Question 4

Let $f(x, y)$ be any smooth function of (x, y), defined for all (x, y) in an open set $D \subset \mathbb{R}^2$.

(a) Show that

$$\sigma(x, y) = (x, y, f(x, y))$$

is a good patch defining a surface in $\mathbb{R}^3$.

(10)

σ is 1-1

$$\sigma_* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$ always 1-1

(b) If $f(x,y) = \sin(\frac{\pi x}{2})\sin(\frac{\pi y}{3})$ and $\sigma(x,y)$ is as in part (a), find the tangent vectors to the parameter curves

$$\sigma_{y_0}(x) = (x, y_0, \sin(\frac{\pi x}{2})\sin(\frac{\pi y_0}{3}))$$
$$\sigma_{x_0}(y) = (x_0, y, \sin(\frac{\pi x_0}{2})\sin(\frac{\pi y}{3}))$$

and hence find a normal vector to the surface at the point $\vec{p} = (1, 1, \sqrt{3}/2)$.

(10)

$$\sigma_{y_0}'(x) = \left(1, 0, \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi y_0}{3}\right)\right)$$

$$\sigma_{x_0}'(y) = \left(0, 1, \frac{\pi}{3}\sin\left(\frac{\pi x_0}{2}\right)\cos\left(\frac{\pi y}{3}\right)\right)$$

Hence at $x_0 = y_0 = 1 = x = y$:

$$\sigma_{y_0}' = (1, 0, 0)$$

$$\sigma_{x_0}' = (0, 1, \pi/6)$$

and $\therefore$ $\vec{n} = \frac{\sigma_{y_0}' \times \sigma_{x_0}'}{|\sigma_{y_0}' \times \sigma_{x_0}'|} = \frac{(0, -\pi/6, 1)}{\sqrt{1 + \pi^2/36}}$

Question 5

Prove that the equation $xy + yz + xz^3 = 1$ defines a surface in $\mathbb{R}^3$.

(10)

Define $g(x,y,z) = xy + yz + xz^3 - 1$

$$g_* = \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right]$$

$$= \left[y + z^3, x + z, y + 3xz^2\right]$$

Hence $g_* = 0 \iff$

$y + z^3 = 0$ —— ①

$x + z = 0$ —— ②

$y + 3xz^2 = 0$ —— ③.

① − ③ : $z^2(z - 3x) = 0$ i.e. $z = 0$ or $z = 3x$.

If $\underline{\underline{z=0}}$ then ② $\Rightarrow$ $\underline{\underline{x=0}}$ and ① $\Rightarrow$ $\underline{\underline{y=0}}$

If $z = 3x$ then ② $\Rightarrow$ $x = -3x$

$\Rightarrow x = 0$ Hence $\underline{\underline{z = -x = 0}}$ and $\underline{\underline{y = -z^3 = 0}}$

Thus the only solution to $g_* = 0$ is $\boxed{(x,y,z) = (0,0,0)}$

But $g(0,0,0) = -1$ i.e. $(0,0,0) \notin g^{-1}(0)$

Hence $g_* \neq 0$ anywhere on $g^{-1}(0)$

i.e. $g(x,y,z) = 0$ defines a surface