1.1 *Problems*

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

1. $y' = 3x^2$; $y = x^3 + 7$
2. $y' + 2y = 0$; $y = 3e^{-2x}$
3. $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$
4. $y'' = 9y$; $y_1 = e^{3x}$, $y_2 = e^{-3x}$
5. $y' = y + 2e^{-x}$; $y = e^x - e^{-x}$
6. $y'' + 4y' + 4y = 0$; $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$
7. $y'' - 2y' + 2y = 0$; $y_1 = e^x \cos x$, $y_2 = e^x \sin x$
8. $y'' + y = 3\cos 2x$, $y_1 = \cos x - \cos 2x$, $y_2 = \sin x - \cos 2x$
9. $y' + 2xy^2 = 0$; $y = \dfrac{1}{1+x^2}$
10. $x^2y'' + xy' - y = \ln x$; $y_1 = x - \ln x$, $y_2 = \dfrac{1}{x} - \ln x$
11. $x^2y'' + 5xy' + 4y = 0$; $y_1 = \dfrac{1}{x^2}$, $y_2 = \dfrac{\ln x}{x^2}$
12. $x^2y'' - xy' + 2y = 0$; $y_1 = x\cos(\ln x)$, $y_2 = x\sin(\ln x)$

In Problems 13 through 16, substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation.

13. $3y' = 2y$
14. $4y'' = y$
15. $y'' + y' - 2y = 0$
16. $3y'' + 3y' - 4y = 0$

In Problems 17 through 26, first verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition.

17. $y' + y = 0$; $y(x) = Ce^{-x}$, $y(0) = 2$
18. $y' = 2y$; $y(x) = Ce^{2x}$, $y(0) = 3$
19. $y' = y + 1$; $y(x) = Ce^x - 1$, $y(0) = 5$
20. $y' = x - y$; $y(x) = Ce^{-x} + x - 1$, $y(0) = 10$
21. $y' + 3x^2y = 0$; $y(x) = Ce^{-x^3}$, $y(0) = 7$
22. $e^y y' = 1$; $y(x) = \ln(x + C)$, $y(0) = 0$
23. $x\dfrac{dy}{dx} + 3y = 2x^5$; $y(x) = \frac{1}{4}x^5 + Cx^{-3}$, $y(2) = 1$
24. $xy' - 3y = x^3$; $y(x) = x^3(C + \ln x)$, $y(1) = 17$
25. $y' = 3x^2(y^2 + 1)$; $y(x) = \tan(x^3 + C)$, $y(0) = 1$
26. $y' + y\tan x = \cos x$; $y(x) = (x + C)\cos x$, $y(\pi) = 0$

In Problems 27 through 31, a function $y = g(x)$ is described by some geometric property of its graph. Write a differential equation of the form $dy/dx = f(x, y)$ having the function g as its solution (or as one of its solutions).

27. The slope of the graph of g at the point (x, y) is the sum of x and y.
28. The line tangent to the graph of g at the point (x, y) intersects the x-axis at the point $(x/2, 0)$.
29. Every straight line normal to the graph of g passes through the point $(0, 1)$.
30. The graph of g is normal to every curve of the form $y = x^2 + k$ (k is a constant) where they meet.
31. The line tangent to the graph of g at (x, y) passes through the point $(-y, x)$.

In Problems 32 through 36, write—in the manner of Eqs. (3) through (6) of this section—a differential equation that is a mathematical model of the situation described.

32. The time rate of change of a population P is proportional to the square root of P.
33. The time rate of change of the velocity v of a coasting motorboat is proportional to the square of v.
34. The acceleration dv/dt of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.
35. In a city having a fixed population of P persons, the time rate of change of the number N of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor.
36. In a city with a fixed population of P persons, the time rate of change of the number N of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not.

In Problems 37 through 42, determine by inspection at least one solution of the given differential equation. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

37. $y'' = 0$
38. $y' = y$
39. $xy' + y = 3x^2$
40. $(y')^2 + y^2 = 1$
41. $y' + y = e^x$
42. $y'' + y = 0$
43. In Example 7 we saw that $y(x) = 1/(C - x)$ defines a one-parameter family of solutions of the differential equation $dy/dx = y^2$. (a) Determine a value of C so that

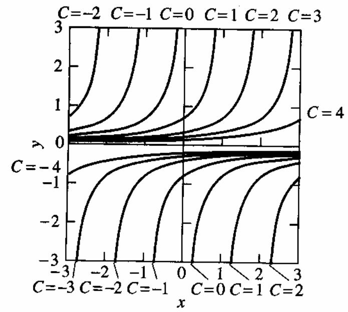

FIGURE 1.1.6. Graphs of solutions of the equation $dy/dx = y^2$.

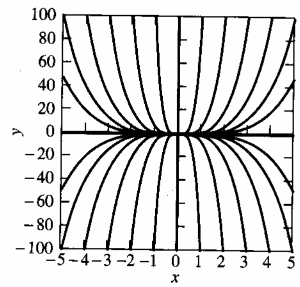

FIGURE 1.1.7. The graph $y = Cx^4$ for various values of C.

$y(10) = 10$. (b) Is there a value of C such that $y(0) = 0$? Can you nevertheless find by inspection a solution of $dy/dx = y^2$ such that $y(0) = 0$? (c) Figure 1.1.6 shows typical graphs of solutions of the form $y(x) = 1/(C - x)$. Does it appear that these solution curves fill the entire xy-plane? Can you conclude that, given any point (a, b) in the plane, the differential equation $dy/dx = y^2$ has exactly one solution $y(x)$ satisfying the condition $y(a) = b$?

44. (a) Show that $y(x) = Cx^4$ defines a one-parameter family of differentiable solutions of the differential equation $xy' = 4y$ (Fig. 1.1.7). (b) Show that

$$y(x) = \begin{cases} -x^4 & \text{if } x < 0, \\ x^4 & \text{if } x \geqq 0 \end{cases}$$

defines a differentiable solution of $xy' = 4y$ for all x, but is not of the form $y(x) = Cx^4$. (c) Given any two real numbers a and b, explain why—in contrast to the situation in part (c) of Problem 43—there exist infinitely many differentiable solutions of $xy' = 4y$ that all satisfy the condition $y(a) = b$.

1.2 Problems

In Problems 1 through 10, find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

1. $\frac{dy}{dx} = 2x + 1$; $y(0) = 3$

2. $\frac{dy}{dx} = (x-2)^2$; $y(2) = 1$

3. $\frac{dy}{dx} = \sqrt{x}$; $y(4) = 0$

4. $\frac{dy}{dx} = \frac{1}{x^2}$; $y(1) = 5$

5. $\frac{dy}{dx} = \frac{1}{\sqrt{x+2}}$; $y(2) = -1$

6. $\frac{dy}{dx} = x\sqrt{x^2+9}$; $y(-4) = 0$

7. $\frac{dy}{dx} = \frac{10}{x^2+1}$; $y(0) = 0$

8. $\frac{dy}{dx} = \cos 2x$; $y(0) = 1$

9. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$; $y(0) = 0$

10. $\frac{dy}{dx} = xe^{-x}$; $y(0) = 1$

In Problems 11 through 18, find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and initial velocity $v_0 = v(0)$.

11. $a(t) = 50$, $v_0 = 10$, $x_0 = 20$

12. $a(t) = -20$, $v_0 = -15$, $x_0 = 5$

13. $a(t) = 3t$, $v_0 = 5$, $x_0 = 0$

14. $a(t) = 2t + 1$, $v_0 = -7$, $x_0 = 4$

15. $a(t) = 4(t+3)^2$, $v_0 = -1$, $x_0 = 1$

16. $a(t) = \frac{1}{\sqrt{t+4}}$, $v_0 = -1$, $x_0 = 1$

17. $a(t) = \frac{1}{(t+1)^3}$, $v_0 = 0$, $x_0 = 0$

18. $a(t) = 50 \sin 5t$, $v_0 = -10$, $x_0 = 8$

19. What is the maximum height attained by the arrow of part (b) of Example 3?

20. A ball is dropped from the top of a building 400 ft high. How long does it take to reach the ground? With what speed does the ball strike the ground?

21. The brakes of a car are applied when it is moving at 100 km/h and provide a constant deceleration of 10 meters per second per second (m/s^2). How far does the car travel before coming to a stop?

22. A projectile is fired straight upward with an initial velocity of 100 m/s from the top of a building 20 m high and falls to the ground at the base of the building. Find (a) its maximum height above the ground; (b) when it passes the top of the building; (c) its total time in the air.

23. A ball is thrown straight downward from the top of a tall building. The initial speed of the ball is 10 m/s. It strikes the ground with a speed of 60 m/s. How tall is the building?

24. A baseball is thrown straight downward with an initial speed of 40 ft/s from the top of the Washington Monument (555 ft high). How long does it take to reach the ground, and with what speed does the baseball strike the ground?

25. A diesel car gradually speeds up so that for the first 10 s its acceleration is given by

$$\frac{dy}{dx} = (0.12)t^2 + (0.6)t \quad (\text{ft/s}^2).$$

If the car starts from rest ($x_0 = 0$, $v_0 = 0$), find the distance it has traveled at the end of the first 10 s and its velocity at that time.

26. A car traveling at 60 mi/h (88 ft/s) skids 176 ft after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration? For how long does the skid continue?

27. The skid marks made by an automobile indicated that its brakes were fully applied for a distance of 75 m before it came to a stop. The car in question is known to have a constant deceleration of 20 m/s^2 under these conditions. How fast—in km/h—was the car traveling when the brakes were first applied?

28. Suppose that a car skids 15 m if it is moving at 50 km/h when the brakes are applied. Assuming that the car has the same constant deceleration, how far will it skid if it is moving at 100 km/h when the brakes are applied?

29. On the planet Gzyx, a ball dropped from a height of 20 ft hits the ground in 2 s. If a ball is dropped from the top of a 200-ft-tall building on Gzyx, how long will it take to hit the ground? With what speed will it hit?

30. A person can throw a ball straight upward from the surface of the earth to a maximum height of 144 ft. How high could this person throw the ball on the planet Gzyx of Problem 29?

31. A stone is dropped from rest at an initial height h above the surface of the earth. Show that the speed with which it strikes the ground is $v = \sqrt{2gh}$.

32. If a woman has enough "spring" in her legs to jump vertically to a height of 2.25 ft on the earth, how high could she jump on the moon, where the surface gravitational acceleration is (approximately) 5.3 ft/s^2?

33. At noon a car starts from rest at point A and proceeds at constant acceleration along a straight road toward point B. If the car reaches B at 12:50 P.M. with a velocity of 60 mi/h, what is the distance from A to B?

34. At noon a car starts from rest at point A and proceeds with constant acceleration along a straight road toward point C, 35 miles away. If the constantly accelerated car arrives at C with a velocity of 60 mi/h, at what time does it arrive at C?

35. If $a = 0.5$ mi and $v_0 = 9$ mi/h as in Example 4, what must the swimmer's speed v_S be in order that he drifts only 1 mile downstream as he crosses the river?

36. Suppose that $a = 0.5$ mi, $v_0 = 9$ mi/h, and $v_S = 3$ mi/h as in Example 4, but that the velocity of the river is given by the fourth-degree function

$$v_R = v_0\left(1 - \frac{x^4}{a^4}\right)$$

rather than the quadratic function in Eq. (18). Now find how far downstream the swimmer drifts as he crosses the river.

1.3 Problems

In Problems 1 through 10, we have provided the direction field of the indicated differential equation, together with one or more solution curves. Sketch likely solution curves through the additional points marked in each direction field. (One method: Photocopy the direction field and draw your solution curves in a second color. Another method: Use tracing paper.)

1. $\dfrac{dy}{dx} = -y - \sin x$

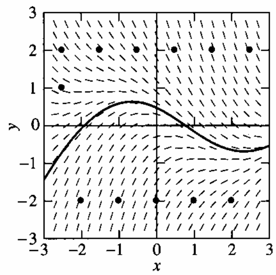

FIGURE 1.3.8.

2. $\dfrac{dy}{dx} = x + y$

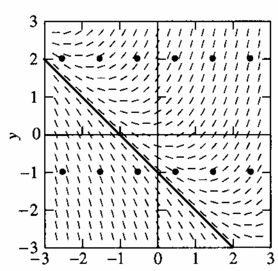

FIGURE 1.3.9.

3. $\dfrac{dy}{dx} = y - \sin x$

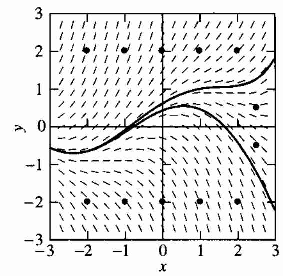

FIGURE 1.3.10.

4. $\dfrac{dy}{dx} = x - y$

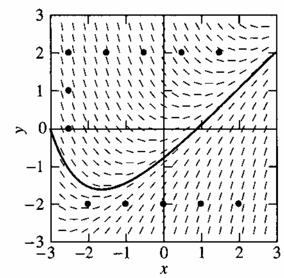

FIGURE 1.3.11.

5. $\dfrac{dy}{dx} = y - x + 1$

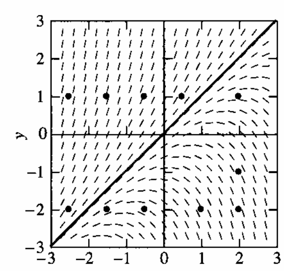

FIGURE 1.3.12.

6. $\dfrac{dy}{dx} = x - y + 1$

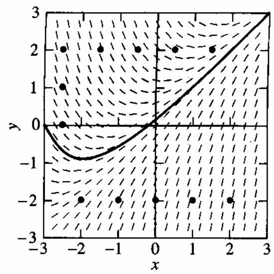

FIGURE 1.3.13.

7. $\dfrac{dy}{dx} = \sin x + \sin y$

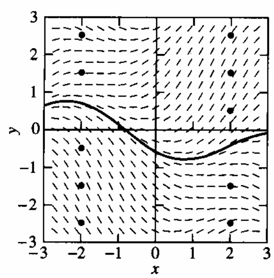

FIGURE 1.3.14.

8. $\dfrac{dy}{dx} = x^2 - y$

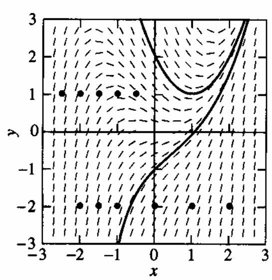

FIGURE 1.3.15.

9. $\dfrac{dy}{dx} = x^2 - y - 2$

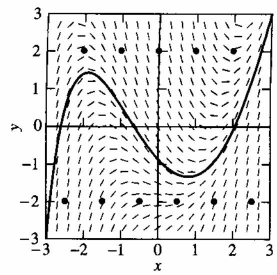

FIGURE 1.3.16.

10. $\dfrac{dy}{dx} = -x^2 + \sin y$

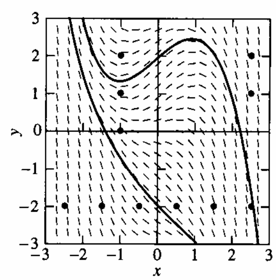

FIGURE 1.3.17.

In Problems 11 through 20, identify the isoclines of the given differential equation. Draw a sketch showing several of these isoclines, each marked with short line segments having the appropriate slope.

11. $\dfrac{dy}{dx} = x - 1$

12. $\dfrac{dy}{dx} = x + y$

13. $\dfrac{dy}{dx} = y^2$

14. $\dfrac{dy}{dx} = \sqrt[3]{y}$

15. $\dfrac{dy}{dx} = \dfrac{y}{x}$

16. $\dfrac{dy}{dx} = x^2 - y^2$

17. $\dfrac{dy}{dx} = xy$

18. $\dfrac{dy}{dx} = x - y^2$

19. $\dfrac{dy}{dx} = y - x^2$

20. $\dfrac{dy}{dx} = ye^{-x}$

In Problems 21 through 30, determine whether Theorem 1 does or does not guarantee existence of a solution of the given initial value problem. If existence is guaranteed, determine whether Theorem 1 does or does not guarantee uniqueness of that solution.

21. $\dfrac{dy}{dx} = 2x^2y^2;\quad y(1) = -1$

22. $\dfrac{dy}{dx} = x\ln y;\quad y(1) = 1$

23. $\dfrac{dy}{dx} = \sqrt[3]{y};\quad y(0) = 1$

24. $\dfrac{dy}{dx} = \sqrt[3]{y};\quad y(0) = 0$

25. $\dfrac{dy}{dx} = \sqrt{x-y};\quad y(2) = 2$

26. $\dfrac{dy}{dx} = \sqrt{x-y};\quad y(2) = 1$

27. $y\dfrac{dy}{dx} = x - 1;\quad y(0) = 1$

28. $y\dfrac{dy}{dx} = x - 1;\quad y(1) = 0$

29. $\dfrac{dy}{dx} = \ln(1+y^2);\quad y(0) = 0$

30. $\dfrac{dy}{dx} = x^2 - y^2;\quad y(0) = 1$

The next six problems illustrate that if the hypotheses of Theorem 1 fail at a point (a, b), then there may be no solutions, finitely many solutions, or infinitely many solutions passing through (a, b).

31. Show that on the interval $[0, \pi]$, the functions $y_1(x) \equiv 1$ and $y_2(x) = \cos x$ both satisfy the initial value problem

$$\frac{dy}{dx} + \sqrt{1-y^2} = 0,\quad y(0) = 1.$$

Why does this fact not contradict Theorem 1? Explain your answer carefully.

32. Find by inspection two different solutions of the initial value problem

$$\frac{dy}{dx} = 3y^{2/3},\quad y(0) = 0.$$

Why does the existence of different solutions not contradict Theorem 1?

33. Use Fig. 1.3.18 as a suggestion for showing that the initial value problem

$$\frac{dy}{dx} = 3y^{2/3},\quad y(-1) = -1$$

has infinitely many solutions. Why does this not contradict Theorem 1?

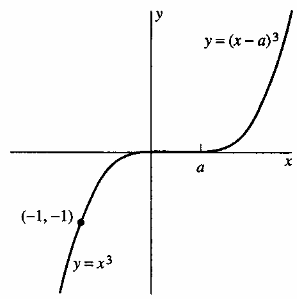

FIGURE 1.3.18. A suggestion for Problem 33.

34. Verify that if k is a constant, then $y = kx$ satisfies the differential equation $xy' = y$. Hence conclude that the initial value problem

$$x\frac{dy}{dx} = y,\quad y(0) = 0$$

has infinitely many solutions on any open interval containing $x = 0$.

35. Use isoclines to construct a direction field for the differential equation $xy' = y$. Explain why this direction field suggests that the initial value problem

$$x\frac{dy}{dx} = y,\quad y(a) = b$$

has (a) a unique solution if $a \neq 0$, (b) no solution if $a = 0$ and $b \neq 0$, (c) infinitely many solutions if $a = b = 0$. Are these results consistent with Theorem 1?

36. Consider the differential equation $dy/dx = 4x\sqrt{y}$ for $y \geqq 0$. Apply Theorem 1 to find those points (a, b) such that a solution of the differential equation *must* exist on an open interval J containing a. Next, find those points (a, b) such that a unique solution exists on an open interval J_0 containing a. Then find how many solutions pass through $(0, 0)$. (*Suggestion:* Note that $y_1(x) \equiv 0$ is a solution, as is $y_2(x) = (x^2 + C)^2$ if $x^2 + C \geqq 0$.)