HOMEWOR 8

p280 #1

If $K = H = 0$ then the principal curvatures k_1, k_2 must satisfy

$$k_1 k_2 = 0 \quad — ①$$

$$k_1 + k_2 = 0 \quad — ②$$

It follows that $k_1 = k_2 = 0$ and hence the Shape Operator satisfies

$$S = 0$$

It now follows from Theorem 3.1 (p274) or by what we did in class, that the surface is part of a plane.

p286 #2

(a) Let $\sigma(u,v)$ be a patch for M. To show

$$F : M \longrightarrow N$$

is an isometry, we know that it's enough to show that

$$\left| F_*\left(\frac{\partial \vec{\sigma}}{\partial u}\right)\right| = \left|\frac{\partial \vec{\sigma}}{\partial u}\right|$$

$$\left| F_*\left(\frac{\partial \vec{\sigma}}{\partial v}\right)\right| = \left|\frac{\partial \vec{\sigma}}{\partial v}\right|$$

$$F_*\left(\frac{\partial \vec{\sigma}}{\partial u}\right)\cdot F_*\left(\frac{\partial \vec{\sigma}}{\partial v}\right) = \frac{\partial \vec{\sigma}}{\partial u}\cdot\frac{\partial \vec{\sigma}}{\partial v}$$

But if $\alpha(t)$ is any curve in M then on the open set covered by the patch σ we can write

$$\vec{\alpha}(t) = \sigma(u(t), v(t))$$

whence $\vec{\alpha}'(t) = u'\frac{\partial \vec{\sigma}}{\partial u} + v'\frac{\partial \vec{\sigma}}{\partial v}$

Also $F(\alpha) = F(\sigma(u(t), v(t)))$

$$\Longrightarrow \overrightarrow{F(\alpha)}' = u' F_*\left(\frac{\partial \vec{\sigma}}{\partial u}\right) + v' F_*\left(\frac{\partial \vec{\sigma}}{\partial v}\right) \quad (*)$$

Hence $\boxed{|F(\alpha)'|^2 = u'^2\left|F_*\left(\frac{\partial \sigma}{\partial u}\right)\right|^2 + 2u'v'\left(F_*\frac{\partial \sigma}{\partial u}\cdot F_*\frac{\partial \sigma}{\partial v}\right) + v'^2\left|F_*\left(\frac{\partial \sigma}{\partial v}\right)\right|^2}$

The result follows from $\circledast$:

Suppose F is an isometry. Then for any $\alpha(t)$ $(*)$ yields

$$|F(\alpha)'|^2 = |\alpha'|^2$$

Suppose $|F(\alpha)'|^2 = |\alpha'|^2$ for all $\alpha(t)$.

Then we can apply $(*)$ to the curves

$$\alpha_1(t) = \sigma(t, v_0)$$
$$\alpha_2(t) = \sigma(u_0, t)$$

to get $\left|F_*\left(\frac{\partial\vec{\sigma}}{\partial u}\right)\right| = \left|\frac{\partial\vec{\sigma}}{\partial u}\right|$ — ①

$$\left|F_*\left(\frac{\partial\vec{\sigma}}{\partial v}\right)\right| = \left|\frac{\partial\vec{\sigma}}{\partial v}\right|, \quad — ②$$

Finally, using ① and ② we can deduce that

$$F_*\left(\frac{\partial\vec{\sigma}}{\partial u}\right) \cdot F_*\left(\frac{\partial\vec{\sigma}}{\partial v}\right) = \frac{\partial\vec{\sigma}}{\partial u} \cdot \frac{\partial\vec{\sigma}}{\partial v}$$