A CRANK ANALOG ON A CERTAIN KIND OF PARTITION FUNCTION ARISING FROM THE CUBIC CONTINUED FRACTION

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Abstract. In a series of papers, H.-C. Chan has studied congruence properties of a certain kind of partition function that arises from Ramanujan's cubic continued fraction. This partition function $a(n)$, is defined by $\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}$. In particular, he proved that $a(3n+2) \equiv 0 \pmod 3$. As Chan mentioned in his paper, it is natural to ask if there exists an analog of the rank or the crank for the ordinary partition function that provides a combinatorial explanation of the above congruence. Here, we will define a crank analog $M'(m,N,n)$ for $a(n)$ and prove that

$$M'(0,3,3n+2) \equiv M'(1,3,3n+2) \equiv M'(2,3,3n+2) \pmod 3,$$

for all nonnegative integers n, where $M'(m,N,n)$ is the number of partitions of n with crank $\equiv m \pmod N$. Next, using the theory of modular forms, we will investigate further congruences of $a(n)$.

1. Introduction and Statement of Results

In a series of papers ([5], [6], [7]) H.-C. Chan has studied congruence properties of a certain kind of partition $a(n)$, which arise from Ramanujan's cubic continued fraction. This partition function $a(n)$ is defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}. \tag{1}$$

Here and in the sequel, we will use the following standard q-series notation:

$$(a;q)_0 := 1,$$
$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),\ n \geq 1,$$

and

$$(a;q)_\infty := \lim_{n\to\infty} (a;q)_n,\ |q|<1.$$

We can interpret $a(n)$ as the number of 2-color partitions of n with colors r and b subject to the restriction that the color b appears only in even parts. For example, there are 3 such partitions of 2:

$$2^r,\ 2^b,\ 1^r + 1^r.$$

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Since $a(n)$ is closely related with Ramanujan's cubic continued fraction (see [5] for the relation.), we will say that $a(n)$ is the number of cubic partitions of n.

In particular, by using identities for the cubic continued fraction, Chan found a result analogous to "Ramanujan's most beautiful identity" (in the words of G.H. Hardy [12, p. xxxv]), namely,

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q)_\infty^6},$$

where $p(n)$ is the number of the ordinary partitions of n. Chan's identity is given by

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty (q^6;q^6)_\infty}{(q;q)_\infty^4 (q^2;q^2)_\infty}.$$

This implies immediately that

$$a(3n+2) \equiv 0 \pmod 3. \tag{2}$$

To give a combinatorial explanation of the famous Ramanujan's partition congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

G.E. Andrews and F.G. Garvan [3] introduced the crank of a partition. For a given partition λ, the crank $c(\lambda)$ of a partition is defined as

$$c(\lambda) := \begin{cases} \ell(\lambda), & \text{if } r = 0, \\ \omega(\lambda) - r, & \text{if } r \geq 1, \end{cases}$$

where r is the number of appearances of 1's in λ, $\omega(\lambda)$ is the number of parts in λ that are strictly larger than r and $\ell(\lambda)$ is the largest part in λ.

Let $M(m,n)$ be the number of ordinary partitions of n with crank m. Andrews and Garvan showed that

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n)x^m q^n = \frac{(q;q)_\infty}{(xq;q)_\infty (x^{-1}q;q)_\infty}, \; n \neq 1. \tag{3}$$

Let $M(k,N,n)$ be the number of ordinary partitions of n with crank $\equiv k \pmod N$. In [3] and [8], Andrews and Garvan showed that for all $n \geq 0$,

$$\begin{aligned} M(i,5,5n+4) &= M(j,5,5n+4), \text{ for all } 0 \leq i \leq j \leq 4, \\ M(i,7,7n+5) &= M(j,7,7n+5), \text{ for all } 0 \leq i \leq j \leq 6, \\ M(i,11,11n+6) &= M(j,11,11n+6), \text{ for all } 0 \leq i \leq j \leq 10. \end{aligned}$$

These identities clearly imply Ramanujan's congruences.

As Chan mentioned in his paper [7], it is natural to seek an analog of the crank of the ordinary partition to give a combinatorial explanation of (2). In light of (3), it is natural to conjecture that

$$F(x,q)=\frac{(q;q)_\infty(q^2;q^2)_\infty}{(xq;q)_\infty(x^{-1}q;q)_\infty(xq^2;q^2)_\infty(x^{-1}q^2;q^2)_\infty} \tag{4}$$

gives an analogous crank for cubic partitions. In Section 2, we will review the crank of Andrews and Garvan of the ordinary partition and after that, by giving a combinatorial interpretation of (4), we will define a crank analog c_a that is analogous to the crank given by Andrews and Garvan. By using basic q-series identities, we will prove our first theorem.

Theorem 1.1. *Let $M'(m,N,n)$ be the number of cubic partitions of n with crank $\equiv m \pmod N$. Then, we have*

$$M'(0,3,3n+2)\equiv M'(1,3,3n+2)\equiv M'(2,3,3n+2) \pmod 3,$$

for all nonnegative integers n.

This immediately implies the following corollary.

Corollary 1.2. *For all nonnegative integers n, we have $a(3n+2)\equiv 0 \pmod 3$.*

In [10], K. Mahlburg proved that there are infinitely many arithmetic progressions $An+B$ such that

$$M(m,\ell^j,An+B)\equiv 0 \pmod{\ell^\tau}$$

simultaneously for every $0\le m\le \ell^j-1$, where $\ell\ge 5$ is a prime and τ, j are positive integers. This implies that $p(An+B)\equiv 0 \pmod{\ell^\tau}$.

In Section 3, we will review some basic properties of modular form. With this equipment, in Section 4, we will prove our second theorem, which is analogous to Mahlburg's result.

Theorem 1.3. *There are infinitely many arithmetic progression $An+B$ such that*

$$M'(m,\ell^j,An+B)\equiv 0 \pmod{\ell^\tau}$$

simultaneously for every $0\le m\le \ell^j-1$, where $\ell\ge 5$ is a prime and τ, j are positive integers.

2. A Crank Analog for $a(n)$

Before defining a crank analog, we need to introduce some notation and review the definition of crank of ordinary partitions. After Andrews and Garvan [3], we define that, for a partition λ, $\#(\lambda)$ is the number of parts in

λ and $\sigma(\lambda)$ is the sum of the parts of λ with the convention $\#(\lambda)=\sigma(\lambda)=0$ for the empty partition λ. Let $\mathcal{P}$ be the set of all ordinary partitions and $\mathcal{D}$ be the set of all partitions into distinct parts. We define

$$V=\{(\lambda_1,\lambda_2,\lambda_3)|\lambda_1\in\mathcal{D},\text{ and }\lambda_2,\lambda_3\in\mathcal{P}\}.$$

For $\lambda=(\lambda_1,\ \lambda_2,\ \lambda_3)$, we define the sum of parts s, a weight w, and a crank t, by

$$\begin{aligned}s(\lambda)&=\sigma(\lambda_1)+\sigma(\lambda_2)+\sigma(\lambda_3),\\ w(\lambda)&=(-1)^{\#(\lambda_1)},\\ t(\lambda)&=\#(\lambda_2)-\#(\lambda_3).\end{aligned}$$

We say λ is a vector partition of n if $s(\lambda)=n$. Let $N_V(m,n)$ denote the number of vector partitions of n (counted according to the weight w) with crank m, so that

$$N_V(m,n)=\sum_{\substack{\lambda\in V\\ s(\lambda)=n\\ t(\lambda)=m}} w(\lambda).$$

Then, we have

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}N_V(m,n)x^mq^n=\frac{(q;q)_\infty}{(xq;q)_\infty(x^{-1}q;q)_\infty}. \tag{5}$$

By putting $x=1$ in (5) we find

$$\sum_{m=-\infty}^{\infty}N_V(m,n)=p(n).$$

Andrews and Garvan showed that this vector crank actually gives a crank for the ordinary partitions.

Theorem 2.1 (Theorem 1 in [3]). *For all* $n>1$, $M(m,n)=N_V(m,n)$.

Now, we are ready to define a crank analog for cubic partitions. For a given cubic partition λ, we define λ^r to be a partition that consists of parts with color r and λ^b to be a partition that is formed by dividing each of the parts with color b by 2. The generating function (4) suggests that it is natural to define a vector crank analog $N_V^a(m,n)$ as

$$N_V^a(m,n)=\sum_{\substack{\lambda^r,\lambda^b\in V\\ s(\lambda^r)+2s(\lambda^b)=n\\ t(\lambda^r)+t(\lambda^b)=m}} w(\lambda^r)w(\lambda^b).$$

Then, we have

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}N_V^a(m,n)x^mq^n=\frac{(q;q)_\infty(q^2;q^2)_\infty}{(xq;q)_\infty(x^{-1}q;q)_\infty(xq^2;q^2)_\infty(x^{-1}q^2;q^2)_\infty}. \tag{6}$$

By putting $x = 1$ in (6), we find

$$\sum_{m=-\infty}^{\infty} N_V^a(m,n) = a(n).$$

From now on, if $\lambda = (1)$, then we will regard λ as an element of V with $s(\lambda) = 1$, and let us define the crank weight $wt(\lambda)$ for $\lambda \in \mathcal{P}$ as

$$wt(\lambda) = \begin{cases} 1, & \text{if } \lambda \neq (1), \\ w(\lambda), & \lambda = ((1),\emptyset,\emptyset),\ (\emptyset,(1),\emptyset) \text{ or } (\emptyset,\emptyset,(1)), \end{cases}$$

and the crank size $cs(\lambda)$ as

$$cs(\lambda) = \begin{cases} c(\lambda), & \text{if } \lambda \neq (1), \\ t(\lambda), & \text{if } \lambda = ((1),\emptyset,\emptyset),\ (\emptyset,(1),\emptyset) \text{ or } (\emptyset,\emptyset,(1)). \end{cases}$$

For a given cubic partition λ, we define a crank analog $c^a(\lambda)$ as

$$c^a(\lambda) = (wt(\lambda^r) \cdot wt(\lambda^b),\ cs(\lambda^r) + cs(\lambda^b)).$$

For example, here are some $c^a(\lambda)$, where λ is a cubic partition.

$$c^a((1^r,1^r,1^r,2^b)) = (1\cdot 1, -3+1),\ \ (1\cdot 1, -3-1),\ \text{and } (1\cdot(-1), -3+0)$$
$$c^a((1^r,1^r,2^r,2^b,2^b)) = (1\cdot 1, -2-2).$$

Let $M'(m,n)$ be the number of cubic partitions of n counted according to the weight, so that

$$M'(m,n) = \sum_{cs(\lambda^r)+cs(\lambda^b)=m} wt(\lambda^r)wt(\lambda^b).$$

Since

$$N_V(m,1) = \begin{cases} 1, & \text{if } m = 1 \text{ or } -1, \\ -1, & \text{if } m = 0, \\ 0, & \text{otherwise,} \end{cases}$$

by Theorem 2.1, we have

Theorem 2.2. *For all $n \geq 1$, we have $M'(m,n) = N_V^a(m,n)$.*

Therefore, we have

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M'(m,n)x^m q^n = F(x,q). \tag{7}$$

By an abuse of notation, we will say that $M'(m,n)$ is the number of cubic partitions of n with crank m. Let $M'(m,N,n)$ be the number of cubic partitions of n with crank $\equiv m \pmod{N}$. Now, we are ready to give a proof for our first theorem.

Proof of Theorem 1.1. By a simple argument, we have

$$F(\zeta,q)=\frac{(q;q)_\infty(q^2;q^2)_\infty}{(\zeta q;q)_\infty(\zeta^{-1}q;q)_\infty(\zeta q^2;q^2)_\infty(\zeta^{-1}q^2;q^2)_\infty}=\sum_{n=0}^{\infty}\sum_{k=0}^{2}M'(k,3,n)\zeta^k q^n,$$

where ζ is a primitive third roof of unity.

To find the coefficient of q^{3n+2} of $F(\zeta,q)$, we multiply the numerator and the denominator by $(q;q)_\infty(q^2;q^2)_\infty$. Then, we have

$$\begin{aligned}F(\zeta,q)&=\frac{(q;q)_\infty^2(q^2;q^2)_\infty^2}{(q^3;q^3)_\infty(q^6;q^6)_\infty}\\&=\frac{(q;q^2)_\infty^2(q^2;q^2)_\infty(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}\\&=\frac{\left(\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}\right)\left(\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{m(m+1)}\right)}{(q^3;q^3)_\infty(q^6;q^6)_\infty}.\end{aligned}$$

For the last equality, we used the Jacobi triple product identity and Jacobi's identity. (See [4, p.12 − 14] for the proof of these identities.) Since $n^2\equiv 0$ or $1 \pmod 3$ and $m(m+1)\equiv 0$ or $2 \pmod 3$, the coefficient of q^{3n+2} of $F(\zeta,q)$ is the same as the coefficient of q^{3n+2} of

$$\frac{\left(\sum_{n=-\infty}^{\infty}(-1)^nq^{9n^2}\right)\left(\sum_{m=0}^{\infty}(-1)^{3m+1}(6m+3)q^{9m^2+9m+2}\right)}{(q^3;q^3)_\infty(q^6;q^6)_\infty}. \tag{8}$$

Note that the coefficients of (8) are multiples of 3. Thus, we have

$$\sum_{k=0}^{2}M'(k,3,3n+2)\zeta^k=3N,$$

for some integer N. Since $1+\zeta+\zeta^2$ is a minimal polynomial in $\mathbf{Z}[\zeta]$, we must have

$$M'(0,3,3n+2)\equiv M'(1,3,3n+2)\equiv M'(2,3,3n+2)\pmod 3.$$

This complete the proof of Theorem 1.1. □

Recall that

$$a(n)=\sum_{m=-\infty}^{\infty}M'(m,n).$$

Therefore, Theorem 1.1 immediately implies Corollary 2.

3. Preliminary Results

This section contains the basic definitions and properties of modular forms that we will use in Section 4. For additional basic properties of modular forms, see [11, Chaps. 1, 2, and 3].

Define $\Gamma = SL_2(\mathbb{Z})$, $\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N}\}$, and $\Gamma_1(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 1 \pmod{N}$ and $c \equiv 0 \pmod{N}\}$. For a meromorphic function f on the complex upper half plane $\mathcal{H}$, define the Slash operator by

$$f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (cz+d)^{-k} f(\frac{az+b}{cz+d}).$$

Let $\mathcal{M}_k(\Gamma)$ (resp. $\mathcal{S}_k(\Gamma)$) denote the vector space of weakly holomorphic forms (resp. cusp forms) of weight k. Let $\mathcal{M}_k(\Gamma_0(N), \chi)$ (resp. $\mathcal{S}_k(\Gamma_0(N), \chi)$) denote the vector space of weakly holomorphic forms (resp. cusp forms) on $\Gamma_0(N)$ with character χ. For a prime p and a positive integer m, we need to define the Hecke operators T_p, the U_m-operator and the V_m-operator on $\mathcal{M}_k(\Gamma_0, \chi)$. If $f(q)$ has a Fourier expansion $f(q) = \sum a(n) q^n$, then

$$\begin{aligned} f|_{T_p} &:= \sum \left(a(pn) + \chi(p) p^{k-1} a(\frac{n}{p}) \right) q^n, \\ f|_{U_m} &:= \sum a(mn) q^n = m^{\frac{k}{2}-1} \sum_{v=0}^{m-1} f|_k \begin{pmatrix} 1 & v \\ 0 & m \end{pmatrix}, \\ f|_{V_m} &:= \sum a(n) q^{mn}. \end{aligned}$$

The following Theorem 3.1 is a slightly modified version of Serre's famous theorem in [14]. Theorem 3.1 is an integer weight analog of Theorem 2.2 of [10] and is proved in K. Ono and S. Ahlgren's paper [1].

Theorem 3.1. *For $0 \leq i \leq r$, let N_i and k_i be positive integers and let $g_i \in \mathcal{S}_{k_i}(\Gamma_1(N_i))$, where the Fourier coefficients of g_i are algebraic integers. If $M \geq 1$, then a positive proportion of primes $p \equiv -1 \pmod{N_1 \cdots N_r M}$ have the property that for every i,*

$$g_i(z)|T_p \equiv 0 \pmod{M}.$$

If $\zeta = \exp(2\pi i/N)$, then for $1 \leq s \leq N-1$, we define the $(0,\, s)$-Klein form by

$$t_{0,s}(z) = \frac{\omega_s}{2\pi i} \frac{(\zeta^s q; q)_\infty (\zeta^{-s} q; q)_\infty}{(q;q)_\infty^2}, \text{ for } 1 \leq s \leq N-1, \tag{9}$$

where $\omega_s := \zeta^{s/2}(1 - \zeta^{-s})$.

The following proposition gives a transformation formula under $\Gamma_0(N)$.

Proposition 3.2 (Proposition 3.2 in [10], eqn. **K2** (p.28) in [9])**.** *If* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$*, then*

$$t_{0,s}(z)|_{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \beta \cdot t_{0,\overline{ds}}(z), \tag{10}$$

where β *is given by* $\exp\left(\frac{cs+(ds-\overline{ds})}{2N} - \frac{cds^2}{2N^2}\right)$.

For certain congruence subgroups, a Klein form is a weakly holomorphic modular form.

Proposition 3.3 (Corollary 3.3 of [10])**.** *If* $1 \le s \le N-1$*, then* $t_{0,s}(z) \in \mathcal{M}_{-1}(\Gamma_1(2N^2))$.

Recall that Dedekind eta function $\eta(z)$ is defined by

$$\eta(z) = q^{\frac{1}{24}}(q)_\infty. \tag{11}$$

The following eta-quotient $E_{\ell,t}(z)$ will play an important role in our proof. Given a prime $\ell \ge 5$ and a positive integer t, we define

$$E_{\ell,t}(z) = \frac{\eta^{\ell^t}(z)}{\eta(\ell^t z)}.$$

The following lemma summarizes necessary and well-known properties of $E_{\ell,t}(z)$.

Lemma 3.4. *The eta-quotient* $E_{\ell,t}$ *satisfies the followings*

(i) *For a prime* $\ell \ge 5$,

$$E_{\ell,t}(z) \in \mathcal{M}_{(\ell^t-1)/2}\left(\Gamma_0(\ell^t), \chi_{\ell,t}\right),$$

where $\chi_{\ell,t} = \left(\frac{(-1)^{(\ell^t-1)/2}\ell^t}{\cdot}\right)$ *denotes the Legendre symbol,*

(ii) $E_{\ell,t}(z)^{\ell^j} \equiv 1 \pmod{\ell^{j+1}}$ *for* $j \ge 0$,

(iii) $E_{\ell,t}(z)$ *vanishes at every cusp* a/c *with* $\ell^t \nmid c$.

4. PROOF OF THEOREM 1.3

Throughout the proof, we fix $N = \ell^j$, where ℓ is a prime ≥ 5, and j is a positive integer. Since our proof follows the works of K. Ono and S. Ahlgren ([1],[11]) and Mahlburg [10], we will not give every detail of each step.

Recall that

$$F(x,q) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M'(m,n) x^m q^n,$$

where $q = \exp(2\pi i z)$ and $z \in \mathcal{H}$. Then, by a simple argument,

$$\sum_{n=0}^{\infty} M'(m,N,n) q^n = \frac{1}{N} \sum_{s=0}^{N-1} F(\zeta^s, z) \zeta^{-ms}, \tag{12}$$

where $\zeta = \exp(2\pi i/N)$.

By (9) and (11), we have

$$F(\zeta^s, z) = \frac{-\omega_s^2 q^{1/8}}{4\pi^2} \frac{1}{\eta(z)\eta(2z)t_{0,s}(z)t_{0,s}(2z)}. \tag{13}$$

Therefore, by (12) and (13),

$$\sum_{n=0}^{\infty} N\cdot M'(m,N,n)q^n = \frac{-1}{4\pi^2}\sum_{s=1}^{N-1} \frac{\omega_s^2 \zeta^{-ms} q^{1/8}}{\eta(z)\eta(2z)t_{0,s}(z)t_{0,s}(2z)} + \sum_{n=0}^{\infty} a(n)q^n. \tag{14}$$

Remark. We have multiplied (12) by N, so as to ensure that the Fourier coefficients of

$$\frac{-1}{4\pi^2}\sum_{s=1}^{N-1} \frac{\omega_s^2 \zeta^{-ms} q^{1/8}}{\eta(z)\eta(2z)t_{0,s}(z)t_{0,s}(2z)}$$

are algebraic integers with a view toward applying Theorem 3.1.

Define $\delta_\ell = \frac{\ell^2-1}{24}$, and $\overline{\delta_\ell} = 3\delta_\ell$. We also define

$$g_m(z) = \left(\sum_{n=0}^{\infty} N\cdot M'(m,N,n)q^{n+\overline{\delta_\ell}}\right)(q^\ell;q^\ell)_\infty^\ell (q^{2\ell};q^{2\ell})_\infty^\ell. \tag{15}$$

Then, we have

$$\begin{aligned} g_m(z) &= \frac{-1}{4\pi^2}\sum_{s=1}^{N-1} \frac{\eta^\ell(\ell z)\eta^\ell(2\ell z)}{\eta(z)\eta(2z)} \frac{\omega_s^2\zeta^{-ms}}{t_{0,s}(z)t_{0,s}(2z)} + \frac{\eta^\ell(\ell z)\eta^\ell(2\ell z)}{\eta(z)\eta(2z)} \\ &=: \frac{1}{4\pi^2}\sum_{s=1}^{N-1} G_{m,s}(z) + P(z). \end{aligned}$$

In [7], Chan proved, for sufficiently large τ,

$$\left(\frac{P(z)|_{U_\ell}}{\eta^\ell(z)\eta^\ell(2z)} E_{\ell,1}^{\ell^\tau}\right)|_{V_8} \in S_k(\Gamma_0(128\ell,\chi), \tag{16}$$

for some positive integer k and Dirichlet character χ. Here, we prove the following similar result.

Theorem 4.1. *For sufficiently large τ, there is a positive integer k' such that*

$$\left(\frac{G_{m,s}(z)|_{U_\ell}}{\eta^\ell(z)\eta^\ell(2z)} E_{\ell,j+1}^{\ell^\tau}\right)|_{V_8} \in S_{k'}(\Gamma_1(128\ell N^2),\ \textit{for all } 1\le s\le N-1. \tag{17}$$

Throughout the proof, we will use the following notation.

$$q_m = e^{2\pi i z/m} = q^{1/m}, \lambda = e^{2\pi i/\ell}.$$

Proof. First, note that $\frac{\eta^\ell(\ell z)}{\eta(z)} \in \mathcal{M}_{(\ell-1)/2}(\Gamma_0(\ell), (\frac{\cdot}{\ell}))$. Thus, $G_{m,s}(z) \in M_{\ell+1}(\Gamma_1(4N^2))$. Since $\eta(z)\eta(2z) \in \mathcal{S}_1(\Gamma_0(128))$, the left side of (17) transforms correctly on $\Gamma_1(128N^2)$. By Lemma 3.4, if τ is sufficiently large, then we only need to show that $\frac{G_{m,s}(z)|_{U_\ell}}{\eta^\ell(z)\eta^\ell(2z)}$ vanishes at each cusp $\frac{a}{c}$ with $\ell N|c$. Since the Fourier expansion of $\eta(z)\eta(2z)$ at such cusps is of the form $B_0 q_2^{\ell/8} + \cdots$, where B_0 is a nonzero constant, it suffices to show that the Fourier expansion of $G_{m,s}|_{U_\ell}$ at such cusps is of the form $B_1 q_2^h + \cdots$, where B_1 is a constant and $h > \ell/8$. Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell N)$. Then,

$$G_{m,s}(z)|_{U_\ell}|_{\ell+1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \ell^{(\ell-1)/2} \sum_{j=0}^{\ell-1} G_{m,s}(z)|_{\ell+1} \begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix} |_{\ell+1} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{18}$$

Note that, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 1 & j' \\ 0 & \ell \end{pmatrix},$$

where

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+cj & (-aj' - cjj' + b + dj)/\ell \\ c\ell & -cj' + d \end{pmatrix}.$$

By choosing $j' \in \{0, 1, \ldots, j-1\}$ such that $-aj' + b + dj \equiv 0 \pmod{\ell}$, we have $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(\ell N)$. Note that as j runs over a complete residue system modulo ℓ, j' does as well. Thus,

$$G_{m,s}(z)|_{U_\ell}|_{\ell+1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \ell^{(\ell-1)/2} \sum_{j'=0}^{\ell-1} G_{m,s}(z)|_{\ell+1} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 1 & j' \\ 0 & \ell \end{pmatrix}.$$

For $f(z) \in M_k(\Gamma_0(\ell), \chi)$, we have

$$\begin{aligned} f(z)|V_2| \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} &= f(z)| \begin{pmatrix} 2a' & -a'v + b' \\ c' & (d' - c'v)/2 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 2 \end{pmatrix} & (19) \\ &= \chi\left(\frac{d' - c'v}{2}\right) f\left(\frac{z+v}{2}\right), & (20) \end{aligned}$$

because

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 2a' & -a'v + b' \\ c' & (d' - c'v)/2 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 2 \end{pmatrix}, \tag{21}$$

where

$$v = \begin{cases} 0 \text{ , if } d' \text{ is even,} \\ 1 \text{ , if } d' \text{ is odd.} \end{cases}$$

Thus, we have, by setting $u = (z+v)/2$,

$$\begin{aligned} G_{m,s}(z)|_{\ell+1} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} &= \left(\frac{\eta^\ell(\ell z)\eta^\ell(2\ell z)}{\eta(z)\eta(2z)} \frac{\omega_s^2 \zeta^{-ms}}{t_{0,s}(z) t_{0,s}(2z)} \right) |_{\ell+1} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \\ &= \chi(d')\chi((d' - c'v)/2) \frac{\eta^\ell(\ell z)}{\eta(z)} \frac{\eta^\ell(\ell u)}{\eta(u)} \frac{\omega_s^2 \zeta^{-ms}}{\beta t_{0,\overline{d's}}(z) \beta' t_{0,\overline{(d'-c'v)s/2}}(u)}, \end{aligned}$$

where β and β' are the roots of unity defined in Proposition 3.2, and $\chi(d) = \left(\frac{\cdot}{\ell}\right)$. Since $\ell N|c$, after some calculation, we can check that β, β', $\chi(d')$ and $\chi((d'-c'v)/2)$ do not depend on j'. In summary, we obtain

$$G_{m,s}(z)|_{\ell+1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = A_1 q_2^{\overline{\delta_\ell}}(-1)^{\overline{\delta_\ell}v}\left(1+\sum_{n\geq 0} c_1(n,j')q_2^n\right), \tag{22}$$

where A_1 is a nonzero constant not depending on j'.

Thus, we finally have

$$\begin{aligned} G_{m,s}(z)|_{U_\ell}|_{\ell+1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= A_1\sum_{j'=0}^{\ell-1}\left(q_2^{\overline{\delta_\ell}}(-1)^{\overline{\delta_\ell}v}\left(1+\sum_{n\geq 0}c_1(n,j')q_2^n\right)\right) | \begin{pmatrix} 1 & j' \\ 0 & \ell \end{pmatrix} \\ &= A_2 q_{2\ell}^{\overline{\delta_\ell}}\sum_{j'=0}^{\ell-1}\lambda^{\overline{\delta_\ell}j'/2}(-1)^{\overline{\delta_\ell}v}\left(1+\sum_{n\geq 1}c_2(n,j')q_{2\ell}^n\right) \\ &= q_{2\ell}^{\overline{\delta_\ell}}(\sum_{n\geq 1}c_3(n)q_{2\ell}^n), \end{aligned}$$

since

$$\sum_{j'=0}^{\ell-1}\lambda^{\overline{\delta_\ell}j'/2}(-1)^{\overline{\delta_\ell}v} = 0,$$

by a simple calculation. Since $1+\overline{\delta_\ell}-\ell^2/8>0$, we are done.

□

Now, we are ready to prove our Theorem 1.3. To that end,

$$g_m(z)|U_\ell = \left(\sum_{n=0}^{\infty} N\cdot M'(m,N,n)q^{n+\overline{\delta_\ell}}\right)|_{U_\ell}(q;q)_\infty^\ell(q^2;q^2)_\infty^\ell$$

and so

$$\frac{g_m(z)|U_\ell}{\eta^\ell(z)\eta^\ell(2z)} = \sum_{n=0}^{\infty} N\cdot M'(m,N,\ell n-\overline{\delta_\ell})q^{n-\frac{\ell}{8}}.$$

Thus, by Theorem 4.1, for sufficiently large t,

$$\left(\frac{g_m(z)|U_\ell}{\eta^\ell(z)\eta^\ell(2z)}E_{\ell,j+1}^{\ell^t}\right)|_{V_8} \equiv \sum_{\substack{n\geq 0\\ \ell n\equiv -1 \pmod 8}} N\cdot M'(m,N,\frac{\ell n+1}{8})q^n. \pmod{\ell^{\tau+j}} \tag{23}$$

$$\equiv H_1+H_2 \pmod{\ell^{\tau+j}}, \tag{24}$$

where $H_1\in\mathcal{S}_{k'}(\Gamma_1(128\ell N^2))$ and $H_2\in\mathcal{S}_k(\Gamma_0(128\ell),\chi)$. Then, by Theorem 3.1, a positive portion of primes $Q\equiv -1 \pmod{128\ell N^2}$ have the property that

$$H_1|_{T_Q} = H_2|_{T_Q}\equiv 0 \pmod{\ell^{\tau+j}}.$$

This implies that

$$N \cdot M'(m, N, \frac{\ell n Q + 1}{8}) \equiv 0 \pmod{\ell^{\tau+j}}, \text{ whenever } (n, Q) = 1.$$

This completes the proof of Theorem 1.3.

5. REMARKS

It would be nice to find a more natural combinatorial interpretation for the coefficients of $F(x,q)$ as in (4). After the author completed writing his paper, F. Garvan informed him that another crank analog for $a(n)$ was also studied by Z. Reti in his unpublished thesis [13].

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