Question 1

Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{bmatrix}$. *Find* A^{-1} *or prove that it does not exist.*

(10)

$$[A : I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & -4 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & -4 & 3 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 3/2 & 1/2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -9/2 & -3/2 \\ 0 & 1 & 0 & 4 & -2 & -1 \\ 0 & 0 & 1 & -2 & 3/2 & 1/2 \end{array}\right] = [I : A^{-1}]$$

i.e. $A^{-1} = \begin{bmatrix} 7 & -9/2 & -3/2 \\ 4 & -2 & -1 \\ -2 & 3/2 & 1/2 \end{bmatrix}$

Question 2

Use row operations to show that the determinant of

$$A = \begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix}$$

is $(x-y)(x-z)(z-y)$ and hence prove that the matrix is invertible whenever x, y and z are distinct real numbers. (10)

$$|A| = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & 0 & z^2-x^2-\left(\frac{z-x}{y-x}\right)(y^2-x^2) \end{vmatrix}$$

$$z^2-x^2-\left(\frac{z-x}{y-x}\right)(y^2-x^2) = z^2-x^2-(z-x)(y+x) = (z-x)(z+x-y-x)$$

Hence $|A| = 1 \cdot (y-x) \cdot (z-x)(z-y) = (x-y)(x-z)(z-y)$.

A is invertible $\iff |A| \neq 0$

$$\iff \begin{cases} x-y \neq 0 \\ x-z \neq 0 \\ z-y \neq 0 \end{cases}$$ i.e. iff x, y, z are distinct.

Question 3

(a) Without computing the entire inverse matrix, find the entry in row 1, column 3 of A^{-1} if

$$A = \begin{pmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

(5)

Use $A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$ to get

$$A^{-1}_{13} = \frac{1}{|A|} C_{31} \quad \text{where} \quad C_{31} = \begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} (-1)^{1+3} = 0$$

hence $\boxed{A^{-1}_{13} = 0}$.

(b) Use Cramer's rule to find x_2 if $A = \begin{pmatrix} 1 & -1 & 4 \\ 2 & -2 & 3 \\ 3 & 1 & 5 \end{pmatrix}$ and $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

[Useful fact: $|A| = 20$.] (5)

$$x_2 = \frac{|A_2(b)|}{|A|} \quad \text{where} \quad b = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

But $A_2(b) = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 0 & 3 \\ 3 & 2 & 5 \end{bmatrix}$, so $|A_2(b)| = (-1)^{3+2} 2 \cdot \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 5 \cdot 2$

Thus $\boxed{x_2 = 10/20 = \frac{1}{2}}$

Question 4

Let H be the subspace of $\mathbb{R}^4$ defined by

$$H = \{(x_1, x_2, x_3, x_4) \mid x_1 - 2x_2 + 7x_3 + x_4 = 0 \text{ and } x_1 + 3x_3 = 0\ \} .$$

(a) Find a 2 x 4 matrix A such that $H = Nul(A)$ (4)

Take

$$A = \begin{bmatrix} 1 & -2 & 7 & 1 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

(Then $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$ is equivalent to the equations which define H).

(b) Find a basis for H.

(6)

$H = \text{Nul}(A)$ so we need a basis for $\text{Nul}(A)$.

$$A \to \begin{bmatrix} 1 & -2 & 7 & 1 \\ 0 & 2 & -4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & -1/2 \end{bmatrix}$$

so x_3, x_4 are free variables and

$$x_2 = 2x_3 + \tfrac{1}{2}x_4$$

$$x_1 = -3x_3$$

ie $$X = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$$

ie. basis $= \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

Question 5

Complete the definitions:

(a) *A set of vectors* $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ *in* V *is* **linearly independent** *if* (5)

see book or notes !

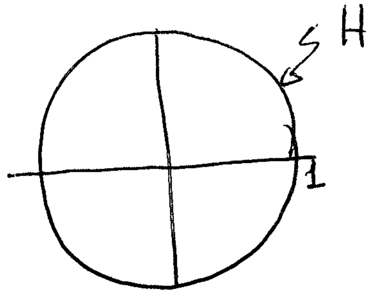
(b) *A set of vectors* $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ *in* V **generates** V *if* (5)

(c) *The set* S *is a* **basis** *for* V *if* (5)

Question 6

For each of the following, draw a sketch of H and explain either why it is or is not a subspace of $\mathbb{R}^2$.

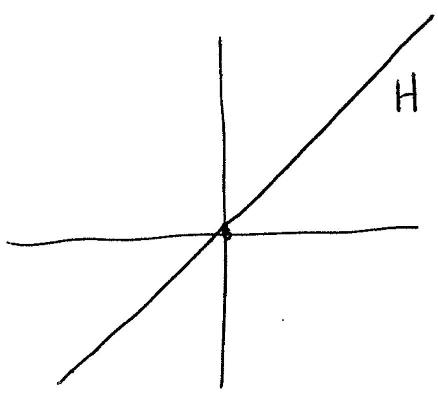
(a) $H = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ (5)



H is not closed under addition or scalar multiplication, and it does not contain (0,0).

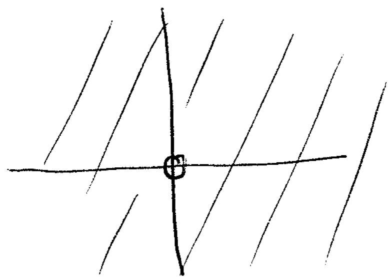
NOT a subspace.

(b) $H = \{(x, y) \in \mathbb{R}^2 \mid y = x\}$ (5)



$H = \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ and hence IS a subspace.

(c) $H = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$ (5)



H does not include (0,0)
and hence is NOT a subspace

(H is also not closed under + or scalar.).

Question 7

If the reduced echelon form of

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 & 2 & 5 \\ -2 & 4 & -1 & -7 & 6 & -9 \\ 1 & -2 & 4 & -7 & 4 & 11 \\ -3 & 6 & -6 & 3 & 0 & 5 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 1 & -2 & 0 & 5 & -4 & 0 \\ 0 & 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

give:

~~(a) The dimension of the null space $Nul(A)$ (4)~~

(b) A basis for $Col(A)$. ~~(7)~~ 9

Take columns of A corresponding to the pivot columns of the reduced echelon form, i.e.

$$\text{basis} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 5 \\ -9 \\ 11 \\ 5 \end{bmatrix} \right\}.$$

Question 8

Indicate True (T) or false (F) [no reasons need be given].

(a) If A and B are invertible $n \times n$ matrices then $A^{-1}B^{-1}$ is the inverse of the productAB.

F $\quad ((AB)^{-1} = B^{-1}A^{-1})$.

~~(b) If vector space V is 23-dimensional and S is a set of 23 non-zero vectors in V, then S is a basis for V.~~

(c) If A is an $n \times n$ matrix and the matrix equation $Ax = 0$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.

T

(d) The columns of a matrix A form a basis for the column space $col(A)$.

F

~~(e) If vector space V contains a set of 9 linearly independent vectors then the dimension of V is at least 9.~~

(f) Switching two rows in an n x n matrix has no affect of the determinant of the matrix.

F

(g) If a vector $\mathbf{y}$ in $\mathbb{R}^m$ is in the column space of an $m \times n$ matrix A, then $\mathbf{y}$ can be written as $A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.

T

(h) If a finite set S of non-zero vectors spans a vector space V, then some subset of S is a basis for V.

T

(i) Any system of n linear equations in n unknowns can be solved using Cramer's rule.

F

(j) If A and B are both 9 x 9 matrices then $det(A + B) = det(A) + det(B)$.

F