

Homework 1 - 3 (selected solutions)

Question 1 on page 67

Compute:

$$\begin{aligned}\beta(s) &= \left(\frac{4}{5} \cos(s), 1 - \sin(s), -\frac{3}{5} \cos(s)\right) \\ \beta'(s) &= \left(-\frac{4}{5} \sin(s), -\cos(s), \frac{3}{5} \sin(s)\right) = \vec{T}(s) \\ \beta''(s) &= \left(-\frac{4}{5} \cos(s), \sin(s), \frac{3}{5} \cos(s)\right) = \vec{T}'(s)\end{aligned}$$

Since $|\vec{T}'| = 1$, we see that

$$\begin{aligned}\kappa(s) &= 1 \\ \vec{N}(s) &= \left(-\frac{4}{5} \cos(s), \sin(s), \frac{3}{5} \cos(s)\right) \\ \vec{B}(s) &= \vec{T}(s) \times \vec{N}(s) = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right)\end{aligned}$$

Since \vec{B} is constant, it follows that the torsion (τ) is zero. Moreover, the curve lies in the plane orthogonal to $\vec{B} = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right)$ and has constant curvature $\kappa = 1$. That is, it's a circle of radius 1. This can also be seen by writing

$$\beta(s) = (0, 1, 0) + \cos(s)\left(\frac{4}{5}, 0, -\frac{3}{5}\right) + \sin(s)(0, -1, 0),$$

which shows furthermore that the center of the circle is at $(0, 1, 0)$.

Question 6 on page 67

Computing $\gamma'(s)$ and $\gamma''(s)$ and evaluating at $s = 0$ gives

$$\begin{aligned}\vec{c} + r\vec{e}_1 &= \beta(0), \\ \vec{e}_2 &= \beta'(0) = \vec{T}(0), \\ -\frac{1}{r}\vec{e}_1 &= \beta''(0) = \kappa(0)\vec{N}(0).\end{aligned}$$

Since \vec{e}_1 and $\vec{N}(0)$ are unit vectors, it follows from the the third equation that

$$r = \frac{1}{\kappa(0)} \text{ and } \vec{e}_1 = -\vec{N}(0) .$$

The first equation thus yields

$$\vec{c} = \beta(0) + \frac{1}{\kappa(0)}\vec{N}(0) .$$

The osculating circle is thus given by

$$\gamma(s) = \beta(0) + \frac{1 - \cos(\kappa(0)s)}{\kappa(0)}\vec{N}(0) + \frac{\sin(\kappa(0)s)}{\kappa(0)}\vec{T}(0) .$$

This lies in the plane perpendicular to $\vec{B}(0)$, i.e. it lies in the osculating plane.

Question 3 on page 106

The inverse to $F = T_a C$ is $F^{-1} = C^{-1}T_{-a}$. Here C^{-1} is the orthogonal transformation inverse to C , i.e. represented by the inverse of the matrix which represents C , and T_{-a} is translation by $-a$, i.e. T_{-a} is the inverse of T_a .

Question 6(b) on page 128

The two curves are both parabolas passing through the origin; one, $\alpha(t)$, lies in the x-y plane and the other, $\beta(t)$ lies in the plane $\{x + y = 0\}$. With respect to appropriately chosen co-ordinates both are graphs of the function

$$y = \frac{1}{2}x^2 .$$

Using just these observations, we can conclude that the curves are related by an isometry which keeps the origin fixed, i.e. by an orthogonal matrix. A little further thought will lead you to this matrix:

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \quad (1)$$

i.e.

$$\begin{pmatrix} -t \\ t \\ t^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}t \\ t^2 \\ 0 \end{pmatrix} \quad (2)$$

Another way to arrive at the same conclusion is to compute the Frenet frame for each curve at $t = 0$ (where $\alpha(0) = \beta(0) = (0, 0, 0)$), and then to find the orthogonal matrix which relates the two frames. (If the curves are related by an orthogonal transformation, then it must be the orthogonal transformation which relates their Frenet frames.)

Using the formulae for arbitrary speed curves, you will find that the Frenet frame for $\alpha(t)$ at $t = 0$ has

$$\begin{aligned} T_\alpha &= (1, 0, 0), \\ N_\alpha &= (0, 1, 0), \\ B_\alpha &= (0, 0, 1) \end{aligned}$$

while the frame for $\beta(t)$ at $t = 0$ has

$$\begin{aligned} T_\beta &= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \\ N_\beta &= (0, 0, 1), \\ B_\beta &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \end{aligned}$$