Question 1

Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{bmatrix}$. *Find* A^{-1} *or prove that it does not exist.*

(10)

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & -4 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & -4 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 3/2 & 1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -9/2 & -3/2 \\ 0 & 1 & 0 & 4 & -2 & -1 \\ 0 & 0 & 1 & -2 & 3/2 & 1/2 \end{array}\right] = [I \,|\, A^{-1}]$$

$$\boxed{A^{-1} = \begin{bmatrix} 7 & -9/2 & -3/2 \\ 4 & -2 & -1 \\ -2 & 3/2 & 1/2 \end{bmatrix}}$$

Question 2

(a) If A is a square matrix with $|A| = 10$, compute $|A^T|$ (2)

$$|A^T| = |A| = 10$$

(b) If A is a square matrix with $|A| = 3$, compute$|A^{-2}|$ (2)

$$|A^{-2}| = \frac{1}{|A|^2} = \frac{1}{9}$$

(c) Use Cramer's rule to find x_2 if $A = \begin{pmatrix} 1 & -1 & 4 \\ 2 & -2 & 3 \\ 0 & 1 & 5 \end{pmatrix}$ and $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. (6)

$$x_2 = \frac{\left| A_2 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & 0 & 4 \\ 2 & 0 & 3 \\ 0 & 2 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 4 \\ 2 & -2 & 3 \\ 0 & 1 & 5 \end{vmatrix}}$$

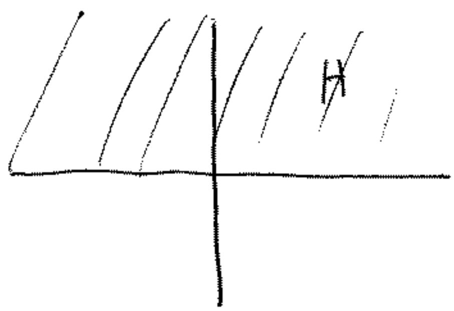
$$= \frac{-2 \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} -2 & 3 \\ 1 & 5 \end{vmatrix} - 2 \begin{vmatrix} -1 & 4 \\ 1 & 5 \end{vmatrix}}$$ (using col. 2) (using col. 1)

$$= \frac{-2(-5)}{-13-2(-9)} = \frac{10}{5} = \underline{\underline{2}}$$

Question 3

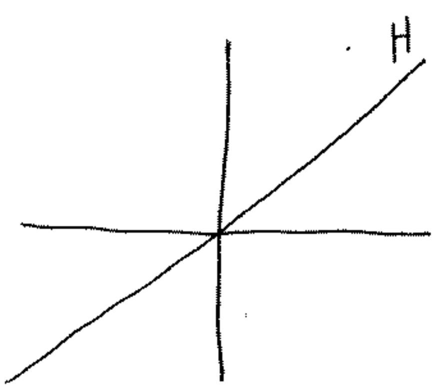
For each of the following, draw a sketch of H and explain either why it is or is not a subspace of $\mathbb{R}^2$.

(a) $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ NO (4)



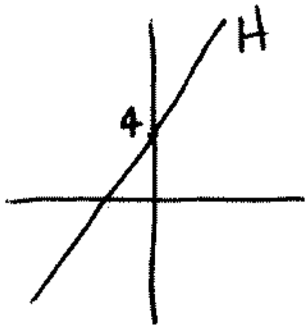
NOT closed under scalar multiplication

(b) $H = \{(x, y) \in \mathbb{R}^2 \mid y = x\}$ YES (4)



$H = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

(c) $H = \{(x, y) \in \mathbb{R}^2 \mid y = 3x + 4\}$ NO (4)



NOT closed under either $+$ or $\cdot$

Question 4

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in a vector space V. Complete the definitions:

(a) The vectors in S are **linearly independent** if

(3)

(b) The set S **generates** V if

(3)

(c)The set S is **a basis** for V if

(3)

Question 5

Given that the reduced echelon form of

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 & 2 & 5 \\ -2 & 4 & -1 & -7 & 6 & -9 \\ 1 & -2 & 4 & -7 & 4 & 11 \\ -3 & 6 & -6 & 3 & 0 & 5 \end{pmatrix} \quad \text{is} \quad \tilde{A} = \begin{pmatrix} 1 & -2 & 0 & 5 & -4 & 0 \\ 0 & 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

write down a basis for

(a) $Col(\tilde{A})$

(5)

Take Columns in $\tilde{A}$ with pivots!

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(b) $Col(A)$. (5)

Take columns in A corresponding to pivot columns in $\tilde{A}$ (ie. col's 1, 3, 6):

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 5 \\ -9 \\ 11 \\ 5 \end{bmatrix} \right\}$$

Question 6

Let H be the subspace of $\mathbb{R}^4$ defined by

$$H = \{(x_1, x_2, x_3, x_4) \mid x_1 - 2x_2 + 7x_3 + x_4 = 0 \text{ and } x_1 + 3x_3 = 0 \}.$$

(a) Find a 2 x ~~3~~ 4 matrix A such that $H = Nul(A)$ (3)

$$A = \begin{bmatrix} 1 & -2 & 7 & 1 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

(then $AX = 0$ is equivalent to

$$\left.\begin{aligned} x_1 - 2x_2 + 7x_3 + x_4 &= 0 \\ x_1 + 3x_3 &= 0 \end{aligned}\right\}$$)

(b) Find a basis for H. (7)

$$A \sim \begin{bmatrix} 1 & -2 & 7 & 1 \\ 0 & 2 & -4 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 7 & 1 \\ 0 & 1 & -2 & -1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & -1/2 \end{bmatrix}$$

So Soltns to $Ax=0$ satisfy: $\begin{cases} x_2 = 2x_3 + \frac{1}{2}x_4 \\ x_1 = -3x_3 \end{cases}$

ie $X = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$

Hence a basis for $Nul(A)$ is

$$\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Question 7

Indicate True (T) or false (F) [no reasons need be given].

(a) If A is an n x n matrix and the equation $A\vec{x} = \vec{0}$ has no non-trivial solutions, then A is invertible. T

(b) If $V = Span\{v_1, v_2, \ldots, v_n\}$, then $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. F

(c) The columns of a 9x11 matrix can never form a basis for $\mathbb{R}^9$. T

(d) Let W denote the set of all upper triangular 3×3 matrices, and let $Mat(3,3)$ denote the set of all 3×3 matrices. Then, using the usual rules for matrix addition and scalar multiplication, $Mat(3,3)$ is a vector space and W is a subspace of $mat(3,3)$. T

(e) If A and B are invertible 5×5 matrices then so is AB. T

(f) If A and B are both 9 x 9 matrices then $det(A + B) = det(A) + det(B)$. F

(g) Switching two rows in an n x n matrix has no affect of the determinant of the matrix. F

(h) Row operations do not change the column space of a matrix. F