Question 1

Parts (a)-(d) refer to the following inhomogeneous system of linear equations:

$$x_1 + x_2 + x_3 + x_4 = 5$$
$$x_1 - x_2 + x_3 - x_4 = 7$$

(a) Rewrite the equations as a single vector equation.
[Hint: the equation should be of the form $x_1\vec{v_1} + \cdots + \vec{v_n} = \vec{b}$] (2)

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

(b) Rewrite the equations as a single matrix equation of the form $A\vec{x} = \vec{b}$. (2)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

(c) Find all the solutions (use whatever form of the equations you prefer most).

(8)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 5 \\ 1 & -1 & 1 & -1 & 7 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 1 & 5 \\ 0 & -2 & 0 & -2 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

Thus $x_2 = -1 - x_4$

$x_1 = 6 - x_3$

So

$$\boxed{\vec{X} = \begin{bmatrix} 6 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$$

(d) Describe the solution set (as a subspace of $\mathbb{R}^4$) for the corresponding *homogeneous* system of equations.

(3)

$$\text{Soln set} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

= plane spanned by the <u>2 vectors</u> $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Question 2

For parts (a) and (b) state (give reasons!) whether the given sets of vectors are linearly **dependent** *or linearly* **independent**.

(a) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 0 \\ 9 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ 4 \end{bmatrix}$ ~~[illegible]~~

(5)

The vectors are linearly dependent since the one is a multiple of the other.

(b) $\begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}$

(10)

$$A = \begin{bmatrix}-1 & 1 & 0\\ 2 & 0 & 1\\ 1 & 3 & -1\end{bmatrix} \to \begin{bmatrix}1 & -1 & 0\\ 2 & 0 & 1\\ 1 & 3 & -1\end{bmatrix}$$

$$\to \begin{bmatrix}1 & -1 & 0\\ 0 & 2 & 1\\ 0 & 4 & -1\end{bmatrix}$$

$$\to \begin{bmatrix}1 & -1 & 0\\ 0 & 2 & 1\\ 0 & 0 & -3\end{bmatrix}$$ This has a pivot in <u>every</u> column

Hence $A\vec{x}=0$ has <u>only</u> trivial solutions

Thus there are <u>no</u> <u>non-trivial</u> linear combinations

$x_1\begin{bmatrix}-1\\2\\1\end{bmatrix} + x_2\begin{bmatrix}1\\0\\3\end{bmatrix} + x_3\begin{bmatrix}0\\1\\-1\end{bmatrix} = \vec{0}$, i.e. the

vectors are $\boxed{\text{linearly independent}}$

Question 3

If $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ *find* A^{-1} *or prove that it does not exist.*

$$[A\ I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right] = [I, A^{-1}]$$

i.e. $\boxed{A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}}$

Question 4

Indicate True (T) or false (F) [no reasons need be given].

(a) If $A\vec{x} = \vec{0}$ has a non-trivial solution then the columns of matrix A form a linearly independent set of vectors.

F

(b) If A is an $n \times m$ matrix which, after being reduced to ecehelon form, has a pivot in every column, then $n \geq m$.

T

(c) If $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ are all non-zero vectors in $\mathbb{R}^3$, then $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ is a linearly dependent set.

T

(d) If A is a 10×4 matrix and B is a 4×27 matrix then $(AB)^T$ is a 10×27 matrix.

F

(e) A set of 19 linear equations in 23 unknowns will never have a unique solution.

T

(f) A set of 19 linear equations in 23 unknowns will always have at least one solution.

F

(g) A homogeneous set of linear equations is always consistent.

T

(h) If A^T is the transpose of an invertible matrix A, then $(A^T)^{-1} = (A^{-1})^T$.

T

(i) The matrix $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ is invertible.

F

$\boxed{\text{T: b, c, e, g, h.}}$

F: a, d, f, i, i

(j) Adding two columns is an allowed operation when reducing a matrix of coefficients to echelon form.

F