## Question 1

Let

$$\alpha(t) = (t, \cosh(t), 0)\ ,\ t > 0$$

(a) Find the speed of $\alpha(t)$. Use this to reparameterize the curve by arclength.

(10)

$$\alpha'(t) = (1, \sinh(t), 0)$$

So $v(t) \equiv |\alpha'(t)| = \sqrt{1+\sinh^2(t)} = \underline{\underline{\cosh(t)}}.$

Let $s = \int_0^t \cosh(t)\, dt = \sinh(t).$

So $t = \sinh^{-1}(s)$

and $\cosh(t) = \sqrt{1+\sinh^2(t)}$

$= \underline{\underline{\sqrt{1+s^2}}}$

Thus $\alpha(t(s)) = (\sinh^{-1}(s), \sqrt{1+s^2}, 0)$

$= \beta(s).$

(b)Without doing any further calculations, write down the unit bi-normal $B$ for $\alpha(t)$ and explain why the torsion is zero.

(10)

$\alpha(t)$ is a planar curve lying in the plane $\{z=0\}$, ie.

$$\boxed{\vec{B} = (0,0,1)}$$

The torsion is zero because

$\underline{\underline{\vec{B} = \text{constant}}}$.

## Question 2

For the curve

$$\alpha(t) = (t\sin(t), \cos(t), t)\ ,\ -\infty < t < \infty$$

(a) Prove that its speed $|\alpha'(t)|$ is never zero. At which $t$, if any, does the curve have unit speed, i.e. when is $|\alpha'(t)| = 1$?

(8)

$$\alpha'(t) = \left(t\cos(t) + \sin(t)\ ,\ -\sin(t)\ ,\ 1\right)$$

So $$|\alpha'(t)|^2 = 1 + \underbrace{\sin^2(t) + \left(t\cos(t) + \sin(t)\right)^2}_{\geq 0}$$

Thus $|\alpha'(t)|^2 \geq 1 > 0$ So $|\alpha'(t)|$ never zero

Also $$|\alpha'(t)| = 1 \iff \begin{cases} \sin(t) = 0 \\ t\cos t + \sin(t) = 0 \end{cases}$$

$$\iff \begin{cases} \sin(t) = 0 \\ t\cos(t) = 0 \end{cases}$$

$$\iff t = 0$$

(b) Compute the Frenet data, i.e. $\vec{T}, \vec{N}, \vec{B}, \kappa$, and $\tau$, at $t = 0$.

(12)

Use $\alpha'(t) = (t \operatorname{Cos} t + \operatorname{Sin} t, -\operatorname{Sin} t,$ [illegible] $)$

$\alpha''(t) = (-t \operatorname{Sin} t + 2 \operatorname{Cos} t, -\operatorname{Cos} t, 0)$.

$\alpha'''(t) = (-t \operatorname{Cos} t - 3 \operatorname{Sin} t, \operatorname{Sin} t, 0)$

So at $t = 0$:

$$\begin{cases} \alpha'(0) = (0, 0, 1) & , \ |\alpha'(0)| = 1. \\ \alpha''(0) = (2, -1, 0) \\ \alpha'''(0) = (0, 0, 0). \end{cases}$$

Hence $\vec{T} = \dfrac{\alpha'}{\|\alpha'\|} = (0, 0, 1)$

$$\vec{B} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} = (1, 2, 0)/\sqrt{5}$$

$$\vec{N} = \vec{B} \times \vec{T} = (2, -1, 0)/\sqrt{5}$$

$$\kappa = \|\alpha' \times \alpha''\| / \|\alpha'\|^3 = \sqrt{5}$$

$$\tau = (\alpha' \times \alpha'') \cdot \alpha''' / \|\alpha' \times \alpha''\|^2 = 0$$

## Question 3

At $s = 1$ a unit-speed curve $\beta(s)$ has Frenet frame

$$\begin{aligned}
\vec{T} &= (1, 0, 0), \\
\vec{N} &= (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \\
\vec{B} &= (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})
\end{aligned}$$

If the curvature and torsion at $s = 1$ are $\kappa = 2$ and $\tau = \sqrt{2}$, find the vectors $\vec{T}', \vec{N}', \vec{B}'$ at $s = 1$.

(10)

Use

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

So

$$\boxed{\begin{aligned}
T' &= 2N = (0, \sqrt{2}, \sqrt{2}). \\
N' &= -2T + \sqrt{2}B = (-2, -1, 1) \\
B' &= -\sqrt{2}N \quad = (0 -1, -1).
\end{aligned}}$$