Question 1

Parts (a)-(d) refer to the following inhomogeneous system of linear equations:

$$
\begin{aligned}
x_1 - x_2 &= 2 \\
3x_1 - 2x_2 + x_3 &= 6 \\
2x_1 - 2x_2 + 4x_3 &= 2
\end{aligned}
$$

(a) Rewrite the equations as a single vector equation. (3)

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

(b) Rewrite the equations as a single matrix equation. (3)

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & -2 & 1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

(c) Find all the solutions (use whatever form of the equations you prefer most).

(9)

Row reduction on augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 3 & -2 & 1 & 6 \\ 2 & -2 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & -2 \end{bmatrix}$$

Back substitution:

$$4x_3 = -2 \implies \boxed{x_3 = -1/2}$$

$$x_2 = -x_3 \implies \boxed{x_2 = 1/2}$$

$$x_1 = 2 + x_2 \implies \boxed{x_1 = 5/2}$$

There is one solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

Question 2

Consider the *homogeneous* system of equations determined by the matrix of coefficients

$$A = \begin{bmatrix} -1 & 0 & 2 & 3 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

(a) Complete the following sentences:

The solutions to this system of equations can be thought of as

vectors in $\mathbb{R}^{\boxed{4}}$.

The set of all solutions can be described as a set spanned by n vectors,

where n is the number of free variables . (4)

(b) Without solving, determine the minimum number of free variables that there must be.

(5)

Max numbe of pivots = 2 (# rows)

$\therefore$ Min numbe of free variables is

$4 - 2 = \underline{\underline{2}}$

(c) Solve the homogeneous system, and describe the set of all solutions as the span of an appropriately chosen set of vectors.

(6)

$$\begin{bmatrix} -1 & 0 & 2 & 3 \\ 1 & 1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Hence: $\boxed{x_2 = -x_3 - 3x_4}$

$-x_1 = -2x_3 - 3X_4$ i.e. $\boxed{x_1 = 2X_3 + 3X_4}$

i.e. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\text{Soln Set} = \text{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

For parts (b)-(d) state (give reasons!) whether the given sets of vectors are linearly **dependent** *or linearly* **independent**.

(b) $\begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ -8 \\ 0 \\ -2 \\ -3 \end{bmatrix}$

(3)

linearly <u>independent</u> since they are <u>not</u> multiples of each other

(c) $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(3)

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Since there is a pivot in every column, the homogeneous system $Ax=0$ has <u>only</u> the trivial soln.

Hence the three vectors are <u>linearly independent</u>

(d) $\begin{bmatrix}-1\\10\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\5\\1\end{bmatrix}$

(3)

We have 4 vectors in $\mathbb{R}^3$.

Since $4 > 3$, the vectors must be linearly <u>dependent</u>

Question 4

If $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, find the value(s) of k for which $\begin{bmatrix} 3 \\ 1 \\ k \end{bmatrix}$ is contained in the set $Span\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$.

We need to be able to solve

$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ k \end{bmatrix}$$

Look at the augmented matrix:

$$\begin{bmatrix} 1 & 3 & 3 \\ -2 & 0 & 1 \\ 1 & 2 & k \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 6 & 7 \\ 0 & -1 & k-3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & k-3+\frac{7}{6} \end{bmatrix}$$

Hence, for consistency, we need

$$k-3+\frac{7}{6}=0$$

i.e. $\boxed{k = \frac{18-7}{6} = 11/6}$

Question 5

Indicate True (T) or false (F) [no reasons need be given].

(a) If $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution then the columns of the matrix A form a linearly independent set of vectors.

F

(b) If the echelon form of an $n \times m$ matrix which has a pivot in every column, then $n \geq m$.

T

(c) If $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ are all non-zero vectors in $\mathbb{R}^3$, then $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ is a linearly dependent set.

F

(d) If A is a 10×6 matrix and B is a 6×19 matrix then the product (AB) is a 10×19 matrix.

T

(e) A set of 13 linear equations in 20 unknowns can never have a unique solution.

T

(f) A set of 13 linear equations in 20 unknowns will always have at least one solution.

F

(g) A homogeneous set of linear equations is always consistent.

T

T: b, d, e, g, i

F: a, c, f, h, j

(h) Adding two columns is an allowed operation when reducing a matrix of coefficients to echelon form.

F

(i) If $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_6}\}$ is a linearly dependent set of vectors in $\mathbb{R}^9$, then at least one of them can be eliminated from any linear combination of them all.

T

(j) If a system of linear equations is consistent, then it always has a finite number of solutions.

F.