p 157 no 7

Let $f(x,y) = xy$.

Then M is given by $z = f(x,y)$. Hence we have a normal vector field

$$\vec{n} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right) = (y, x, -1).$$

We get tangent vectors

$$\vec{\sigma_y}' = \left(1, 0, \frac{\partial f}{\partial x}\right) = (1, 0, y)$$
$$\vec{\sigma_x}' = \left(0, 1, \frac{\partial f}{\partial y}\right) = (0, 1, x).$$

NOTE: ① $\vec{\sigma_y}' \cdot \vec{n} = 0 = \vec{\sigma_x}' \cdot \vec{n}$ (as we expect!)

② $\vec{\sigma_y}'$ and $\vec{\sigma_x}'$ are linearly independent but not necessarily orthogonal since

$$\vec{\sigma_y}' \cdot \vec{\sigma_x}' = xy.$$

We can ~~[illegible]~~ get an orthonormal basis for T_pM by taking

$$\vec{e}_1 = \frac{\vec{\sigma_y}'}{|\vec{\sigma_y}'|} = (1, 0, y)/\sqrt{1+y^2}$$

$$\vec{e}_2 = \frac{\vec{\sigma_x}' - (\vec{\sigma_x}' \cdot \vec{e}_1)\vec{e}_1}{|\vec{\sigma_x}' - (\vec{\sigma_x}' \cdot \vec{e}_1)\vec{e}_1|} = \ldots\ldots \text{(you do the math!)}$$

p157 10

If M is defined by $g(x,y,z)=0$ then

$$\vec{n} = \frac{\vec{\nabla g}}{|\vec{\nabla g}|} \text{ is a unit normal.}$$

The ~~the~~ tangent plane at $\vec{p} \in M$ is thus the plane

- Through $\vec{p}$
- orthogonal to $\vec{n} = \frac{\vec{\nabla g}}{|\vec{\nabla g}|}$.

(b) $\quad g(x,y,z) = \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{18} - 1$

$$\Rightarrow \quad \vec{\nabla g} = \left(\frac{x}{2}, \frac{y}{8}, \frac{z}{9} \right)$$

$\therefore$ At $\vec{p} = (1,-2,3)$ $\quad \vec{n} = \left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}\right) \Big/ \sqrt{\frac{1}{4}+\frac{1}{16}+\frac{1}{9}}$

The equation defining T_pM is thus

$$\frac{x}{2} - \frac{y}{4} + \frac{z}{3} = d$$

with $\quad d = \left(\frac{1}{2}\right) - \left(\frac{-2}{4}\right) + \left(\frac{3}{3}\right) = 2$

So eqn is $\quad \boxed{6x - 3y + 4z = 24}$

p173 #2

Stereographic Projection is defined by (see p167).

$$p(x,y,z) = \left(\frac{2x}{2-z}, \frac{2y}{2-z} \right)$$

Hence if $p^{-1}(u,v) = \dfrac{(4u, 4v, 2f)}{f+4}$ $\quad f = u^2+v^2$

then:

$$pp^{-1}(u,v) = \left(\frac{8u/f+4}{2-\left(\frac{2f}{f+4}\right)}, \frac{8v/f+4}{2-\left(\frac{2f}{f+4}\right)} \right) = (u,v). \quad \text{— (1)}$$

and $p^{-1}p(x,y,z) = \dfrac{\left(\frac{8x}{2-z}, \frac{8y}{2-z}, 2f\right)}{f+4}$ —— (*)

In (*) $f = \left(\frac{2x}{2-z}\right)^2 + \left(\frac{2y}{2-z}\right)^2 = \dfrac{4(x^2+y^2)}{(2-z)^2}$.

~~Thus~~ But if (x,y,z) lies on the unit sphere with center $(0,0,1)$, then

$$x^2+y^2+(z-1)^2 = 1$$

$$\iff x^2+y^2 = z(2-z)$$

Hence $f = \dfrac{8z}{2-z}$, i.e. $\boxed{f+4 = \frac{8}{2-z}}$, and

(*) $\Rightarrow$ $p^{-1}p(x,y,z) = (x,y,z)$ —— (2).

(1) + (2) $\Rightarrow$ p^{-1} is the inverse of p

p207 No 3

Use $S = \text{Hess} f = \begin{bmatrix} f_{xx}^{0} & f_{xy}^{0} \\ f_{xy}^{0} & f_{yy}^{0} \end{bmatrix}$ (at origin)

(a) $f(x,y) = xy \implies S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Since $\det S \neq 0$, S is invertible $\implies$ $\text{Rank}\, S = 2$.

(b) $f(x,y) = 2x^2 + y^2 \implies S = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

$\det S \neq 0 \implies$ $\text{Rank}\, S = 2$

(c) $f(x,y) = (x+y)^2 \implies S = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$= x^2 + 2xy + y^2$

$\det S = 0 \implies \text{Rank}\, S < 2$

In fact $S \begin{bmatrix} a \\ b \end{bmatrix} = 2(a+b) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so the image of

$S: \mathbb{R}^2 \to \mathbb{R}^2$ is $\text{Im}\, S = \left\{ \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \lambda \in \mathbb{R} \right\}$.

$\text{Rank}\, S = 1$

(d) $f(x,y) = xy^2 \Rightarrow S = \begin{bmatrix} 0 & 2y \\ 2y & 2x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ at $(0,0)$.

so Rank $S = 0$