HWK 6 – Selected Solutions

$\boxed{\text{page 207 \# 6}}$

If $f(x,y) = xy$ then the unit normal for the surface $z = f(x,y)$ is

$$\vec{n} = (-f_x, -f_y, 1) \,/\, (1 + f_x^2 + f_y^2)$$

$$= (-y, -x, 1) \,/\, \sqrt{1 + x^2 + y^2}$$

The Gauss Map is thus

$$G(x, y, xy) = (-y, -x, 1) \,/\, \sqrt{1 + x^2 + y^2}$$

If we fix $y = y_0$, we get the curve

$$\gamma(t) \equiv G(t, y_0, t y_0) = (-y_0, t, 1) \,/\, \sqrt{1 + t^2 + y_0^2}$$

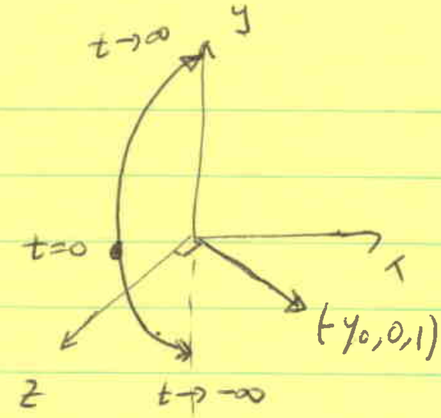
NOTICE THAT $\gamma(t) = (x(t), y(t), z(t))$ satisfies

$$x(t) + y_0 z(t) = 0$$

ie. the curve $\gamma(t)$ lies in the plane $\boxed{x + y_0 z = 0}$.

It is, in fact, on the <u>great circle</u> given by the <u>intersection</u> of this plane with the unit circle.

We get $\quad \gamma(0) = \left(\dfrac{-y_0}{\sqrt{1+y_0^2}}, 0, \dfrac{1}{\sqrt{1+y_0^2}} \right)$



$$\lim_{t \to \pm\infty} \gamma(t) = (0, \pm 1, 0).$$

As y_0 varies, we get half "line of longitude" (with $(0,1,0)$ as the North Pole). Since these cove all of S^2, we see that <u>the image of G</u> is the <u>entire</u> <u>sphere</u>.

Page 216 No. 4

(a) With $f(x,y) = e^{x^2+y^2} - 1$, $z = f(x,y)$ and hence

$$S = \text{Hess} f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{at } (0,0).$$

i.e. $k_1 = k_2 = 2$ at the origin and hence the quadratic approximation is

$$z = x^2 + y^2$$

(b) $f(x,y) = \log \cos x - \log \cos y$

$$\Rightarrow S = \begin{bmatrix} -\sec^2 x & 0 \\ 0 & -\sec^2 y \end{bmatrix} = -I \quad \text{at } (0,0)$$

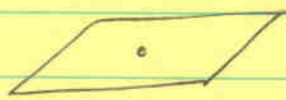
$\Rightarrow$ quadratic approx. is $z = -\frac{1}{2}(x^2+y^2)$.

(c) $f(x,y) = (x+3y)^3$

$$\Rightarrow \frac{\partial f}{\partial x} = 3(x+3y)^2 \, ; \quad \frac{\partial f}{\partial y} = 9(x+3y)^2$$

$$\Rightarrow S = \begin{bmatrix} 6(x+3y) & 18(x+3y) \\ 18(x+3y) & 54(x+3y) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ at } (0,0).$$

So the quadratic approx. is $z = 0$



$\boxed{\text{p222 \#5}}$

Case (i): Write $\boxed{k_1 = a^2,\ k_2 = b^2}$

(The case $k_1 < 0$ and $k_2 < 0$ is analogous)

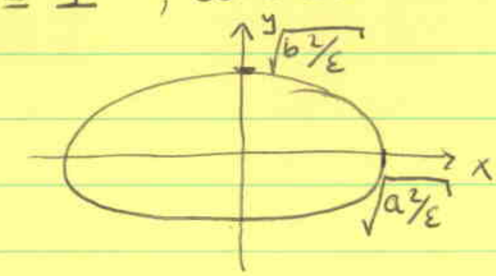
C_0: $a^2x^2 + b^2y^2 = 0$ has solution $\boxed{x = y = 0}$

Thus the curve is one point viz. $(0,0)$.

C_ε: $a^2x^2 + b^2y^2 = \varepsilon$

$\iff \dfrac{x^2}{(\varepsilon/a^2)} + \dfrac{y^2}{(\varepsilon/b^2)} = 1$, which describes

an ellipse

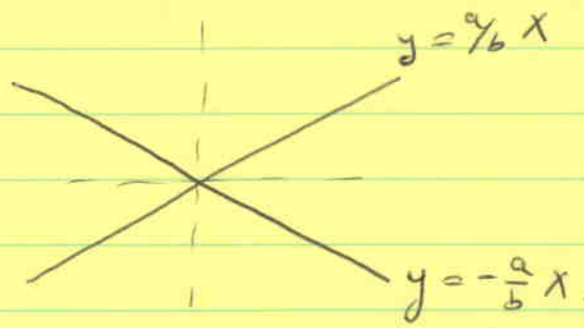


$C_{-\varepsilon}$: No solutions.

Case (ii): $\boxed{\text{Write } k_1 = a^2,\ k_2 = -b^2.}$

C_0: $a^2x^2 - b^2y^2 = 0$

$\iff y = \pm\left(\frac{a}{b}\right)x$ i.e.

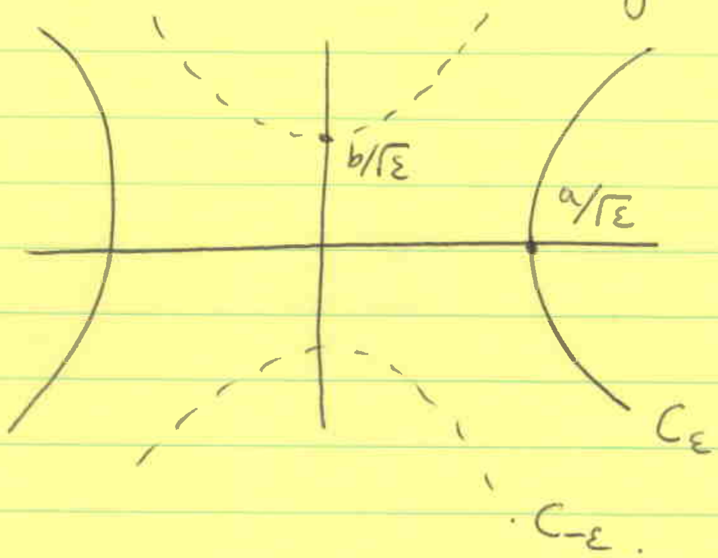


C_ε: $a^2x^2 - b^2y^2 = \varepsilon$

$\iff \dfrac{x^2}{\varepsilon/a^2} - \dfrac{y^2}{\varepsilon/b^2} = 1$, which describes a hyperbola which intersects the x-axis.

$C_{-\varepsilon}$: This gives $\frac{-x^2}{\varepsilon/a^2} + \frac{y^2}{\varepsilon/b^2} = 1$, which is

a hyperbola that intersects the y-axis.

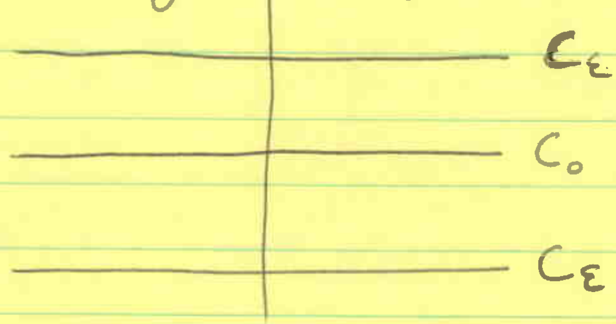


<u>Case (iii)</u>. Take $k_1 = 0$, $k_2 \neq 0$.

C_0: $\quad y^2 = 0 \implies y = 0$ i.e. x-axis

$C_{\pm\varepsilon}$: $\quad y^2 = \pm\varepsilon/k_2 \implies y = \begin{cases} \pm\sqrt{\varepsilon/k_2} & \text{if } k_2 > 0 \\ \pm\sqrt{-\varepsilon/k_2} & \text{if } k_2 < 0. \end{cases}$

Thus if $k_2 > 0$ we get 2 lines $\left(y = \pm\sqrt{\varepsilon/k_2}\right)$ for C_ε and <u>nothing for $C_{-\varepsilon}$</u>:



(Similarly for $k_2 < 0$)