SOLUTIONS

Math 225

Midterm 1

February 18, 2010

- *Answer all questions.*
- *Be as clear as possible. Unless otherwise stated, give reasons for your answers.*
- *Use both sides of the paper, if necessary.*
- *Good luck!*

Question	Maximum	Your Score
1	15	
2	10	
3	15	
4	10	
5	15	
6	10	
TOTAL	75	

Question 1

Parts (a)-(c) refer to the following inhomogeneous system of linear equations:

$$
\begin{aligned}
x_1 - x_2 + 2x_3 &= 1 \\
x_1 + x_3 &= 3 \\
3x_1 + 3x_2 &= 15
\end{aligned}
$$

(a) Rewrite the equations as a single matrix equation. (3)

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 15 \end{bmatrix}.$$

(b) Rewrite the equations as a single vector equation. (3)

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

(c) Find all the solutions (use whatever form of the equations you most prefer).

(9)

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & 0 & 1 & 3 \\ 3 & 3 & 0 & 15 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 6 & -6 & 12 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So x_3 is free and $x_2 = 2 + x_3$

$x_1 = 3 - x_3$

ie. $$\boxed{X = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

Question 2

Find the value(s) of k for which $\begin{bmatrix} 3 \\ 6 \\ k \end{bmatrix}$ is contained in $Span\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ if

$$\overrightarrow{v_1} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \text{ and } \overrightarrow{v_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

(10)

Must Solve $x_1 \vec{v_1} + x_2 \vec{v_2} = \begin{bmatrix} 3 \\ 6 \\ k \end{bmatrix}$

ie. Reduce: $\begin{bmatrix} 4 & 1 & 3 \\ 0 & -1 & 6 \\ 2 & 1 & k \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 1 & 3 \\ 0 & -1 & 6 \\ 0 & 1/2 & k - 3/2 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 4 & 1 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & k + 3/2 \end{bmatrix}$

Hence System has a solution $\Leftrightarrow k + 3/2 = 0$

ie. $k = -3/2$

Question 3

(a) Complete the definition:

A set of vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}\}$ *in* $\mathbb{R}^n$ *is linearly independent if*

$$c_1\vec{v_1} + \cdots + c_k\vec{v_k} = 0$$
$$\Rightarrow c_1 = \cdots = c_k = 0$$

It is linearly dependent if

(6)

$\exists$ a <u>non-trivial</u> choice of $c_1, \ldots c_k$

s.t. $c_1\vec{v_1} + \cdots + c_k\vec{v_k} = 0$

For parts (b)-(d) state (give reasons!) whether the given sets of vectors are linearly **dependent** *or linearly* **independent**.

(b) $\begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ -8 \\ 0 \\ -2 \end{bmatrix}$ (3)

$\vec{v}_1 \quad \vec{v}_2$

$\vec{v}_2 = -2\vec{v}_1$ i.e. $\vec{v}_2 + 2\vec{v}_1 = 0$

dependent

(c) $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}-1\\1\\0\end{bmatrix}$

(3)

Reduce:

$$A = \begin{bmatrix}1 & -1 & -1\\1 & 1 & 1\\1 & -1 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & -1 & -1\\0 & 2 & 2\\0 & 0 & 1\end{bmatrix}$$

There are 3 pivots, i.e. one in each column, + hence the system $Ax=0$ has <u>only</u> the <u>trivial</u> solution.

Thus the vectors are linearly <u>independent</u>

(d) $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}-1\\1\\0\end{bmatrix}$, $\begin{bmatrix}5\\6\\7\end{bmatrix}$. (3)

4 vectors in $\mathbb{R}^3$ have to be

lin. dependent

Question 4

In a combustion reaction

x_1 molecules of $C_{10}H_8$ combine with
x_2 molecules of O_2 to form
x_3 molecules of CO_2 and
x_4 molecules of H_2O, i.e.

$$x_1 C_{10}H_8 + x_2 O_2 \longrightarrow x_3 CO_2 + x_4 H_2O \ .$$

(a) Using vectors in $\mathbb{R}^3$ to represent molecules made of carbon (C), hydrogen (H) and oxygen (O), write down a vector equation which describes the above chemical reaction. (4)

$$x_1 \begin{bmatrix} 10 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

or

$$\begin{bmatrix} 10 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Hence find integers x_1, x_2, x_3, x_4 to balance the chemical reaction. (6)

$$\begin{bmatrix} 10 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 10 & 0 & -1 & 0 \\ 0 & 0 & \frac{4}{5} & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 10 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 4/5 & -2 \end{bmatrix}.$$

Sol'n: x_4 free $\quad x_3 = \frac{5}{2} x_4$

$$x_2 = \frac{x_4 + 2x_3}{2} = 3x_4$$

$$x_1 = \frac{x_3}{10} = \frac{1}{4} x_4.$$

So with $\boxed{x_4 = 4}$ we get $x_1 = 1$

$x_2 = 12$

$x_3 = 10$

ie $\boxed{C_{10}H_8 + 12\,O_2 \longrightarrow 10\,CO_2 + 4\,H_2O}$

Question 5

Let $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 0 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Compute the following or say why they are not defined.

(a) AB (3)

$$= \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 \\ -2 & 5 \\ 0 & -4 \end{bmatrix}$$

(b) BC (3)

$$\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

(c) $A + B$ (3)

not defined : A is 3×2

B is 2×2.

(d) CA (3)

$$\underset{2\times 2}{C}\ \underset{3\times 2}{A}$$

Not defined

(e) $B + C$ (3)

$$\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

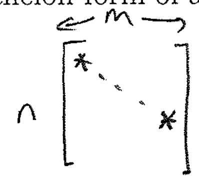
Question 6

Indicate True (T) or false (F) [no reasons need be given].

(a) If $A\mathrm{x} = \mathbf{0}$ has a non-trivial solution then the columns of the matrix A form a linearly independent set of vectors.

F

(b) If the echelon form of an $n \times m$ matrix has a pivot in every column, then $n \geq m$.



T

(c) If the echelon form of an $n \times m$ matrix A has a pivot in every row, then the equation $Ax = \mathbf{b}$ is consistent for every choice of $\mathbf{b}$.

T

(d) If A is a 10×6 matrix and B is a 6×19 matrix then the product (AB) is a 10×19 matrix.

T

(e) Any set of 13 linear equations in 20 unknowns has at least one solution.

F

(f) If a set of 13 linear equations in 20 unknowns has one solution then it has an infinite number of solutions.

T (there have to be free variables!)

(g) A homogeneous set of linear equations is always consistent.

T. ($x = 0$ is always a soln)

(h) Adding two columns is an allowed operation when reducing a matrix of coefficients to echelon form.

F.

(i) Every elementary row operation is reversible.

T

(j) If a system of linear equations is consistent, then it always has a finite number of solutions.

F.

(10)