

Math 510 - Fall 2006

Algebraic Curves and Riemann Surfaces

Problem Set 1

Question 1

Writing $z = re^{i\theta}$ with $0 \leq \theta < 2\pi$, define a square root function by

$$\sqrt{z} = \sqrt{r}e^{i\theta/2} .$$

Prove that \sqrt{z} is not continuous as a map from \mathbb{C} to \mathbb{C} .

Question 2

Let $f(x, y)$ be a non-constant polynomial in two complex variables and let $C_f \subset \mathbb{C}^2$ be the curve defined by $f(x, y) = 0$. Prove (correctly - unlike in class!) that C_f is not a compact subset of \mathbb{C}^2 .

Question 3

(a) Show, by explicitly describing a diffeomorphism, that as a smooth manifold $\mathbb{C}\mathbb{P}^1$ is diffeomorphic to S^2 .

(b) Let

$$\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$$

be the lattice defined by $\omega_1, \omega_2 \in \mathbb{C}$, and let $T_\Lambda = \mathbb{C}/\Lambda$ be the corresponding torus. Construct an atlas of coordinate charts on T_Λ and hence prove that it is a (compact) one-dimensional complex manifold, i.e. prove that it is a Riemann surface.

Question 4

Let $f(x, y)$ and $g(x, y)$ be polynomials with complex coefficients. Show that the following are equivalent:

- f and g have the same irreducible factors (possibly with different multiplicities)
- There are positive integers m and n such that f divides g^m and g divides f^n .

Question 5

(a) Show that if $f(x, y)$ is a (non-trivial) homogeneous polynomial of degree d then it factors as a product of linear polynomials

$$f(x, y) = \prod_{i=1}^d (\alpha_i x + \beta_i y)$$

for some $\alpha_i, \beta_i \in \mathbb{C}$.

(b) Let $F(x, y, z)$ be a non-trivial homogeneous polynomial of degree d , and let C_F be the curve in \mathbb{CP}^2 defined by $F(x, y, z) = 0$. If $\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1$, where $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is the ‘line at infinity’ defined by $z = 0$, show that $C_F \cap \mathbb{CP}^1$ is either a finite set of points or the entire \mathbb{CP}^1 .

(c) Let $f(x, y) = y^2 - 2x^2 - xy + 5x$, and let $C_f = f^{-1}(0)$ be the algebraic curve in \mathbb{C}^2 defined by f . Let C_F be the corresponding curve in \mathbb{CP}^2 , i.e. let C_F be the compactification of C_f obtained by the usual compactification $\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1$. Find all the points in $C_F - C_f$ and hence all the lines in \mathbb{C}^2 on which the intersection with C_f has less than two points.