is no change in the number of parts of ν during this process. Therefore, we obtain the desirable partition λ counted by $A(n,m,N)$. Since the above steps are easily reversible, we are done.

□

5. Concluding Remarks

First of all, our proof for the identity (1.1) is not direct. We do some algebraic manipulation to reach an identity that is actually the q-Chu-Vandermonde summation formula. Can we find a more direct bijective proof for (1.1)? The author has been able to give a direct bijective proof only in the case $a = b = 1$.

Secondly, A. Berkovich [5] recently noted that (1.1) is a special case of the Gasper-Rahman product formula:

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} a, b; q, z \\ c \end{matrix}\right) {}_2\phi_1\left(\begin{matrix} a, aq/c; q, z \\ aq/b \end{matrix}\right) \\
&= \frac{(az, abz/c)_\infty}{(z, bz/c)_\infty}\, {}_6\phi_5\left(\begin{matrix} a, c/b, \sqrt{ac/b}, -\sqrt{ac/b}, \sqrt{acq/b}, -\sqrt{acq/b}; q, q \\ aq/b, c, ac/b, az, cq/bz \end{matrix}\right) \\
&+ \frac{(a, c/b, az, bz, azq/c)_\infty}{(c, zq/b, z, z, c/bz)_\infty}\, {}_6\phi_5\left(\begin{matrix} z, abz/c, z\sqrt{ab/c}, -z\sqrt{ab/c}, z\sqrt{abq/c}, -z\sqrt{abq/c}; q, q \\ az, bz, azq/c, bzq/c, abz^2 \end{matrix}\right),
\end{aligned}
$$

where

$$
{}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \dots, a_{r+1}; q, z \\ b_1, b_2, \dots, b_r \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{r+1})_n}{(q)_n (b_1)_n \cdots (b_r)_n} z^n.
$$

It would be nice to find a combinatorial proof for this identity.

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