Question 1

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in a vector space V. Complete the definitions:

(a) The vectors in S are **linearly independent** if

(3)

LOOK IN ...

(b) The set S **generates** V if

(3)

... THE BOOK ...

(c)The set S is a **basis** for V if

(3)

... OR CLASS NOTES !

Question 2

If A is a 5 x 5 matrix and $|A| = 10$, compute

(a) $|A^T|$ $= |A|$ (3)

$= 10$

(b) $|A^2|$ $= |A|^2$ (3)

$= 10^2$

$= 100$

(c) $|A^{-1}|$ $= |A|^{-1}$ (3)

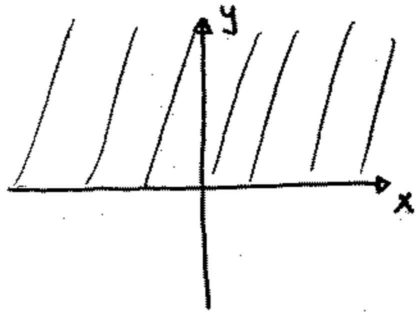
$= 10^{-1}$

$= 1/10$

Question 3

For each of the following, draw a sketch of H and explain why it is or is not a subspace of $\mathbb{R}^2$.

(a) $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ (3)

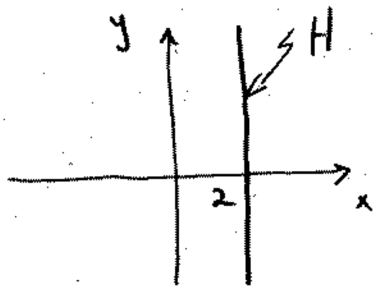


H is NOT closed under scalar multiplication

[e.g. $-1 \cdot (1,1) = (-1,-1)$ which is not in H]

Thus H is NOT a subspace

(b) $H = \{(x, y) \in \mathbb{R}^2 \mid x = 2\}$ (3)

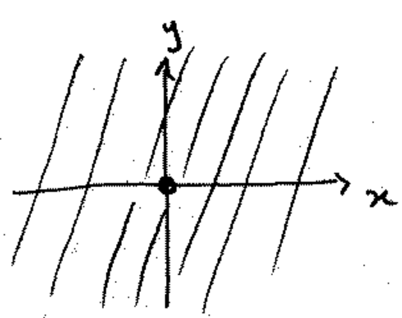


H is not closed under either $+$ or $\cdot$ [e.g. $(2,1)+(2,2)=(4,3)$]

Also H does not contain $\vec{0}$

Thus H is NOT a subspace

(a) $H = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$ (3)



H fails all 3 tests:

- not closed under $+$ [e.g. $(1,1)+(-1,-1)=(0,0)$]
- not closed under $\cdot$ [$0 \cdot (x,y) = (0,0)$]
- does not contain $\vec{0}$.

Thus H is NOT a subspace.

Question 4

Let $T : V \longrightarrow W$ be a linear map between vector spaces. If $T(v_1) = w_1$ and $T(v_2) = w_2$, find

(a) $T(0)$ (3)

For any linear map

$$\boxed{T(0) = 0}.$$

(b) $T(3v_1 - 5v_2) = 3T(v_1) - 5T(v_2)$ (3)

$= 3w_1 - 5w_2$

Question 5

Find a basis for $Col(A)$ if $A = \begin{pmatrix} 1 & -2 & 3 & -4 & 2 & 5 \\ -2 & 4 & -1 & -7 & 6 & -9 \\ 1 & -2 & 4 & -7 & 4 & 11 \\ -3 & 6 & -6 & 3 & 0 & 5 \end{pmatrix}$.

What is the dimension of $Col(A)$?

Reduce to echelon form and identify pivot columns:

$$A \to \begin{pmatrix} 1 & -2 & 3 & -4 & 2 & 5 \\ 0 & 0 & 5 & -15 & 10 & 1 \\ 0 & 0 & 1 & -3 & 2 & 6 \\ 0 & 0 & 3 & -9 & 6 & 20 \end{pmatrix} \quad \begin{array}{l} \text{row 2 + 2 row 1} \\ \text{row 3 } - \text{ row 1} \\ \text{row 4 + 3 row 1} \end{array}$$

$$\to \begin{pmatrix} 1 & -2 & 3 & -4 & 2 & 5 \\ 0 & 0 & 1 & -3 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & -29 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \begin{array}{l} \text{switch rows 2 and 3. Then:} \\ \text{row 3 - 5 row 2} \\ \text{row 4 - 3 row 2.} \end{array}$$

$$\to \begin{pmatrix} 1 & -2 & 3 & -4 & 2 & 5 \\ 0 & 0 & 1 & -3 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & -29 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence pivot columns are col's 1, 3, 6

Thus $Col(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 5 \\ -9 \\ 11 \\ 5 \end{bmatrix} \right\}$

basis for Col(A).

The dimension of Col(A) is $\underline{\underline{3}}$ since the basis has 3 elements.

Question 6

Let H be the subspace of $\mathbb{R}^4$ defined by

$$H = \{(x_1, x_2, x_3, x_4) \mid x_1 - x_3 + 2x_4 = 0 \text{ and } x_2 - x_3 = 0\}.$$

(a) Find a 2 x 4 matrix A such that $H = Nul(A)$ (4)

Set $\boxed{A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}}$

Then $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \iff \begin{cases} x_1 - x_3 + 2x_4 = 0 \\ x_2 - x_3 = 0 \end{cases}$

(b) Find a basis for H and compute its dimension.

(6)

A is in reduced echelon form. Hence we see that

$A\vec{x} = 0$ iff $\begin{cases} x_2 = x_3 \\ x_1 = x_3 - 2x_4 \end{cases}$

ie. $\vec{x} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Thus a basis for Nul(A) (ie. for H) is $\underline{\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}}$.

and Dimension $\underline{H = 2}$.

Question 7

Use Cramer's rule to find x_2 if $A = \begin{pmatrix} 1 & -1 & 4 \\ 2 & -2 & 3 \\ 3 & 1 & 5 \end{pmatrix}$ and $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

[Useful fact: $|A| = -44$.]

$$X_2 = \begin{vmatrix} 1 & 0 & 4 \\ 2 & 0 & 3 \\ 3 & 2 & 5 \end{vmatrix} \Big/ |A|$$

$$= -2 \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} \Big/ -44$$

$$= \frac{-2(3-8)}{-44}$$

$$= \underline{\underline{\frac{5}{22}}}$$

Question 8

Indicate True (T) or false (F) [no reasons need be given].

(a) If A is an n x n matrix and the equation $A\vec{x} = \vec{0}$ has no non-trivial solutions, then the columns of A are linearly independent as vectors in $\mathbb{R}^n$. T

(b) If $V = Span\{v_1, v_2, \ldots, v_n\}$, then $\{v_1, v_2, \ldots, v_n\}$ must be a basis for the vector space V. F

(c) The columns of a 10x17 matrix can never be linearly independent vectors in $\mathbb{R}^{10}$. T

(d) The area of the triangle with vertices at (0,0), (1,3), and (4,2) is $\left|\frac{1}{2}det\begin{pmatrix}1 & 4\\ 3 & 2\end{pmatrix}\right|$ T

(e) If $\{u_1, u_2, \ldots, u_m\}$ is a linearly independent set of vectors in a vector space V, and $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, then $m \leq n$. T

(f) Let $Nul(A)$ be the null space of a 5 x 5 matrix A. If the dimension of $Nul(A)$ is one, then the homogeneous equation $A\vec{x} = \vec{0}$ has only one solution. F

(g) If A and B are both 9 x 9 matrices then $det(A + B) = det(A) + det(B)$. F

(h) If $T : V \longrightarrow W$ is a linear map between vector spaces, then

$$Ker(T) = \{v \in V \mid T(v) = 0\}$$

F

is a sub*set* of V and a sub*space* of W.

(i)Switching two rows in an n x n matrix has no affect of the determinant of the matrix. F

(j) Row operations do not change the null space of a matrix. T