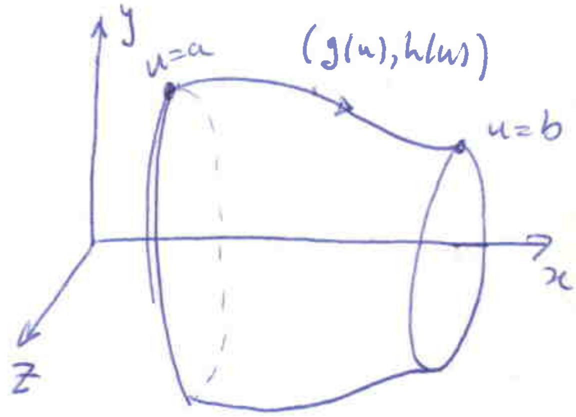
FOR HOMEWORK 10

Background for Ex 5 p 312



Patch:

$$\sigma(u,v) = (g(u), h(u)\cos v, h(u)\sin v). \quad (*)$$

Parallel at $u=a$:

$$\alpha(t) = (g(a), h(a)\cos t, h(a)\sin t).$$

$$\Longrightarrow \quad \alpha'(t) = (0, -h(a)\sin t, h(a)\cos t).$$

NOTE: This is not unit speed. To get unit speed parametrization take $\alpha(s) = \left(g(a), h(a)\cos\left(\frac{s}{h(a)}\right), h(a)\sin(s/h(a))\right)$.

Then $\boxed{\alpha'(s) = (0, -\sin(s/h(a)), \cos(s/h(a))) = T(s)}$

Unit normal: Using $(*)$ we get

$$\vec{n} = \frac{1}{\sqrt{g'^2+h'^2}}(h', -g'\cos v, -g'\sin v).$$

whence $\boxed{J(T) = \vec{n}\times\vec{T} = -\frac{h}{\sqrt{g'^2+h'^2}}(g', h'\cos v, h'\sin v)}$

(at $\sigma(a, s/h(a))$ i.e. at $u=a$, $v = s/h(a)$)

Direct computation yields k_g, where

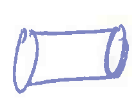
$$\alpha''(s) = k_g J(T) + k\vec{n}.$$

We get $\boxed{k_g(a,v) = \frac{h'(a)}{\sqrt{g'(a)^2+h'(a)^2}}.}$

p312 #5.

We use Gauss-Bonet on Z_{ab}. Since the boundary of Z_{ab} is smooth, we have

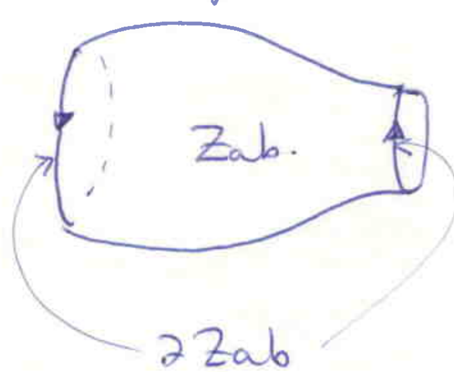
$$\iint_{Z_{ab}} K\, dA + \int_{\partial Z_{ab}} k_g = \chi(Z_{ab}).$$

But • $\chi(Z_{ab}) = \chi($  $) = 0$

cylinder

• ∂Z_{ab} has 2 pieces, the parallel through $(g(a), h(a), 0)$ and the parallel through $(g(b), h(b), 0)$.

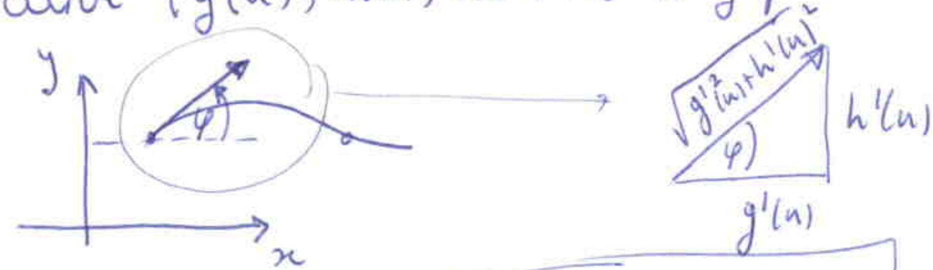
Notice that the 2 pieces have opposite orientation.



(In both cases we must pick the orientation so that Z_{ab} is on the left as you follow the curve).

Hence $$\iint_{Z_{ab}} K\,dA = -\int_{\partial Z_{ab}} k_g = -\left\{ -\int_0^{2\pi} \frac{h'(a)}{\sqrt{g'(a)^2 + h'(a)^2}}\, dv + \int_0^{2\pi} \frac{h'(b)\, dv}{\sqrt{g'(b)^2 + h'(b)^2}} \right\}$$

Look at the curve $(g(u), h(u))$ in the x-y plane:



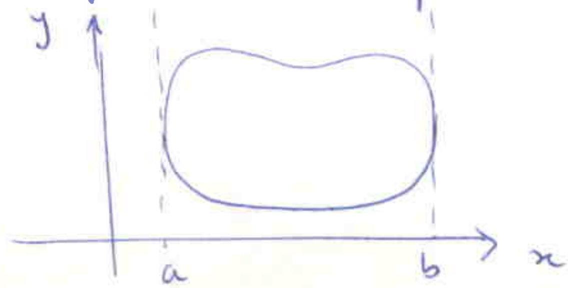
The diagram shows that $\boxed{\dfrac{h'(u)}{\sqrt{g'^2+h'^2}} = \sin\varphi}$.

Hence we get from Gauss-Bonet that

$$\boxed{\iint_{Z_{ab}} K\,dA = 2\pi\left\{\sin\varphi_a - \sin\varphi_b\right\}}$$

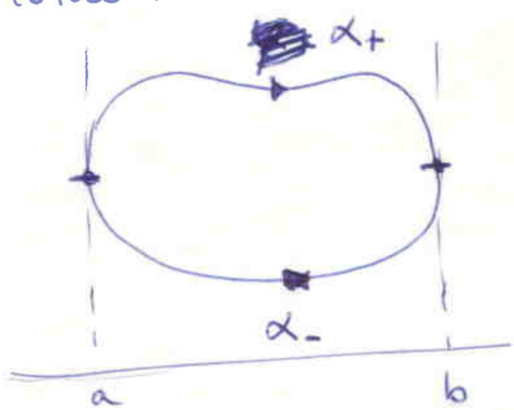
p313 #6

(a) We assume that the profile curve does not intersect the axis of rotation. If the curve is closed, ie. is a loop, then the picture is like this:



We can find boundary lines, say at $x=a$ and $x=b$, (as in the diagram) which are tangent to the curve.

Divide the curve into top and bottom:



The surface of revolution is then divided into 2 pieces, one generated by α_+ and one by α_-. Call these M_+ and M_-

Apply Problem 5 to each of M_+ and M_- and use the fact that α_+ and α_- have vertical tangents at $x=a,b$. Notice though that α_+ has slope angle $+\pi/2$ at $x=a$ and $(-\pi/2)$ at $x=b$, ~~while α_- has slope angle $(-\pi/2)$~~

while α_- has slope angle $(-\pi/2)$ at $x=a$ and $(\pi/2)$ at $x=b$.

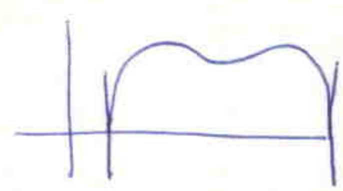
Hence

$$\iint_{M_+} k\,dA = 2\pi\left(\sin\pi/2 - \sin(-\pi/2)\right) = 4\pi \qquad (1)$$

$$\iint_{M_-} k\,dA = 2\pi\left(\sin(-\pi/2) - \sin\pi/2\right) = -4\pi. \qquad (2)$$

So $$\iint_M k\,dA = \iint_{M_+} k\,dA + \iint_{M_-} k\,dA = 0.$$

If the ~~Bo~~ profile curve intersects the axis at 2 places, and does so at right angles then the same calculation as we did to get (1) shows that



$$\iint_M k\,dA = 4\pi \quad \text{in this case.}$$

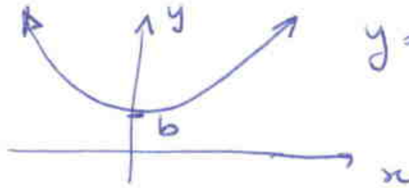
Remark: You could, alternatively, argue that the profile curve is a loop and hence the result of problem 5 can be applied with $a=b$, i.e.

$$\iint_M K\,dA = 2\pi\{\sin\varphi(a)-\sin\varphi(a)\}=0$$

(but then you have to work harder to prove the result for augmented surfaces of revolution!).

(b)

(i) Profile curve:



$y = ax^2 + b$.

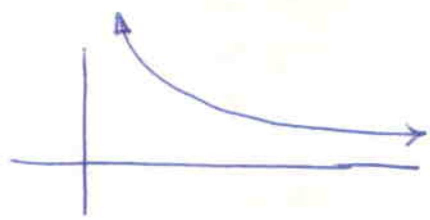
As $x \to -\infty$, the slope angle $\to -\pi/2$.

$x \to +\infty$ --- $\to +\pi/2$.

Hence $\iint_M \kappa \, dA = 2\pi\left(-1 - (+1)\right) = \underline{\underline{-4\pi}}$.

(ii) Similarly for the catenoid, $\iint_M \kappa \, dA = -4\pi$.

(iii) For the bugle surface



the slope angle $\to 0$ as $x \to \infty$

So $\iint_M \kappa \, dA = 2\pi(-1) = \underline{\underline{-2\pi}}$.