page 145 § 4.2

1 (a)

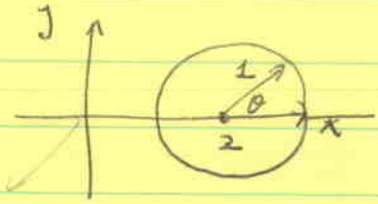
Curve in x-y plane:

$$\alpha(t) = (t, \cosh t) \qquad -\infty < t < \infty$$

Surface of revolution about x-axis; using s as angle of rotation we get

$$\boxed{\begin{gathered}\sigma(s,t) = (t, \cosh t \cos s, \cosh t \sin s) \\ -\infty < t < \infty \\ 0 < s < 2\pi .\end{gathered}}$$

(b) Curve in x-y plane:



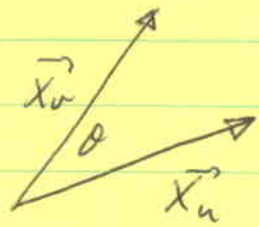
Parameterize by θ = angle to x-axis:

$$\boxed{\alpha(\theta) = (2+\cos\theta, \sin\theta)}$$

Surface of revolution about y axis: use ϕ as angle of rotation, to get

$$\boxed{\begin{gathered}\sigma(\phi,\theta) = ((2+\cos\theta)\cos\phi, \sin\theta, (2+\cos\theta)\sin\phi) \\ 0 < \phi < 2\pi \\ 0 < \theta < 2\pi .\end{gathered}}$$

② Here's one way:



$$\| X_u \times X_v \|^2 = \| X_u \|^2 \| X_v \|^2 \sin^2\theta \quad — ①$$

but $$X_u \cdot X_v = \| X_u \| \| X_v \| \cos\theta \quad — ②$$

Hence, by ②, $$X_u \cdot X_v^{\,2} = \| X_u \|^2 \| X_v \|^2 \cos^2\theta \quad — ③$$

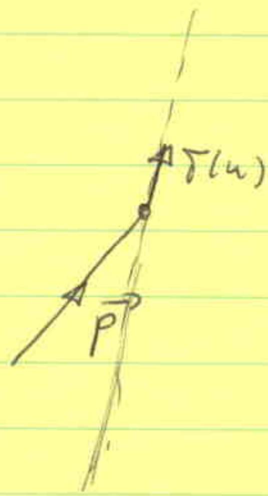
① + ③ gives $$\| X_u \times X_v \|^2 = \| X_u \|^2 \| X_v \|^2 - (X_u \cdot X_v)^2$$

Using the notation in the book:

$$\boxed{\| X_u \times X_v \|^2 = EG - F^2}$$

③ We assume $v > 0$

Let $\vec{\delta}(u) = (x(u), y(u), z(u))$



Then $$X_* = \begin{bmatrix} x'(u)\,v & x(u) \\ y'(u)\,v & y(u) \\ z'(u)\,v & z(u) \end{bmatrix}$$

which is regular iff the columns are linearly independent

i.e. iff the vectors $v\vec{\delta}'(u)$ and $\vec{\delta}(u)$ are linearly indep.

i.e. iff $v\vec{\delta}'(u) \times \vec{\delta}(u) \neq 0$.

i.e. iff $\boxed{v \neq 0 \text{ and } \vec{\delta}'(u) \times \vec{\delta}(u) \neq 0}$

(5) $$X_* = \begin{bmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \\ 0 & b \end{bmatrix}$$

We can check this is 1-1 by checking the columns are linearly independent. Denote the columns by X_v, X_u respectively. Check linear independence by computing

$$\vec{X}_v \times \vec{X}_u = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & b \end{vmatrix}$$

$= (b\sin v, -b\cos v, u)$ ← Never Zero!

(b) The curves $\underline{u = \text{constant}}$ are

$$\alpha_u(t) = X(u,t) = (u\cos t, u\sin t, bt)$$

which are <u>helices</u> of radius u and pitch b

The curves $v =$ constant are

$$\alpha_v(t) = ((\cos v)t, (\sin v)t, bv)$$

which are straight line segments in the plane $z = bv$ at angle $(\cos v)$ with respect to the x-axis.

(c) On H we have

$$\tan(z/b) = y/x$$

i.e. $$z - b\tan^{-1}(y/x) = 0$$

Define $\boxed{g(x,y,z) = z - b\tan^{-1}(y/x)}$

Then $H = g^{-1}(0)$

10 b

If $X(u,v) = (a(u+v), b(u-v), 4uv)$

Then $\frac{x^2}{a^2} - \frac{y^2}{b^2} = (u+v)^2 - (u-v)^2$

$$= 4uv$$

$$= z$$

Hence the image of $\mathbb{R}^2$ under X lies <u>in</u> M.

To see that the image includes <u>all</u> of M we must show that for any $(x,y,z) \in M$ we can solve

$$a(u+v) = x \quad \text{—} ①$$
$$b(u-v) = y \quad \text{—} ②$$
$$4uv = z \quad \text{—} ③$$

From ① + ② we get $\begin{cases} u = \frac{1}{2}(x/a + y/b) \\ v = \frac{1}{2}(x/a - y/b) \end{cases}$

This satisfies ③ $\iff 4 \cdot \frac{1}{4} \cdot \left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = z$

$$\iff \frac{x^2}{a^2} - \frac{y^2}{b^2} = z$$

$\iff (x,y,z)$ lies on M !