NAME: SOLUTIONS.

Math 423 - Midterm 2

November 5, 2008

- *Answer all questions.*
- *Be as clear as possible. Unless otherwise stated, give reasons for your answers.*
- *Use both sides of the paper, if necessary.*
- *Good luck!*

Question	Maximum	Your Score
1	15	
2	15	
3	20	
4	20	
5	15	
6	15	
TOTAL	100	

Question 1

(a) Let

$$\vec{Z}(x, y, z) = (yz, xz, xy)$$

be a vector field on $\mathbf{R}^3$. Compute the directional derivative of $\vec{Z}$ at the point $\vec{p} = (p_1, p_2, p_3)$ along the vector $\vec{V} = (1, 2, 3)$. That is, compute $\nabla_{\vec{V}} \vec{Z}(\vec{p})$.

(7)

$$\nabla_{\vec{V}} Z = \left(\nabla_{\vec{V}}(yz), \nabla_{\vec{V}}(xz), \nabla_{\vec{V}}(x \cdot y) \right)$$

$$= (v_2 z + v_3 y, v_1 z + v_3 x, v_1 y + v_2 x)$$

$$= (2z + 3y, z + 3x, y + 2x)$$

at $\vec{p}$ we get

$$\boxed{\nabla_{\vec{V}} \vec{Z}(\vec{p}) = (2p_3 + 3p_2, p_3 + 3p_1, p_2 + 2p_1).}$$

(b) Let $\vec{\alpha}(t)$ be a curve on a surface $M \subset \mathbf{R}^3$ and let $\vec{n}$ be a unit normal vector field defined along $\vec{\alpha}(t)$. Show that

$$\nabla_{\vec{\alpha}'(t)}\vec{n}(t) \cdot \vec{\alpha}'(t) = -\vec{n}(t) \cdot \vec{\alpha}''(t)$$

where $\vec{\alpha}'(t)$ and $\vec{\alpha}''(t)$ denote the velocity and acceleration vectors to the curve at the point $\vec{\alpha}(t)$, and $\vec{n}(t)$ denotes the value of the vector field at the point $\vec{\alpha}(t)$, i.e. $\vec{n}(t) = \vec{n}(\vec{\alpha}(t))$.

(8)

If $\vec{\alpha}'(t)$ is on M and $\vec{n}(t)$ is normal, then

$\vec{\alpha}'(t) \cdot \vec{n}(t) = 0 \quad \forall t.$

Take $\nabla_{\vec{\alpha}'(t)}$: $\quad (\nabla_{\vec{\alpha}'}\alpha') \cdot \vec{n} + \vec{\alpha}' \cdot (\nabla_{\alpha'}\vec{n}) = 0$

or $(\nabla_{\alpha'}\vec{n}) \cdot \vec{\alpha}' = - \vec{n} \cdot (\nabla_{\vec{\alpha}'}\vec{\alpha}')$

But $\nabla_{\vec{\alpha}'}\vec{\alpha}' = \vec{\alpha}''(t)$

Hence $\boxed{\nabla_{\alpha'}\vec{n} \cdot \vec{\alpha}' = -\vec{n} \cdot \alpha''}$

Question 2

(a) Show that the equation

$$\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{16} = 1$$

defines a surface

(5)

If $g(x, y, z) = \frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{16} - 1$

the $\vec{\nabla} g = \left(\frac{x}{2}, -\frac{2y}{9}, \frac{z}{8} \right)$

Hence $\vec{\nabla} g = 0 \iff x = y = z = 0$

But $(0,0,0)$ is not on the surface $g^{-1}(0)$.

Hence on $g^{-1}(0)$ we have $\vec{\nabla} g \neq 0$

i.e. g defines a surface

(b) Find a nowhere zero normal vector field for the surface defined in (a).

(5)

$\vec{z} = \vec{\nabla} g$ is nowhere zero and normal.

ie $\boxed{\vec{z} = \left(\frac{x}{2}, \frac{-2y}{9}, \frac{z}{8}\right)}$

(c) Show that $\vec{V}(x, y, z) = (-z, 0, 4x)$ is a tangent vector field on M .

(5)

$\vec{V}$ is tangent $\Leftrightarrow \vec{V} \cdot \vec{Z} = 0$

where $\vec{Z} = \left(\frac{x}{2}, \frac{-2y}{9}, \frac{z}{8} \right)$

But $\vec{V} \cdot \vec{Z} = -\frac{xz}{2} + 0 + \frac{xz}{2} = 0$ ✓

Question 3

Fill in the blanks:

(a) If $\vec{n}$ is a unit normal vector field on a neighborhood of point $\vec{p}$ on surface M, the **shape operator** at $\vec{p}$ is a linear map from T_pM to T_pM defined by

$$S_{\vec{p}}(\vec{v}) = -\nabla_{\vec{v}}\vec{n}$$

(4)

(b) At a point $\vec{p}$ on a surface M, a tangent direction $\vec{u} \in T_{\vec{p}}M$ is called a principal direction if $\vec{u}$ is an eigenvector of S_p (3)

(c) If k_1 and k_2 are the principal curvatures of a surface M at the point $\vec{p} \in M$, then the Gauss and mean curvatures at $\vec{p}$ are given by

$K =$ $k_1 k_2$ and $H =$ $\frac{1}{2}(k_1+k_2)$ (4)

(d) A point $\vec{p}$ on a surface M is called umbilic if $k_1(p) = k_2(p)$, i.e. $k(\vec{u}) = \text{const} \ \forall\, \vec{u} \in T_pM$. (3)

(e) A surface is called flat if $K=0$ (3)

(f) A surface is called a minimal surface if $H=0$ (3)

Question 4

Let M be the surface defined by

$$z = x^2 + 3xy - 5y^2$$

(a) Show that at $\vec{p} = (0,0,0)$ the unit vectors $\vec{u}_1 = (1,0,0)$ and $\vec{u}_2 = (0,1,0)$ define a basis for the tangent space to M.

(4)

With $f(x,y) = x^2+3xy-5y^2$, we get a basis for $T_{\vec{p}}M$ ~~at~~ at $\vec{p}=(x,y,f(x,y))$ from:

$$\vec{u}_1 = (1,0,f_x) = (1,0,2x+3y)$$

$$\vec{u}_2 = (0,1,f_y) = (0,1,3x-10y)$$

Thus at $\vec{p}=(0,0,0)$ we have

$$\vec{u}_1 = (1,0,0)$$

$$\vec{u}_2 = (0,1,0).$$