$\boxed{6}$ (a) By Exercise 8 (p 68) – or by exercise 7 (p 128) –

a unit speed plane curve with curvature $\tilde{\kappa}(s)$ is given by

$$\beta(s) = \left(\int \cos \varphi(t)\, dt,\ \int \sin(\varphi(t))\, dt \right)$$

where $\varphi(t) = \int \tilde{\kappa}(x)\, dx$.

The ambiguity in these formulae is due to the constants of integration. Thus if $\alpha(s)$ and $\beta(s)$ are unit speed curves with the same curvature, then

$$\boxed{\varphi_\beta(t) = \varphi_\alpha(t) + \delta}$$ for some fixed δ

Now use
$$\int \cos(\varphi_\beta(t))\, dt = \int \cos(\varphi_\alpha(t) + \delta)\, dt$$
$$= \cos\delta \left(\int \cos(\varphi_\alpha(t))\, dt \right) - \sin\delta \int \sin \varphi_\alpha(t)\, dt$$

and
$$\int \sin(\varphi_\beta(t))\, dt = \sin\delta \left(\int \cos \varphi_\alpha(t)\, dt \right) + \cos\delta \left(\int \sin \varphi_\alpha(t)\, dt \right)$$

Hence, if $\alpha(s) = (x_\alpha(s), y_\alpha(s))$, then
$$\beta(s) = \Big(\cos\delta\, (x_\alpha(s) + a) - \sin\delta\, (y_\alpha(s) + b),$$
$$\sin\delta\, (x_\alpha(s) + a) + \cos\delta\, (y_\alpha(s) + b) \Big)$$

where a, b are constants of integration from $\int \cos\varphi(t)\,dt$ and $\int \sin\varphi(t)\,dt$ respectively.

In "column notation":

$$\begin{bmatrix} x_\beta \\ y_\beta \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix}}_{\text{Rotation thru' } \delta} \underbrace{\begin{bmatrix} x_\alpha + a \\ y_\alpha + b \end{bmatrix}}_{\text{Translation by } (a,b).}$$

Thus curves α and β are related by an isometry, i.e. they are congruent!

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For a unit speed plane curve, say

$$\beta(s) = (x(s), y(s)) \quad s \in I$$

we can write

$$T(s) = (x'(s), y'(s))$$
$$= (\cos\varphi(s), \sin\varphi(s))$$

Then $$\boxed{\kappa(s) = \varphi'(s)}$$

Hence to have $\kappa(s) = f(s)$ we must have

$$\boxed{\varphi(s) = \int_0^t f(t)\,dt}$$

Then $$x'(s) = \cos\varphi(s) \Rightarrow x(s) = \int \cos\varphi(t)\,dt$$

$$y'(s) = \sin\varphi(s) \Rightarrow y(s) = \int \sin\varphi(t)\,dt.$$

Thus the required curve is

$$\boxed{\begin{gathered}\beta(s) = \left(\int \cos\varphi(t)\,dt, \int \sin\varphi(t)\,dt\right) \\ \text{where } \varphi(s) = \int_0^t f(t)\,dt.\end{gathered}}$$